Delay-Coupled Mathieu Equations in Synchrotron Dynamics Revisited: Delay Terms in the Slow Flow

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Abstract

In a previous work, we applied perturbation methods to a system of two delay-coupled Mathieu equations, resulting in a slow flow which contains delayed variables. This previous treatment involved a convenient approximation which involved replacing delay terms in the slow flow by non-delay terms. The current paper explores the effect of keeping delay terms in the slow flow with the hope of illustrating what is lost in making such an approximation. Analytic results are shown to compare favorably with numerical integration of the slow flow itself.

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1 Introduction

In this paper we investigate the dynamics of the following system of two coupled Mathieu equations:

$$\ddot{x} + (\delta + \varepsilon \cos t)x + \varepsilon \gamma x^3 + \varepsilon \mu \dot{x} = \varepsilon \beta (x_d + y_d) \tag{1}$$

$$\ddot{y} + (\delta + \varepsilon \cos t)y + \varepsilon \gamma y^3 + \varepsilon \mu \dot{y} = \varepsilon \beta (x_d + y_d) + \varepsilon \alpha x \tag{2}$$

Here, $x_d \equiv x(t-T)$ and $y_d \equiv y(t-T)$. T is a constant which we will refer to as the delay of the system, and any term containing T in its argument, e.g. x_d, we will refer to as a delay term.

Coupled Mathieu equations without delay have been studied previously [1], [2]. In our previous paper [3], we explored the effect of including delay in eqs. (1), (2), but used a convenient approximation to replace the delay terms in the resulting slow flow with non-delay terms. Recent research [4], [5] suggests that keeping the delay terms in the slow flow can result in more accurate analytic solutions.

In this paper, we will use the method of two-variable expansion to derive the slow flow of the system. We will investigate both the case when delay terms are replaced with non-delay terms and the case

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when the delay terms are kept in the slow flow, and compare these results with each other as well as numerical integration of the slow flow. Percent error calculations will then be used to measure the error associated with approximating the delay terms with non-delay terms.

1.1 Application

Motivation for studying this system comes from the field of accelerator physics [6]. A synchrotron is a type of particle accelerator that uses magnets arranged periodically in a circle to maintain the velocity of the charged particles moving along the circular trajectory. The charged particles are in collections, called bunches, and they leave wake fields behind them that can persist through entire orbits.

The system (1), (2) models the dynamics of two bunches, where the parametric excitation represents the forcing of the magnets, the α terms represent the coupling between one bunch and the next, and the β terms represent the delayed self-feedback of the bunches as they pass through their wake fields from one orbit ago.

2 Two-Variable Expansion

We use the two-variable expansion method [7], [8] to study the dynamics of eqs. (1), (2). We set

$$\xi(t) = t, \quad \eta(t) = \varepsilon t$$

where ξ is the time t and η is the slow time.

Since x and y are functions of ξ and η, the derivative with respect to time t is expressed through the chain rule:

$$\dot{x} = x_\xi + \varepsilon x_\eta, \quad \dot{y} = y_\xi + \varepsilon y_\eta$$

Similarly, for the second derivative we obtain:

$$\ddot{x} = x_{\xi\xi} + 2\varepsilon x_{\xi\eta} + \varepsilon^2 x_{\eta\eta}, \quad \ddot{y} = y_{\xi\xi} + 2\varepsilon y_{\xi\eta} + \varepsilon^2 y_{\eta\eta}$$

In this paper we only perturb up to $O(\varepsilon)$, and so we will ignore the ε^2 terms.

We then expand x and y in a power series in ε:

$$x(\xi,\eta) = x_0(\xi,\eta) + \varepsilon x_1(\xi,\eta) + O(\varepsilon^2), \quad y(\xi,\eta) = y_0(\xi,\eta) + \varepsilon y_1(\xi,\eta) + O(\varepsilon^2) \tag{3}$$

In addition, we detune off of the 2:1 subharmonic resonance by setting:

$$\delta = \frac{1}{4} + \varepsilon \delta_1 + O(\varepsilon^2) \tag{4}$$

Substituting (3), (4) into (1), (2) and collecting terms in ε, we arrive at the following equations:

$$x_{0,\xi\xi} + \frac{1}{4}x_0 = 0 \tag{5}$$

$$y_{0,\xi\xi} + \frac{1}{4}y_0 = 0 \tag{6}$$

$$x_{1,\xi\xi} + \frac{1}{4}x_1 = -2x_{0,\xi\eta} - \mu x_{0,\xi} - \delta_1 x_0 - x_0 \cos\xi - \gamma x_0^3 + \beta(x_{0_d} + y_{0_d}) \tag{7}$$

$$y_{1,\xi\xi} + \frac{1}{4}y_1 = -2y_{0,\xi\eta} - \mu y_{0,\xi} - \delta_1 y_0 - y_0 \cos\xi - \gamma y_0^3 + \beta(x_{0_d} + y_{0_d}) + \alpha x_0 \tag{8}$$

The solutions to (5) and (6) are simply:

$$x_0 = A(\eta)\cos\left(\frac{\xi}{2}\right) + B(\eta)\sin\left(\frac{\xi}{2}\right) \tag{9}$$

$$y_0 = C(\eta)\cos\left(\frac{\xi}{2}\right) + D(\eta)\sin\left(\frac{\xi}{2}\right) \tag{10}$$

We then substitute (9), (10) into (7), (8). Note that:

$$x_{0_d} = A(\eta - \varepsilon T)\cos\left(\frac{\xi}{2} - \frac{T}{2}\right) + B(\eta - \varepsilon T)\sin\left(\frac{\xi}{2} - \frac{T}{2}\right) \tag{11}$$

$$y_{0_d} = C(\eta - \varepsilon T)\cos\left(\frac{\xi}{2} - \frac{T}{2}\right) + D(\eta - \varepsilon T)\sin\left(\frac{\xi}{2} - \frac{T}{2}\right) \tag{12}$$

Using trigonometric identities, these equations can be written in terms of $\cos\frac{\xi}{2}$ and $\sin\frac{\xi}{2}$. We set the coefficients of such terms equal to zero in order to remove the secular terms and avoid resonance. This results in four delay-differential equations in four unknowns:

$$A' = -\beta\sin\left(\frac{T}{2}\right)(A_d + C_d) - \beta\cos\left(\frac{T}{2}\right)(B_d + D_d) - \frac{\mu}{2}A + \left(\delta_1 - \frac{1}{2}\right)B + \frac{3\gamma B}{4}\left(A^2 + B^2\right) \tag{13}$$

$$B' = -\beta\sin\left(\frac{T}{2}\right)(B_d + D_d) + \beta\cos\left(\frac{T}{2}\right)(A_d + C_d) - \frac{\mu}{2}B - \left(\delta_1 + \frac{1}{2}\right)A - \frac{3\gamma A}{4}\left(A^2 + B^2\right) \tag{14}$$

$$C' = -\beta\sin\left(\frac{T}{2}\right)(A_d + C_d) - \beta\cos\left(\frac{T}{2}\right)(B_d + D_d) - \frac{\mu}{2}C + \left(\delta_1 - \frac{1}{2}\right)D + \frac{3\gamma D}{4}\left(C^2 + D^2\right) - \alpha B \tag{15}$$

$$D' = -\beta\sin\left(\frac{T}{2}\right)(B_d + D_d) + \beta\cos\left(\frac{T}{2}\right)(A_d + C_d) - \frac{\mu}{2}D - \left(\delta_1 + \frac{1}{2}\right)C - \frac{3\gamma C}{4}\left(C^2 + D^2\right) + \alpha A \tag{16}$$

It is common in the literature [9] to employ the following approximation here:

$$A_d = A(\eta - \varepsilon T) = A(\eta) - \varepsilon T A' + O(\varepsilon^2) = A(\eta) + O(\varepsilon)$$

We will briefly go over this case in the next section to demonstrate its results. However, the point of this paper is to keep the delay terms A_d, B_d, C_d, D_d in the slow flow and observe the difference in slow flow dynamics.

3 Analytic Results

The present authors are interested in analyzing the stability of the origin of the original system (1), (2), which also corresponds to the origin of the slow flow (13), (14), (15), (16). To help accomplish this goal, we will linearize the slow flow around the origin and analyze the stability of that system, since the stability of the linearized system will be the same as the stability of the nonlinear system.

Thus, the slow flow becomes:

$$A' = -\beta \sin\left(\frac{T}{2}\right)(A_d + C_d) - \beta \cos\left(\frac{T}{2}\right)(B_d + D_d) - \frac{\mu}{2}A + \left(\delta_1 - \frac{1}{2}\right)B \tag{17}$$

$$B' = -\beta \sin\left(\frac{T}{2}\right)(B_d + D_d) + \beta \cos\left(\frac{T}{2}\right)(A_d + C_d) - \frac{\mu}{2}B - \left(\delta_1 + \frac{1}{2}\right)A \tag{18}$$

$$C' = -\beta \sin\left(\frac{T}{2}\right)(A_d + C_d) - \beta \cos\left(\frac{T}{2}\right)(B_d + D_d) - \frac{\mu}{2}C + \left(\delta_1 - \frac{1}{2}\right)D - \alpha B \tag{19}$$

$$D' = -\beta \sin\left(\frac{T}{2}\right)(B_d + D_d) + \beta \cos\left(\frac{T}{2}\right)(A_d + C_d) - \frac{\mu}{2}D - \left(\delta_1 + \frac{1}{2}\right)C + \alpha A \tag{20}$$

As eqs. (17), (18), (19), (20) are linear in A, B, C, and D, we know that the general solution will be a linear combination of exponential functions. Thus, for instance, $A = C_1 e^{\lambda \eta}$ and its derivative becomes:

$$A' = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(C_1 e^{\lambda \eta}\right) = C_1 \lambda e^{\lambda \eta}$$

In addition, the delay term becomes:

$$A_d = A(\eta - \varepsilon T) = C_1 e^{\lambda \eta - \varepsilon \lambda T} = C_1 e^{\lambda \eta} e^{-\varepsilon \lambda T}$$

Substituting these expressions into eqs. (17), (18), (19), (20) and expressing the system in matrix form, we obtain:

$$\begin{bmatrix} -\beta v \mathscr{S} - \frac{\mu}{2} - \lambda & -\beta v \mathscr{C} + \delta_1 - \frac{1}{2} & -\beta v \mathscr{S} & -\beta v \mathscr{C} \\ \beta v \mathscr{C} - \delta_1 - \frac{1}{2} & -\beta v \mathscr{S} - \frac{\mu}{2} - \lambda & \beta v \mathscr{C} & -\beta v \mathscr{S} \\ -\beta v \mathscr{S} & -\beta v \mathscr{C} - \alpha & -\beta v \mathscr{S} - \frac{\mu}{2} - \lambda & -\beta v \mathscr{C} + \delta_1 - \frac{1}{2} \\ \beta v \mathscr{C} + \alpha & -\beta v \mathscr{S} & \beta v \mathscr{C} - \delta_1 - \frac{1}{2} & -\beta v \mathscr{S} - \frac{\mu}{2} - \lambda \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

Where $\mathscr{S} = \sin\left(\frac{T}{2}\right), \mathscr{C} = \cos\left(\frac{T}{2}\right), v = e^{-\varepsilon \lambda T}$.

Our goal is to compare two approaches: 1) Replacing delay terms by non-delay terms, e.g. A_d by A, versus 2) Analyzing the system with the delay terms. Replacing delayed terms with non-delayed terms is equivalent to setting $v = 1$, and this can be achieved by setting $\varepsilon = 0$ in the slow flow.

To obtain nontrivial solutions we set the determinant of this matrix equal to zero, resulting in a characteristic equation of the form:

$$\lambda^4 + p\lambda^3 + q\lambda^2 + r\lambda + s = 0 \tag{22}$$

where

$$p = 4\beta v \mathscr{S} + 2\mu \tag{23}$$

$$q = 6\beta \mu v \mathscr{S} - 4\beta \delta_1 v \mathscr{C} + 2\alpha \beta v \mathscr{C} + 4\beta^2 v^2 + \frac{3\mu^2}{2} + 2\delta_1^2 - \frac{1}{2} \tag{24}$$

$$r = 4\beta \delta_1^2 v \mathscr{S} + 4\alpha \beta \delta_1 v \mathscr{S} + 3\beta \mu^2 v \mathscr{S} - \beta v \mathscr{S} - 4\beta \delta_1 \mu v \mathscr{C} + 2\alpha \beta \mu v \mathscr{C} + 4\beta^2 \mu v^2 + \frac{\mu^3}{2} + 2\delta_1^2 \mu - \frac{\mu}{2} \tag{25}$$

$$s = 2\beta\delta_1^2\mu v\mathscr{S} + 2\alpha\beta\delta_1\mu v\mathscr{S} + \frac{\beta\mu^3 v\mathscr{S}}{2} - \frac{\beta\mu v\mathscr{S}}{2}$$
$$- 4\beta\delta_1^3 v\mathscr{C} - 2\alpha\beta\delta_1^2 v\mathscr{C} - \beta\delta_1\mu^2 v\mathscr{C} + \beta\delta_1 v\mathscr{C} + \frac{\alpha\beta\mu^2 v\mathscr{C}}{2} - \frac{\alpha\beta v\mathscr{C}}{2}$$
$$+ 4\beta^2\delta_1^2 v^2 + 4\alpha\beta^2\delta_1 v^2 + \beta^2\mu^2 v^2 + \alpha^2\beta^2 v^2 - \beta^2 v^2 + \frac{\mu^4}{16} + \frac{\delta_1^2\mu^2}{2} - \frac{\mu^2}{8} + \delta_1^4 - \frac{\delta_1^2}{2} + \frac{1}{16} \qquad (26)$$

Note that p, q, r, and s all depend on λ appearing in exponential form, so eq. (22) is not a polynomial equation. In the previous paper the Routh-Hurwitz criterion [10] was used to determine the stable regions of the characteristic polynomial, but since eq. (22) is a transcendental equation a different technique will have to be used here.

A necessary condition for stability is for the real part all of the eigenvalues λ_i to be nonpositive, and the transition curves between stable and unstable regions in parameter space occur when the real part of λ_i is exactly zero. The approach we'll take to find the transition curves is to set $\lambda = i\omega$; the case when $\omega = 0$ corresponds to a saddle node bifurcation, and all nonzero values of ω correspond to a possible Hopf bifurcation in the nonlinear system.

Substituting $\lambda = i\omega$ in (22) turns it into a complex equation, and to solve it we set the real and imaginary parts equal to zero:

$$0 = 4\beta^2\mu\omega\sin(2\varepsilon\omega T) - 4\beta^2\omega^2\cos(2\varepsilon\omega T) + 4\beta^2\delta_1^2\cos(2\varepsilon\omega T) + 4\alpha\beta^2\delta_1\cos(2\varepsilon\omega T) + \beta^2\mu^2\cos(2\varepsilon\omega T)$$
$$+ \alpha^2\beta^2\cos(2\varepsilon\omega T) - \beta^2\cos(2\varepsilon\omega T) - 4\beta\omega^3\mathscr{S}\sin(\varepsilon\omega T) + 4\beta\delta_1^2\omega\mathscr{S}\sin(\varepsilon\omega T) + 4\alpha\beta\delta_1\omega\mathscr{S}\sin(\varepsilon\omega T)$$
$$+ 3\beta\mu^2\omega\mathscr{S}\sin(\varepsilon\omega T) - \beta\omega\mathscr{S}\sin(\varepsilon\omega T) - 4\beta\mu\delta_1\omega\mathscr{C}\sin(\varepsilon\omega T) + 2\alpha\beta\mu\omega\mathscr{C}\sin(\varepsilon\omega T) - 6\beta\mu\omega^2\mathscr{S}\cos(\varepsilon\omega T)$$
$$+ 2\beta\mu\delta_1^2\mathscr{S}\cos(\varepsilon\omega T) + 2\alpha\beta\mu\delta_1\mathscr{S}\cos(\varepsilon\omega T) + \frac{\beta\mu^3\mathscr{S}\cos(\varepsilon\omega T)}{2} - \frac{\beta\mu\mathscr{S}\cos(\varepsilon\omega T)}{2} + 4\beta\delta_1\omega^2\mathscr{C}\cos(\varepsilon\omega T)$$
$$- 2\alpha\beta\omega^2\mathscr{C}\cos(\varepsilon\omega T) - 4\beta\delta_1^3\mathscr{C}\cos(\varepsilon\omega T) - 2\alpha\beta\delta_1^2\mathscr{C}\cos(\varepsilon\omega T) - \beta\mu^2\delta_1\mathscr{C}\cos(\varepsilon\omega T) + \beta\delta_1\mathscr{C}\cos(\varepsilon\omega T)$$
$$+ \frac{\alpha\beta\mu^2\mathscr{C}\cos(\varepsilon\omega T)}{2} - \frac{\alpha\beta\mathscr{C}\cos(\varepsilon\omega T)}{2} + \omega^4 - 2\delta_1^2\omega^2 - \frac{3\mu^2\omega^2}{2} + \frac{\omega^2}{2} + \delta_1^4 + \frac{\mu^2\delta_1^2}{2} - \frac{\delta_1^2}{2} + \frac{\mu^4}{16} - \frac{\mu^2}{8} + \frac{1}{16} \quad (27)$$
$$0 = 4\beta^2\omega^2\sin(2\varepsilon\omega T) - 4\beta^2\delta_1^2\sin(2\varepsilon\omega T) - 4\alpha\beta^2\delta_1\sin(2\varepsilon\omega T) - \beta^2\mu^2\sin(2\varepsilon\omega T) - \alpha^2\beta^2\sin(2\varepsilon\omega T)$$
$$+ \beta^2\sin(2\varepsilon\omega T) + 4\beta^2\mu\omega\cos(2\varepsilon\omega T) + 6\beta\mu\omega^2\mathscr{S}\sin(\varepsilon\omega T) - 2\beta\mu\delta_1^2\mathscr{S}\sin(\varepsilon\omega T) - 2\alpha\beta\mu\delta_1\mathscr{S}\sin(\varepsilon\omega T)$$
$$- \frac{\beta\mu^3\mathscr{S}\sin(\varepsilon\omega T)}{2} + \frac{\beta\mu\mathscr{S}\sin(\varepsilon\omega T)}{2} - 4\beta\delta_1\omega^2\mathscr{C}\sin(\varepsilon\omega T) + 2\alpha\beta\omega^2\mathscr{C}\sin(\varepsilon\omega T) + 4\beta\delta_1^3\mathscr{C}\sin(\varepsilon\omega T)$$
$$+ 2\alpha\beta\delta_1^2\mathscr{C}\sin(\varepsilon\omega T) + \beta\mu^2\delta_1\mathscr{C}\sin(\varepsilon\omega T) - \beta\delta_1\mathscr{C}\sin(\varepsilon\omega T) - \frac{\alpha\beta\mu^2\mathscr{C}\sin(\varepsilon\omega T)}{2} + \frac{\alpha\beta\mathscr{C}\sin(\varepsilon\omega T)}{2}$$
$$- 4\beta\omega^3\mathscr{S}\cos(\varepsilon\omega T) + 4\beta\delta_1^2\omega\mathscr{S}\cos(\varepsilon\omega T) + 4\alpha\beta\delta_1\omega\mathscr{S}\cos(\varepsilon\omega T) + 3\beta\mu^2\omega\mathscr{S}\cos(\varepsilon\omega T) - \beta\omega\mathscr{S}\cos(\varepsilon\omega T)$$
$$- 4\beta\mu\delta_1\omega\mathscr{C}\cos(\varepsilon\omega T) + 2\alpha\beta\mu\omega\mathscr{C}\cos(\varepsilon\omega T) - 2\mu\omega^3 + 2\mu\delta_1^2\omega + \frac{\mu^3\omega}{2} - \frac{\mu\omega}{2} \qquad (28)$$

These equations are very messy, and it is not clear if a closed form solution $T(\delta_1, \alpha, \beta, \mu, \varepsilon)$ can be found by eliminating ω. To simplify matters, we will start by setting $\varepsilon = 0$ in eqs. (27), (28), which is equivalent to approximating the delayed terms in the slow flow as non-delayed terms.

3.1 The Non-Delayed Case

Setting $\varepsilon = 0$ in eqs. (27), (28) yields:

$$0 = -4\beta^2\omega^2 + 4\beta^2\delta_1^2 + 4\alpha\beta^2\delta_1 + \beta^2\mu^2 + \alpha^2\beta^2 - \beta^2 - 6\beta\mu\omega^2\mathscr{S} + 2\beta\mu\delta_1^2\mathscr{S} + 2\alpha\beta\mu\delta_1\mathscr{S}$$
$$+ \frac{\beta\mu^3\mathscr{S}}{2} - \frac{\beta\mu\mathscr{S}}{2} + 4\beta\delta_1\omega^2\mathscr{C} - 2\alpha\beta\omega^2\mathscr{C} - 4\beta\delta_1^3\mathscr{C} - 2\alpha\beta\delta_1^2\mathscr{C} - \beta\mu^2\delta_1\mathscr{C} + \beta\delta_1\mathscr{C}$$
$$+ \frac{\alpha\beta\mu^2\mathscr{C}}{2} - \frac{\alpha\beta\mathscr{C}}{2} + \omega^4 - 2\delta_1^2\omega^2 - \frac{3\mu^2\omega^2}{2} + \frac{\omega^2}{2} + \delta_1^4 + \frac{\mu^2\delta_1^2}{2} - \frac{\delta_1^2}{2} + \frac{\mu^4}{16} - \frac{\mu^2}{8} + \frac{1}{16} \qquad (29)$$
$$0 = 4\beta^2\mu\omega - 4\beta\delta_1\omega^2\mathscr{C} - 4\beta\omega^3\mathscr{S} + 4\beta\delta_1^2\omega\mathscr{S} + 4\alpha\beta\delta_1\omega\mathscr{S} + 3\beta\mu^2\omega\mathscr{S} - \beta\omega\mathscr{S}$$
$$- 4\beta\mu\delta_1\omega\mathscr{C} + 2\alpha\beta\mu\omega\mathscr{C} - 2\mu\omega^3 + 2\mu\delta_1^2\omega + \frac{\mu^3\omega}{2} - \frac{\mu\omega}{2} \qquad (30)$$

This system is simple enough to have a closed form solution. We start by solving eq. (30) for ω, resulting in the trivial solution:

$$\omega = 0$$

and the nontrivial solution:

$$\omega^2 = \frac{(8\beta\delta_1^2 + 8\alpha\beta\delta_1 + 6\beta\mu^2 - 2\beta)\mathscr{S} + (4\alpha\beta\mu - 8\beta\mu\delta_1)\mathscr{C} + 4\mu\delta_1^2 + \mu^3 + (8\beta^2 - 1)\mu}{8\beta\mathscr{S} + 4\mu} \qquad (31)$$

The saddle node transition curves are obtained by substituting $\omega = 0$ in eq. (29):

$$0 = (32\beta\mu\delta_1^2 + 32\alpha\beta\mu\delta_1 + 8\beta\mu^3 - 8\beta\mu)\mathscr{S}$$
$$+ (16\beta\delta_1 - 64\beta\delta_1^3 - 32\alpha\beta\delta_1^2 - 16\beta\mu^2\delta_1 + 8\alpha\beta\mu^2 - 8\alpha\beta)\mathscr{C}$$
$$+ 16\delta_1^4 + 8\mu^2\delta_1^2 + 64\beta^2\delta_1^2 - 8\delta_1^2 + 64\alpha\beta^2\delta_1 + \mu^4 + 16\beta^2\mu^2 - 2\mu^2 + 16\alpha^2\beta^2 - 16\beta^2 + 1 \qquad (32)$$

The Hopf transition curves are obtained by substituting (31) in eq. (29):

$$0 = 128\alpha\beta^3\mu\delta_1\mathscr{S}^3 + 64\beta^3\mu^3\mathscr{S}^3 - 16\alpha^2\beta^3\mu\mathscr{S}^3 - 16\beta^3\mu\mathscr{S}^3 - 128\beta^3\mu^2\delta_1\mathscr{C}\mathscr{S}^2 + 32\alpha^2\beta^3\delta_1\mathscr{C}\mathscr{S}^2$$
$$+ 64\alpha\beta^3\mu^2\mathscr{C}\mathscr{S}^2 + 64\beta^2\mu^2\delta_1^2\mathscr{S}^2 - 16\alpha^2\beta^2\delta_1^2\mathscr{S}^2 + 32\alpha\beta^2\mu^2\delta_1\mathscr{S}^2 + 48\beta^2\mu^4\mathscr{S}^2 + 128\beta^4\mu^2\mathscr{S}^2$$
$$- 4\alpha^2\beta^2\mu^2\mathscr{S}^2 - 20\beta^2\mu^2\mathscr{S}^2 - 16\alpha^2\beta^4\mathscr{S}^2 + 64\alpha\beta^2\mu\delta_1^2\mathscr{C}\mathscr{S} - 64\beta^2\mu^3\delta_1\mathscr{C}\mathscr{S} - 128\beta^4\mu\delta_1\mathscr{C}\mathscr{S}$$
$$+ 32\alpha\beta^2\mu^3\mathscr{C}\mathscr{S} + 64\alpha\beta^4\mu\mathscr{C}\mathscr{S} + 32\beta\mu^3\delta_1^2\mathscr{S} + 64\beta^3\mu\delta_1^2\mathscr{S} - 128\alpha\beta^3\mu\delta_1\mathscr{S} + 12\beta\mu^5\mathscr{S}$$
$$+ 64\beta^3\mu^3\mathscr{S} - 8\beta\mu^3\mathscr{S} + 64\beta^5\mu\mathscr{S} + 16\alpha\beta\mu^2\delta_1^2\mathscr{C} - 8\beta\mu^4\delta_1\mathscr{C} - 32\beta^3\mu^2\delta_1\mathscr{C} + 4\alpha\beta\mu^4\mathscr{C}$$
$$+ 16\alpha\beta^3\mu^2\mathscr{C} + 4\mu^4\delta_1^2 + 16\beta^2\mu^2\delta_1^2 - 32\alpha\beta^2\mu^2\delta_1 + \mu^6 + 8\beta^2\mu^4 - \mu^4 + 16\beta^4\mu^2 \qquad (33)$$

Figure 1 shows both the saddle node bifurcation curves and the Hopf bifurcation curves for the non-delayed system, as well as the stable regions. Stable regions are determined by selecting a representative point from each disjoint region and testing that point for stability.

We now return to eqs. (27), (28) and employ a perturbation approach to calculate the Hopf bifurcation.

3.2   The Delayed Case

Since we now have a solution when $\varepsilon = 0$, the next reasonable course of action would be to look for a series solution in ε with our result being the zeroth order solution. Unfortunately, the result when $\varepsilon = 0$ is still too complicated to be written explicitly in a closed form solution. In order to determine
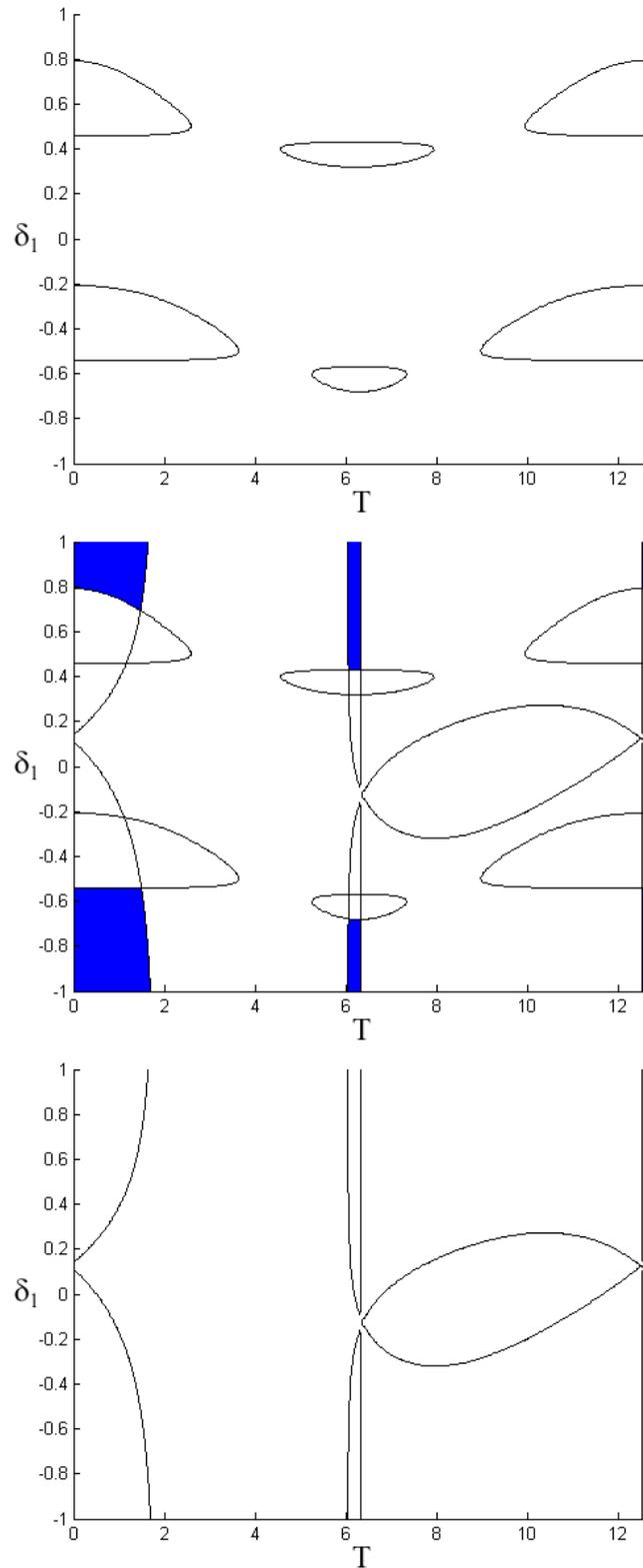
Fig. 1 The top graph shows the transition curves for the saddle node bifurcations of the non-delayed system ($\varepsilon = 0$). The bottom graph shows the transition curves for the Hopf bifurcations of the non-delayed system ($\varepsilon = 0$). The middle graph shows both sets of transition curves and has the stable regions shaded in. Parameter values are: $\alpha = 0.1$; $\beta = 0.125$; $\mu = 0.01$.

the effect of ε on the system, we also need to perturb off of α and μ, resulting in a series expansion in three variables.

Fortunately, the saddle node bifurcations are the same for both the delayed and non-delayed systems, so we do not need to investigate the case when $\omega = 0$ as this solution is already known exactly.

We begin by expanding T and ω in the following series:

$$T = T_{000} + T_{100}\alpha + T_{010}\varepsilon + T_{001}\mu + T_{200}\alpha^2 + T_{110}\alpha\varepsilon + T_{101}\alpha\mu + T_{020}\varepsilon^2 + T_{011}\varepsilon\mu + T_{002}\mu^2 + \cdots \quad (34)$$

$$\omega = \omega_{000} + \omega_{100}\alpha + \omega_{010}\varepsilon + \omega_{001}\mu + \omega_{200}\alpha^2 + \omega_{110}\alpha\varepsilon + \omega_{101}\alpha\mu + \omega_{020}\varepsilon^2 + \omega_{011}\varepsilon\mu + \omega_{002}\mu^2 + \cdots (35)$$

We first calculate the zeroth order terms T_{000} and ω_{000} and use those results to calculate higher order terms.

Substituting (34), (35) into eqs. (27), (28) and setting $\alpha = 0$, $\varepsilon = 0$, and $\mu = 0$, eqs. (27), (28) become:

$$0 = \frac{\left(4\omega_{000}^2 - 4\delta_1^2 + 1\right)\left(16\beta\delta_1\cos\left(\frac{T_{000}}{2}\right) + 4\omega_{000}^2 - 4\delta_1^2 - 16\beta^2 + 1\right)}{16} \quad (36)$$

$$0 = -\beta\omega_{000}\left(4\omega_{000}^2 - 4\delta_1^2 + 1\right)\sin\left(\frac{T_{000}}{2}\right) \quad (37)$$

Looking at eq. (37), we see there are three distinct cases to examine: $\omega_{000} = 0$; $4\omega_{000}^2 - 4\delta_1^2 + 1 = 0$; and $\sin\left(\frac{T_{000}}{2}\right) = 0$. As we are not interested in the case when $\omega = 0$, we will only focus on the second and third cases.

In the second case when $4\omega_{000}^2 - 4\delta_1^2 + 1 = 0$, eq. (36) is also identically zero. Thus, this single condition satisfies both eq. (36) and eq. (37). In this case, we get $\omega_{000} = \sqrt{\delta_1^2 - \frac{1}{4}}$, which undergoes a Hopf bifurcation when $|\delta_1| > 1/2$.

In the third case when $\sin\left(\frac{T_{000}}{2}\right) = 0$, the condition is satisfied when $T_{000} = 2n\pi$ for integer values of n. In this paper we will examine the smallest nonzero delay at which a Hopf bifurcation occurs, which is when $T_{000} = 2\pi$.

Substituting $T_{000} = 2\pi$ into (36) yields the expression:

$$-16\beta\delta_1 + 4\omega_{000}^2 - 4\delta_1^2 - 16\beta^2 + 1 = 0$$

Solving for ω_{000} gives us the solution:

$$\omega_{000} = \frac{\sqrt{4\delta_1^2 + 16\beta\delta_1 + 16\beta^2 - 1}}{2}$$

Since $\omega > 0$ for a Hopf bifurcation, the radicand must be positive. The radicand is positive for $\delta_1 < -2\beta - 1/2$ and $\delta_1 > -2\beta + 1/2$, which fully determines the Hopf bifurcation for the case of $T_{000} = 2\pi$.

This means that the Hopf curve represents a necessary condition for a Hopf bifurcation, but not a sufficient one. The points where the Hopf curve intersect the saddle node curves divide the Hopf

curve into disjoint regions; the additional conditions $\delta_1 < -2\beta - 1/2$ and $\delta_1 > -2\beta + 1/2$ are used to determine which of those disjoint regions represent actual Hopf bifurcations.

The third case is the solution we will focus on in this paper, and so we will pick $T_{000} = 2\pi$ and $\omega_{000} = \sqrt{4\delta_1^2 + 16\beta\delta_1 + 16\beta^2 - 1}/2$ as our zeroth order solutions in the perturbation method.

By substituting the zeroth order solutions back into eqs. (27), (28), we are then able to solve for higher order terms. At each step in this process, ω_{ijk} and T_{ijk} are solved simultaneously, just as ω_{000} and T_{000} were in the zeroth order case. The final result for (34), (35) is:

$$T = 2\pi + \frac{\mu}{2\beta} - 4\pi(\delta_1 + 2\beta)\varepsilon + 2\pi\alpha\varepsilon - \frac{\delta_1 + 2\beta}{\beta}\varepsilon\mu + 8\pi(\delta_1 + 2\beta)^2\varepsilon^2 + HOT \tag{38}$$

$$\omega = \omega_{000} - \frac{1}{2\omega_{000}}(\delta_1 + 2\beta)\alpha - \frac{1}{32\beta\omega_{000}^3}\left(4\delta_1^3 + 24\beta\delta_1^2 + 48\beta^2\delta_1 - \delta_1 + 32\beta^3 - \beta\right)\alpha^2$$
$$- \frac{1}{\omega_{000}}\left(\pi^2\beta\delta_1\right)\varepsilon^2 + \frac{\pi}{4\omega_{000}}\varepsilon\mu - \frac{1}{16\beta\omega_{000}}(\delta_1 + 2\beta)\mu^2 + HOT \tag{39}$$

Figure 2 shows eq. (38) and eq. (32).

Since we are mostly concerned with the area around $T = 2\pi$, the graphs shown here will zoom in on that region. A blowup of Figure 1 is shown in Figure 1 for reference.

4 Numerical Results

The numerical computations use DDE23 in MATLAB [11] to numerically integrate the slow flow (13), (14), (15), (16).

These numerical results will be compared to the analytical results presented earlier in the paper. We note that the analytical results are approximate due to the perturbation method, which truncates the solution, neglecting higher order terms; in this way both the numerical and the analytic approaches are approximate. Additionally, the slow flow only captures the behavior of the system on the slow time scale, and thus ignores some of the structure of the original system. We find much better agreement when comparing the analytical results to the numerical integration of the slow flow instead of the numerical integration of the original system, and this stems from the fact that the slow flow is itself only an approximation. We will only integrate the slow flow in this paper.

The code we used determines stability of the origin by taking the initial condition as a point close to the origin and checking if the amplitude grows without bound. Since nonlinear terms will trap unstable trajectories in a limit cycle of finite amplitude, the techniques outlined here only work with the linearized system.

We utilized a combination of two techniques to accomplish this goal: first, we select an upper bound on amplitude size and quit out of integration if that amplitude is reached; second, for all other cases we check if the maximum amplitude over the entire time interval is equivalent to the maximum amplitude over a subsection of the time interval. This latter method helps capture edge cases that the former doesn't catch.
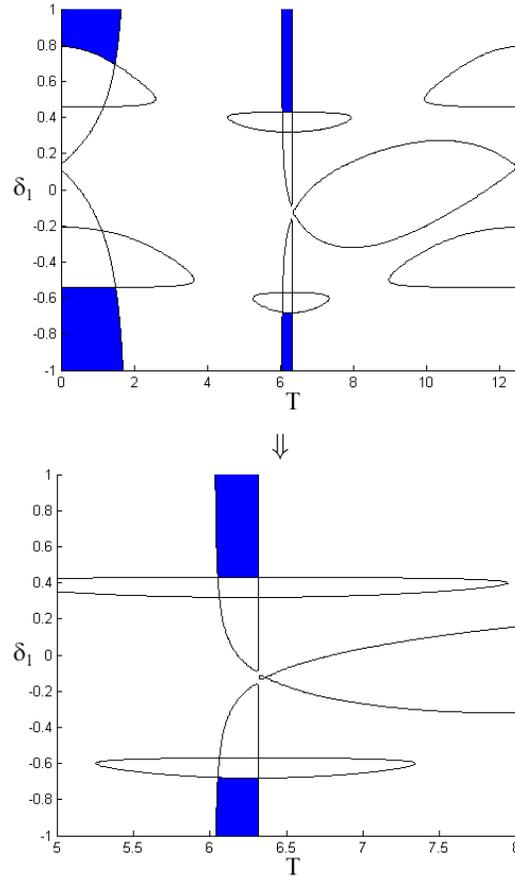
Fig. 1 The top graph is Figure 1. The bottom graph zooms in on the region for T between 5 and 8, while keeping the range of δ_1 values the same.

In all computations we used $\alpha = 0.1$, $\beta = 0.125$ and $\mu = 0.01$.

In Figure 1 we see the effect of including delay purely from the standpoint of numerical integration. The bottom graph in the figure also compares the analytic results to the results of numerical integration.

5 Conclusion

In this paper, we investigated the dynamics of two coupled Mathieu equations with delay. In particular we analyzed the stability of the origin and the effect of delay and damping on stability. We used the method of two variable expansion to calculate a characteristic equation of the system's slow flow, and used power series to analyze the Hopf bifurcations around $T = 2\pi$; these results were then compared with numerical integration.

The numerical results of the slow flow closely matched the analytical results for small values of ε, α and μ. Furthermore, within the range of parameter values for which agreement held, we found significant variation in the Hopf bifurcation transition curve. These results demonstrate that including delay terms in the slow flow can be very important when approximating numerical results.

To measure the significance of replacing delay terms by non-delay terms, we calculate the error between the numerical integration of T and the series solution, for both the entire series and just the zeroth order terms. The percent error is calculated as follows:
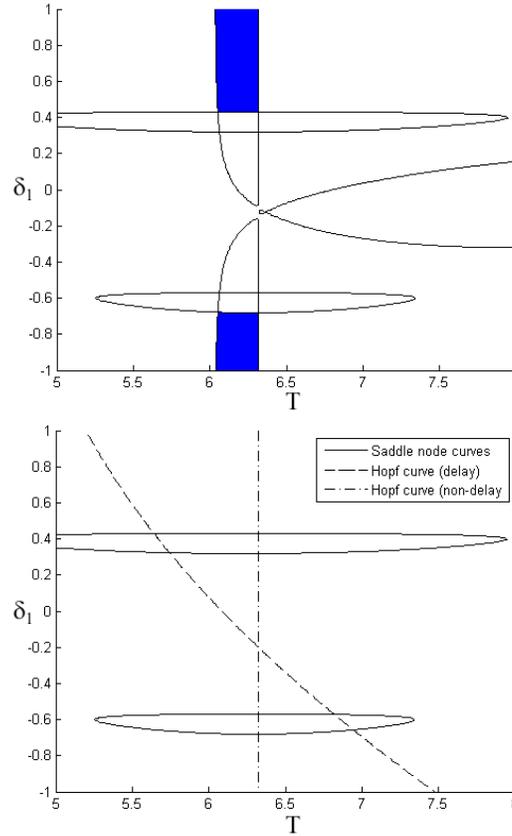
Fig. 2 The top graph is the blowup of Figure 1. The bottom graph shows the series solutions for the Hopf curves for both the delayed and non-delayed systems, as well as the saddle node bifurcation curves.

$$\text{error} = \frac{|T_{num} - T_{series}|}{T_{num}} \times 100\%$$

For fixed values of δ_1 between -1 and 1, different values of T_{num} and T_{series} are obtained; the Table below shows the average and maximum values of the error in this range.

	Omitting delay terms ($\varepsilon = 0$)	Including delay terms ($\varepsilon = 0.1$)
Average	15.04%	0.60%
Maximum	19.34%	1.88%

In the case of the maximum error, including delay terms results in a more accurate solution by a factor of 10. When looking at the average error, the accuracy is even better. This demonstrates what is lost by omitting the delay terms in the slow flow.
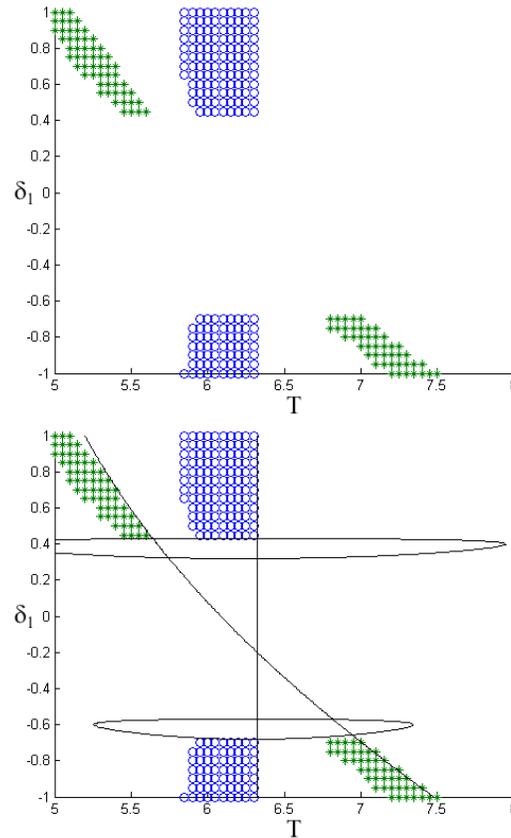
6 Acknowledgments

Fig. 1 The top graph shows the stable points near the Hopf curve as computed through numerical integration of the slow flow, with the circles representing the non-delayed system and the asterisks representing the delayed system. The bottom graph shows the Hopf and saddle node transition curves on top of these stable regions.

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