

Abstract for

International Conference on Structural Nonlinear Dynamics and Diagnosis, Marrakech 2016.

## Parametric Excitation of a Delay Limit Cycle Oscillator by Periodically Varying the Duration of the Delay

Lauren Lazarus, Matthew Davidow and Richard Rand  
Cornell University

A recent study [1] of dynamical systems with delayed terms has considered the following “delay limit cycle oscillator” in the form of a differential–delay equation (DDE):

$$\dot{x} = -x(t - T_0) - \epsilon x^3 \quad (1)$$

This system exhibits a supercritical Hopf bifurcation at delay  $T_0 = \pi/2$  such that the equilibrium point at the origin  $x = 0$  is stable for  $T_0 < \pi/2$  and unstable otherwise. The stable limit cycle for  $T_0 > \pi/2$  is created with natural frequency 1 [2],[3].

This work considers a system of the same form as eq.(1), but with a periodically time–varying delay  $T(t) = \pi/2 + \epsilon k + \epsilon \cos \omega t$ :

$$\dot{x} = -x(t - T(t)) - \epsilon x^3 = -x\left(t - \frac{\pi}{2} - \epsilon k - \epsilon \cos \omega t\right) - \epsilon x^3 \quad (2)$$

The delay  $T$  is taken to be time–dependent such that the system may periodically cross the Hopf bifurcation exhibited by the constant  $T$  case. This causes the stability of the  $x = 0$  equilibrium to regularly alternate between stable and unstable. We would anticipate the equilibrium being stable if it is in the stable region for more than half of the forcing period, and unstable otherwise. However, it turns out that the effect of this forcing produces unexpected behavior due to resonance between the forcing frequency  $\omega$  and the frequency of the limit cycle created in the Hopf in the neighborhood of 2:1 parametric excitation when  $\omega = 2$ . Thus we may say that we have a form of parametric excitation which has been previously unstudied, where the periodic forcing term appears in the delay.

We investigate the dynamics by expanding the system about the  $\epsilon = 0$  solution, using two time variables, fast time  $\xi$  and slow time  $\eta$ :

$$\omega t = 2\xi = 2(1 + \epsilon\Delta/2)t \quad \eta = \epsilon t \quad x = x_0 + \epsilon x_1 + O(\epsilon^2) \quad (3)$$

where  $\omega = 2 + \epsilon\Delta$ . Here  $x_0$  satisfies the  $\epsilon = 0$  equation:

$$x_{0\xi} + x_0\left(\xi - \frac{\pi}{2}, \eta\right) = 0 \quad \Rightarrow \quad x_0(\xi, \eta) = A(\eta) \cos \xi + B(\eta) \sin \xi \quad (4)$$

Removal of secular terms from the  $x_1$  equation gives the following slow flow on  $A(\eta)$  and  $B(\eta)$ :

$$(2\pi^2 + 8)A' = (8k + 4)A + (2\pi - 4\pi k - \pi^2\Delta - 4\Delta)B + (-6A + 3\pi B)(A^2 + B^2) \quad (5)$$

$$(2\pi^2 + 8)B' = (2\pi + 4\pi k + \pi^2\Delta + 4\Delta)A + (8k - 4)B + (-6B - 3\pi A)(A^2 + B^2) \quad (6)$$

This system of slow flow equations exhibits an assortment of bifurcation phenomena. Its steady–state solutions includes equilibrium points, representing periodic motions in  $x(t)$ , and limit cycles, corresponding to quasiperiodic behavior of the original system. Analytical computations and numerical simulations have resulted in the discovery of numerous bifurcation curves in eqs.(5) and (6), see Fig.1.

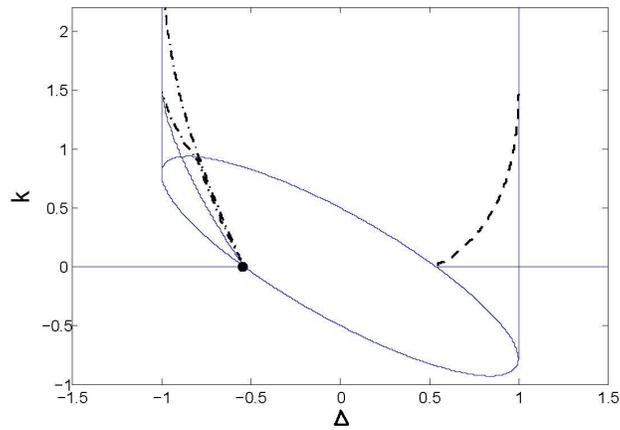


Figure 1: Local bifurcation curves (solid lines) obtained analytically, and global bifurcation curves (dashed lines) found numerically.

For most forcing frequencies of the delay, the system exhibits quasiperiodic behavior due to the coappearance of forcing–frequency oscillations with the frequency of the limit cycle of the oscillator. However, the system has a 2:1 resonance which results in a small region of parameter space about  $\omega = 2$  where the oscillator behaves periodically. Within this region, the system is entrained to oscillate at half of the forcing frequency.

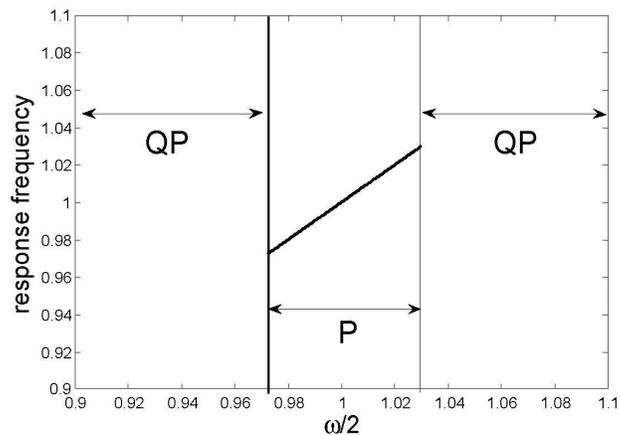


Figure 2: The system is entrained to periodic motion (P) at frequency  $\omega/2$  within the resonance region (shown for  $k = 0.05$ ). It exhibits quasiperiodic motion (QP) with multiple frequencies everywhere else.

## References

- [1] Lazarus, L., Davidow, M., Rand, R.: Dynamics of a delay limit cycle oscillator with self–feedback. *Nonlinear Dynamics* 82 (2015) 481–488.
- [2] Rand, R.H.: Lecture notes in nonlinear vibrations published online by the Internet–First University Press: <http://ecommons.library.cornell.edu/handle/1813/28989> (2012)
- [3] Rand, R.: Differential–delay equations. In: Luo, A.C.J., Sun, J.–Q. (eds.) Chapter 3 in “Complex Systems: Fractionality, Time–delay and Synchronization”, pp. 83117. Springer, Berlin (2011)