Proceedings of DETC'97
1997 ASME Design Engineering Technical Conferences
September 14-17, 1997, Sacramento, California

DETC97/VIB-4103

THE DYNAMICS OF THE CELT WITH SECOND ORDER AVERAGING AND COMPUTER ALGEBRA

A.D. Blackowiak
Intel Corporation
Santa Clara, CA 95052

R.H. Rand
Dept.Theoretical &
Applied Mechanics
Cornell University
Ithaca, NY 14853

H. Kaplan
Dept. Physics
Syracuse University
Syracuse, NY 13244

ABSTRACT

Starting with a no-slip, dissipation-free model of the celt developed by Kane and Levinson in 1982, we obtain a three-dimensional slow flow using second order averaging. The coefficients of the slow flow are obtained in symbolic form through the use of computer algebra, thus permitting a bifurcation analysis to be performed. It is shown that for all physically relevant parameters the celt is predicted to exhibit an infinite number of spin reversals. The analysis assumes small energy and small inertial asymmetry.

INTRODUCTION

The celt, also known as a wobblestone or rattleback, is a body which, when spun like a top on a flat surface, has a characteristic tendency to prefer only one direction of spin. If a celt is spun in the opposite direction, it slows down and reverses its direction of spin, to the surprise and delight of the uninitiated observer.

Although celts have been known since early times due to their occurrence in nature in the form of smooth stones, nowadays they are inexpensively manufactured from plastic and are commonly sold as scientific novelties. The first scientific paper on celts appeared in 1896 (Walker, 1896). A *Scientific American* article (Walker, 1979) in 1979 stimulated interest in the dynamics of the celt with the result that many articles have appeared since then.

The celt is perhaps the simplest dynamics problem which is difficult to explain in simple terms. Although the physical feature leading to celt-like behavior is readily identified (see below), an explanation in the tradition of modern dynamics, involving a qualitative picture of the geometry of phase space, is more elusive.

In this paper we report on our use of second order averaging and computer algebra to provide a relatively simple geometrical picture of celt dynamics based on a slow flow in the phase space of the averaged variables. In addition we present the results of a bifurcation analysis showing that there is only one qualitatively distinct phase portrait for physically realizable parameters. Our discussion is limited to motions which occur in the neighborhood of static equilibrium, i.e., small displacements and velocities. We also assume the dynamic imbalance is small. The model we use omits dissipation of energy and assumes no slip between the celt and the surface it rolls on.

LITERATURE REVIEW

Before presenting a mathematical description of the celt, we briefly report on a number of related research papers. See (Garcia and Hubbard, 1988) for a more detailed literature review.

(Walker, 1896) presented the first dynamical analysis, based on a model with rolling without slipping, and involving a linearization about a uniform rotation about a vertical axis. He showed that it is the presence of an inertial asymmetry (i.e. the non-coincidence of the principal axes of the moment of inertia ellipsoid with the principal axes of the body surface ellipsoid) that is responsible for the characterstic celt-like behavior. (Bondi, 1986) extended the linear analysis of a rolling–without–slipping model and concluded that three distinct types of behavior can occur, depending on the parameters and initial kinetic energy. He found that a celt can behave 1) as if one direction of spin is "stable" and one "unstable" (in which case there is only one change of spin direction), 2) as if neither direction of spin is "stable" (in which case the direction of spin continues to change as long as the celt has energy), or 3) as if it has a stable spin only about a non-vertical axis. The analysis of this no-slip model also shows that at low energy levels all celts without damping experience type 2) behavior, i.e., they exhibit multiple changes of direction of spin.

(Magnus, 1974) included the possibility of both rolling and slipping in his celt model and concluded that in this case a celt can exhibit only one change of direction, even at low energy levels. (Caughey, 1980) presented a nonlinear analysis of a model which, while capturing certain dynamical features, was relatively unrealistic. (Kane and Levinson, 1982) presented a realistic model which was based on rolling without slipping. Numerical integration of the full equations of motion showed many spin reversals. Damping was invoked to reduce the number of reversals to one or two. Bondi, on the other hand, showed that in some cases only one spin reversal occurred with no damping. (Lindberg and Longman, 1983) also numerically integrated the equations of motion and showed that due to the presence of gyroscopic terms, no change of variables will uncouple the linearized equations of motion into two independent oscillatory modes. (Garcia and Hubbard, 1988) made a careful analysis of the effects of dissipation, including aerodynamic dissipation, dry friction, and slipping. Aerodynamic effects were found to be generally weaker than dry friction effects for real celts. Both calculations and experiments showed that dissipation plays an important role in the spin reversals of real celts. Garcia and Hubbard also presented a simplified model of the spin dynamics (in section 5 of (Garcia and Hubbard, 1988)) which used an average vertical reaction torque. In comparison, the present work uses the method of averaging to systematically include all low order nonlinear effects.

(Markeev, 1983) and (Pascal, 1984), (Pascal, 1986) presented a perturbation analysis, valid to lowest order in a small parameter. Their results are similar to the order ϵ^2 analysis presented in the present work.

FORMULATION AND AVERAGING

We use the formulation of (Kane and Levinson, 1982) to describe the celt's kinematics and to obtain the equations of motion. The surface of the celt is modeled as a portion of an ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

The center of mass of the celt is assumed to lie on the z axis, at a distance $h < c$ from the origin, see Fig.1.

The celt's configuration is described by three angles, α, β and γ. Starting with the $x-y-z$ axes (fixed in the celt) in a reference configuration in which z is vertically downward, the celt's general position is obtained by rotating by angle γ about the z axis, then by α about the x axis, and finally by β about the y axis.

The moment of inertia matrix relative to the center of mass in the directions of the $x - y - z$ axes is assumed to be given by

$$\begin{bmatrix} A & D & 0 \\ D & B & 0 \\ 0 & 0 & C \end{bmatrix} \tag{2}$$

(Kane and Levinson, 1982) used Kane's equations to obtain the equations of motion. These may be written in the form

$$\begin{aligned} \ddot{\alpha} &= F_1(\alpha, \beta, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) \\ \ddot{\beta} &= F_2(\alpha, \beta, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) \\ \ddot{\gamma} &= F_3(\alpha, \beta, \dot{\alpha}, \dot{\beta}, \dot{\gamma}) \end{aligned} \tag{3}$$

where the F_i are extremely complicated and are here derived using MACSYMA. Note that γ is an ignorable coordinate, so that the motion occurs in a 5-dimensional phase space. Moreover, the celt is assumed to roll without slipping, so the constraint force does no work and energy is conserved. Thus the motion remains on a codimension 1 energy manifold, i.e., a locally 4-dimensional phase space.

These equations exhibit the exact solution $\alpha \equiv 0, \beta \equiv 0, \gamma = \omega t$ representing a uniform spin about a vertical axis.

In order to use a perturbation method to simplify eqs.(3), we assumed the variables $\alpha, \beta, \dot{\gamma}$ and the product of inertia D to be proportional to a small parameter ϵ:

$$\alpha = \epsilon\hat{\alpha}, \beta = \epsilon\hat{\beta}, \dot{\gamma} = \epsilon\hat{p}, D = \epsilon\hat{D} \tag{4}$$

Expanding eqs.(3) in a power series in ϵ, we find

$$\begin{aligned} \ddot{\hat{\alpha}} + \Omega_1^2\hat{\alpha} &= \epsilon G_1 + \epsilon^2 H_1 + \cdots \\ \ddot{\hat{\beta}} + \Omega_2^2\hat{\beta} &= \epsilon G_2 + \epsilon^2 H_2 + \cdots \\ \dot{\hat{p}} &= \epsilon G_3 + \epsilon^2 H_3 + \cdots \end{aligned} \tag{5}$$

where the G_i and H_i are complicated functions of the phase variables (see (Blackowiak, 1996) for a listing of these functions). Here Ω_1 and Ω_2 are the linearized frequencies of free vibration of the α and β normal modes, representing "rocking" motions without rotation about the γ direction:

$$\Omega_1^2 = \frac{mg(b^2 + c(h-c))}{c(m(h-c)^2 + A)}, \quad \Omega_2^2 = \frac{mg(a^2 + c(h-c))}{c(m(h-c)^2 + B)} \quad (6)$$

Note that when ϵ is zero, eqs.(5) are uncoupled. This desirable feature avoids the necessity of a linear eigentransformation and is due to our assumption that D is of order ϵ, i.e., that the angle between the principal axes of the inertia ellipsoid and the principal axes of the body surface ellipsoid is small. (Markeev, 1983) and (Pascal, 1984), (Pascal, 1986) did not make this assumption, and therefore had to uncouple the corresponding equations by performing a linear eigentransformation, thereby complicating their analysis significantly. However, the price we must pay for the convenience of starting with uncoupled equations is that we have to go to order ϵ^2 in our averaging procedure, whereas they they only had to go to lowest order in their small parameter. It is remarkable that in spite of this difference, both approaches give a similar final form of the averaged equations.

In order to use averaging, we make the usual transformation (see (Rand, 1994)):

$$\hat{\alpha} = \hat{r_1}\cos(\Omega_1 t + \hat{\theta_1}),\ \dot{\hat{\alpha}} = -\hat{r_1}\Omega_1 \sin(\Omega_1 t + \hat{\theta_1})$$
$$\hat{\beta} = \hat{r_2}\cos(\Omega_2 t + \hat{\theta_2}),\ \dot{\hat{\beta}} = -\hat{r_2}\Omega_2 \sin(\Omega_2 t + \hat{\theta_2}) \quad (7)$$

where $\hat{r_i}$ and $\hat{\theta_i}$ are slowly varying functions of t. Substituting eqs.(7) into eqs.(5) gives eqs. on $\dot{\hat{r_1}}, \dot{\hat{r_2}}, \dot{\hat{p}}, \dot{\hat{\theta_1}}, \dot{\hat{\theta_2}}$. Second order averaging involves simplifying these equations by transforming to new variables via a near-identity transformation (Rand, 1994):

$$\begin{aligned} \hat{r_1} &= r_1 + \epsilon W_1 + \epsilon^2 V_1 + \cdots \\ \hat{r_2} &= r_2 + \epsilon W_2 + \epsilon^2 V_2 + \cdots \\ \hat{p} &= p \ + \epsilon W_3 + \epsilon^2 V_3 + \cdots \\ \hat{\theta_1} &= \theta_1 + \epsilon W_4 + \epsilon^2 V_4 + \cdots \\ \hat{\theta_2} &= \theta_2 + \epsilon W_5 + \epsilon^2 V_5 + \cdots \end{aligned} \quad (8)$$

where the *generating functions* W_i and V_i are chosen as appropriate functions of the transformed variables. This calculation is accomplished using a previously published MACSYMA program "AVERAGE" given in Chapter 5 of (Rand and Armbruster, 1987) . The resulting averaged equations on the transformed variables are computed to be:

$$\dot{r_1} = \epsilon^2 K_1 r_1 p \quad (9)$$
$$\dot{r_2} = -\epsilon^2 K_2 r_2 p \quad (10)$$
$$\dot{p} = \epsilon^2(-K_3 r_1^2 + K_4 r_2^2) \quad (11)$$
$$\dot{\theta_1} = \epsilon^2(C_1 r_1^2 + C_2 r_2^2 + C_3 p^2 + C_4) \quad (12)$$
$$\dot{\theta_2} = \epsilon^2(C_5 r_1^2 + C_6 r_2^2 + C_7 p^2 + C_8) \quad (13)$$

The parameters K_i are computed to be:

$$K_1 = \frac{m^2 g \hat{D}(a^2-b^2)(h-c)(c(h-c)+b^2)}{2c^2(\Omega_2^2-\Omega_1^2)(m(h-c)^2+A)^2(m(h-c)^2+B)}$$

$$K_2 = \frac{m^2 g \hat{D}(a^2-b^2)(h-c)(c(h-c)+a^2)}{2c^2(\Omega_2^2-\Omega_1^2)(m(h-c)^2+A)(m(h-c)^2+B)^2}$$

(14)

$$K_3 = \frac{m^3 g^2 \hat{D}(a^2-b^2)(h-c)(c(h-c)+b^2)^2}{2c^3 C(\Omega_2^2-\Omega_1^2)(m(h-c)^2+A)^2(m(h-c)^2+B)}$$

$$K_4 = \frac{m^3 g^2 \hat{D}(a^2-b^2)(h-c)(c(h-c)+a^2)^2}{2c^3 C(\Omega_2^2-\Omega_1^2)(m(h-c)^2+A)(m(h-c)^2+B)^2}$$

As a check on the averaging computation, we compare the numerical integration of the averaged equations with numerical integration of the original equations of motion for the following parameters and initial conditions:

$$a = 0.2\ meter,\ b = 0.03\ meter,\ c = 0.02\ meter \quad (15)$$

$$h = 0.01\ meter,\ m = 1\ kg,\ g = 9.81\ meter/sec^2 \quad (16)$$

$$A = 0.0002\ kg\ meter^2,\ B = 0.0016\ kg\ meter^2, \quad (17)$$

$$C = 0.0017\ kg\ meter^2, D = -0.000005\ kg\ meter^2, \quad (18)$$

$$\alpha(0) = 0.05\,\text{deg},\ \beta(0) = 0.05\,\text{deg} \quad (19)$$

$$\dot{\alpha}(0) = 0,\ \dot{\beta}(0) = 0,\ \dot{\gamma}(0) = -0.005\ rad/sec \quad (20)$$

$$\epsilon = 0.1 \quad (21)$$

The parameters in eqs.(15)-(18) are the same as those chosen by (Kane and Levinson, 1982). The initial conditions in eqs.(19)-(20) have been chosen smaller in magnitude than those in (Kane and Levinson, 1982) in order to be close to equilibrium.

For these parameters, the averaged eqs. become:

$$\begin{aligned} \dot{r_1} &= 0.0010608\ r_1 p \\ \dot{r_2} &= -0.010644\ r_2 p \\ \dot{p} &= -.21426\ r_1^2 + 122.23\ r_2^2 \quad (22) \\ \dot{\theta}_1 &= -0.56711\ r_1^2 - 5.2775\ r_2^2 + 0.000126\ p^2 - 0.00009179 \\ \dot{\theta}_2 &= -.30747\ r_1^2 - 647.26\ r_2^2 + 0.000888\ p^2 + 0.0029172 \end{aligned}$$

After numerically integrating these eqs., the results are substituted into eqs.(7) (without using the near-identity transformation (8), a step which is valid to lowest order in ϵ). The results are displayed in Fig.2.

ANALYSIS OF THE AVERAGED EQUATIONS

Note that eqs.(9),(10),(11) are uncoupled from eqs.(12),(13). From eqs.(7) we see that eqs.(9),(10),(11) govern the amplitudes of the α, β, γ motions, respectively, whereas eqs.(12),(13) govern the phases of the α, β motions. We shall direct our attention to eqs.(9),(10),(11) in what follows. Eqs. of the form of eqs.(9),(10),(11) have been derived by (Markeev, 1983) and (Pascal, 1984), (Pascal, 1986) using a different averaging scheme. Our expressions (14) for the coefficients K_i are algebraically simpler than those derived by these authors because our scaling (4) uncouples the governing eqs.(5) when ϵ is zero, producing simpler expressions for the eigenfrequencies (6).

As shown by (Markeev, 1983) and (Pascal, 1984), (Pascal, 1986), eqs.(9),(10),(11) possess two first integrals. Dividing eq.(9) by eq.(10) gives

$$r_1^{K_2} r_2^{K_1} = constant \quad (23)$$

Multiplying eq.(9) by $\frac{K_3}{K_1} r_1$, eq.(10) by $\frac{K_4}{K_2} r_2$, and eq.(11) by p, and adding them together gives

$$\frac{K_3}{K_1} r_1^2 + \frac{K_4}{K_2} r_2^2 + p^2 = constant \quad (24)$$

For the parameters of eqs.(15)-(21), each of the K_i's is positive. In this case, eq.(23) is hyperbola-like in the $r_1 - r_2$ plane, and eq.(24) is an ellipsoid in $r_1 - r_2 - p$ phase space. The intersection of these two surfaces will in general be a closed curve representing a periodic motion in phase space. See Fig.3 in which the constants in eqs.(23), (23) have been chosen to correspond to the initial conditions and parameters in eqs.(15)-(21). Note the difference in scale between the r_1, r_2 and p axes.

Fig.4 shows the periodic motion in phase space, which lies on the intersection of the two first integrals (23),(24). The direction of the motion in Fig.4 is from A to B to C to D to A. Since the frequency of this motion is in general incommensurate with the frequencies Ω_1, Ω_2, we see from eq.(7) that the motion of the celt in the unaveraged variables α, β, γ is in general quasiperiodic.

The motion may be described in words as follows (cf.Figs.2 and 4): If the celt starts spinning in a counterclockwise direction just below point A in Fig.4 (which lies on the equator $p = 0$), it soon reverses direction amidst a mixture of α- and β-rocking (corresponding to relatively small r_1 and large r_2). Its clockwise spin rate increases until point B after which it begins to slow down. When it gets to just above point C on the equator, it again reverses direction this time with mostly α-rocking, and then proceeds to increase its counterclockwise spin rate until point D, after which it slows down and reverses direction at A once again. The model (which omits dissipation) predicts this motion to continue forever.

In the exceptional case in which the constant in eq.(23) is zero, the motion corresponds to one of the following two possibilities:

(i) one of two equilibria in the slow-flow variables $r_1 - r_2 - p$ which occurs when $r_1 = r_2 = 0$ and which corresponds to a uniform spin of the celt in which the z-axis remains vertical, in either a clockwise or a counterclockwise direction. Note that the p-axis is filled with a continuum of such equilibria, two of which correspond to a given value of the constant in eq.(24).

(ii) one of two heteroclinic motions, i.e. saddle connections between these two equilibria. One heteroclinic motion lies in the plane $r_1 = 0$, the other in $r_2 = 0$. Both of these two planes are filled with a continuum of such heteroclinic motions, one of which in each plane corresponds to a given value of the constant in eq.(24).

Another exceptional case occurs if the surface of eq.(23) is tangent to the ellipsoid of eq.(24), in which case the motion corresponds to an equilibrium in the slow-flow variables $r_1 - r_2 - p$ for which $p = 0$ and which corresponds to a "rocking" motion with no spin in the celt.

BIFURCATION ANALYSIS

Using the analytical expressions (14) for the constants K_i in the slow flow (9),(10),(11), it may be concluded that the foregoing qualitative dynamics persists for all physically realistic parameters of the celt, as follows: The nature of the first integrals (23),(24) depends upon the signs of the quantities K_i. Now the denominators of each of the K_i's is positive, and the only terms in the numerators of K_1 and K_2 which could produce a change in sign as the physical parameters are varied are $c(h-c)+b^2$ and $c(h-c)+a^2$, respectively. This assumes that $a > b, h < c$ and $D < 0$, assumptions which can be made without loss of generality, since if e.g. $a < b$ then the x and y axes could be relabeled from the start, switching the roles of a and b throughout. Now the terms $c(h-c)+b^2$ and $c(h-c)+a^2$, are, from eqs.(6), necessarily positive, in order that the unperturbed equilibrium position be stable, and therefore the signs of the constants K_i are positive for all physically relevant parameter values.

CONCLUSIONS

In order to fully understand the dynamics of the celt, we would like to have a complete qualitative picture of the geometry of the flow in the associated five dimensional phase space. The present work has moved towards this goal by providing a set of averaged equations which are sufficiently simple to allow a qualitative understanding of the phase portrait in a reduced three dimensional space, valid for small energy and small inertial asymmetry. We have checked our averaging calculation by comparing the behavior of the averaged equations with that of the original equations of motion. By using computer algebra, we have obtained the coefficients of our averaged equations in symbolic form, thus permitting us to perform a bifurcation analysis which has shown that no bifurcations occur near equilibrium in this no-slip model of the celt.

REFERENCES

A. Donald Blackowiak, *The Dynamics of the Celt with Second Order Averaging and Computer Algebra*, M.S.thesis, Cornell University (1996)

Sir Hermann Bondi, *The Rigid Body Dynamics of Unidirectional Spin*, Proc. R. Soc. Lond. A 405:265–274, (1986)

T.K. Caughey, *A Mathematical Model of the Rattleback*, Int. J. Non-linear Mech. 15:293–302, (1980)

A. Garcia and M. Hubbard, *Spin Reversal of the Rattleback: Theory and Experiment*, Proc. R. Soc. Lond. A 418:165–197 (1988)

Thomas R. Kane and David A. Levinson, *Realistic Mathematical Modeling of the Rattleback*, Int. J. Non-linear Mech. 17:175–186 (1982)

R.E. Lindberg, Jr. and R.W. Longman, *On the Dynamic Behavior of the Wobblestone*, Acta Mechanica 49:81–94 (1983)

Kurt Magnus, *Zur Theorie der Keltischen Wackelsteine*, Zeitschrift fuer angewandte Mathematik und Mechanik 54:54–55 (1974)

A.P. Markeev, *On the Dynamics of a Solid on an Absolutely Rough Plane*, PMM U.S.S.R. 47:473–478 (1983)

M. Pascal, *Asymptotic Solution of the Equations of Motion for a Celtic Stone*, PMM U.S.S.R. 47:269–276 (1984)

M. Pascal, *The Use of the Method of Averaging to Study Non-linear Oscillations of the Celtic Stone*, PMM U.S.S.R. 50:520–522 (1986)

Richard H. Rand, *Topics in Nonlinear Dynamics with Computer Algebra*, Gordon and Breach, Langhorne, PA (1994)

Richard H. Rand and Dieter Armbruster, *Perturbation Methods, Bifurcation Theory, and Computer Algebra*, Springer-Verlag, New York (1987)

G.T. Walker, *On a Dynamical Top*, Q. J. Pure App. Math. 28:175–184 (1896)

Jearl Walker, *The Amateur Scientist*, Scientific American 241:172–184 (1979)

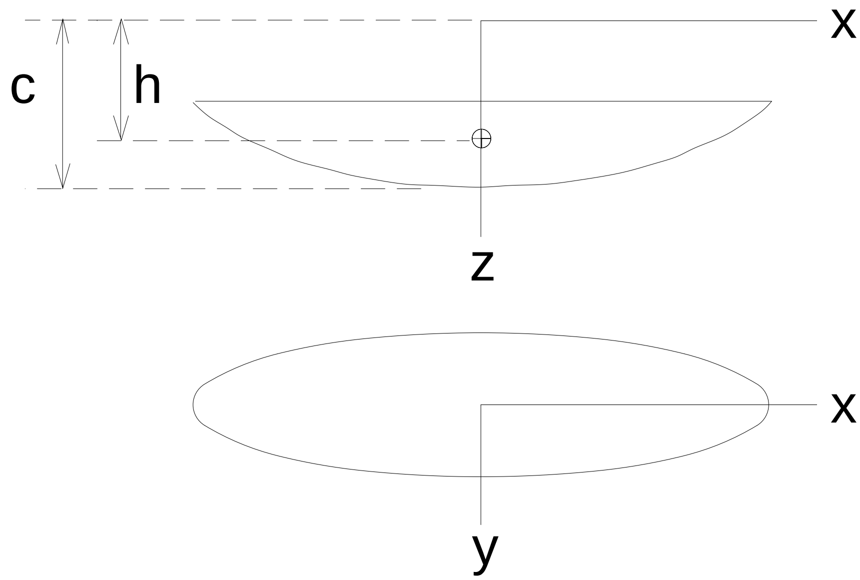

Fig.1. The surface of the celt is modeled as a portion of an ellipsoid.

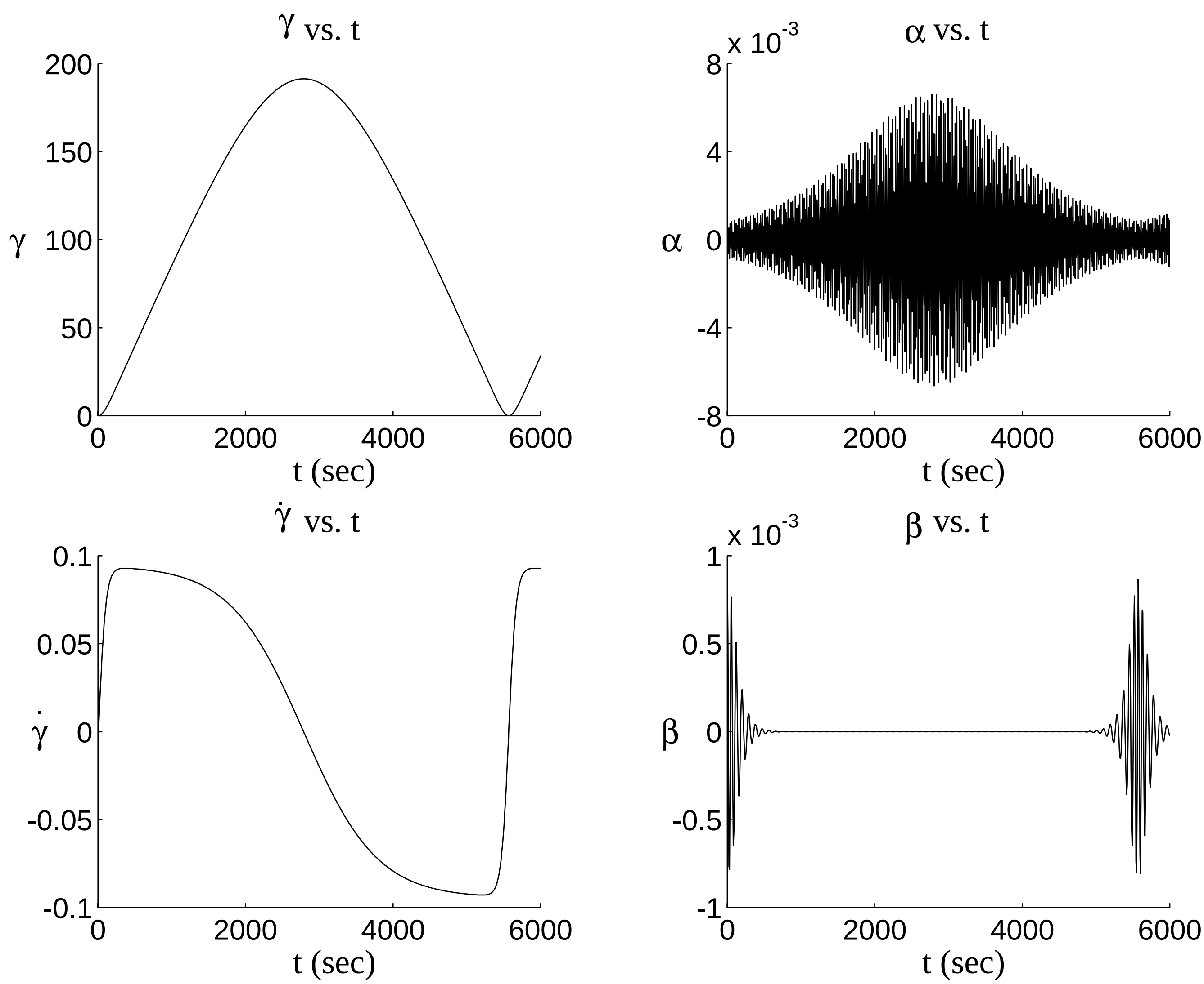

Fig.2a. Numerical integration of the averaged equations for parameters and initial conditions given in eqs.(15)-(21).

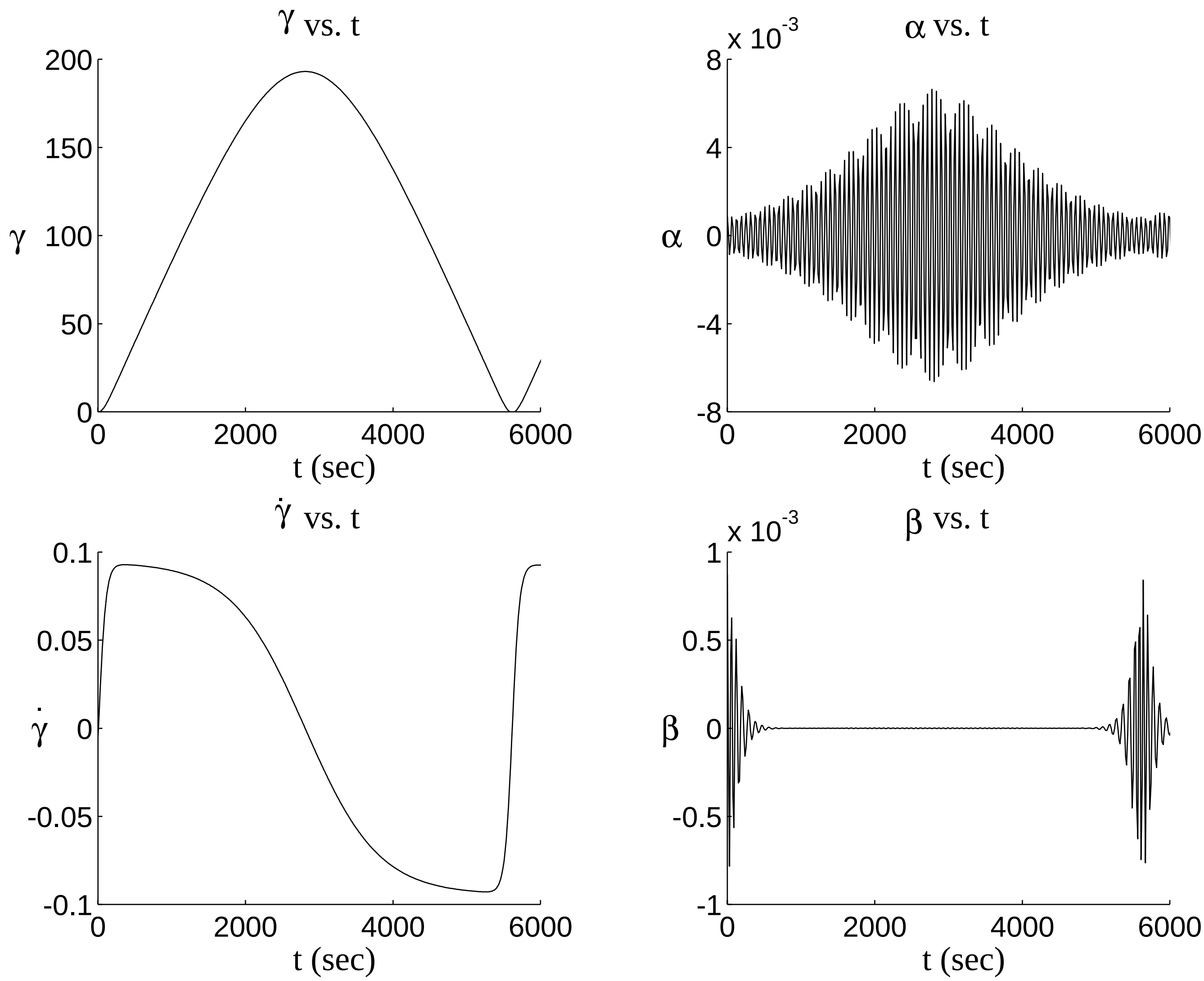

Fig.2b. Numerical integration of the original (unaveraged) equations for parameters and initial conditions given in eqs.(15)-(21).

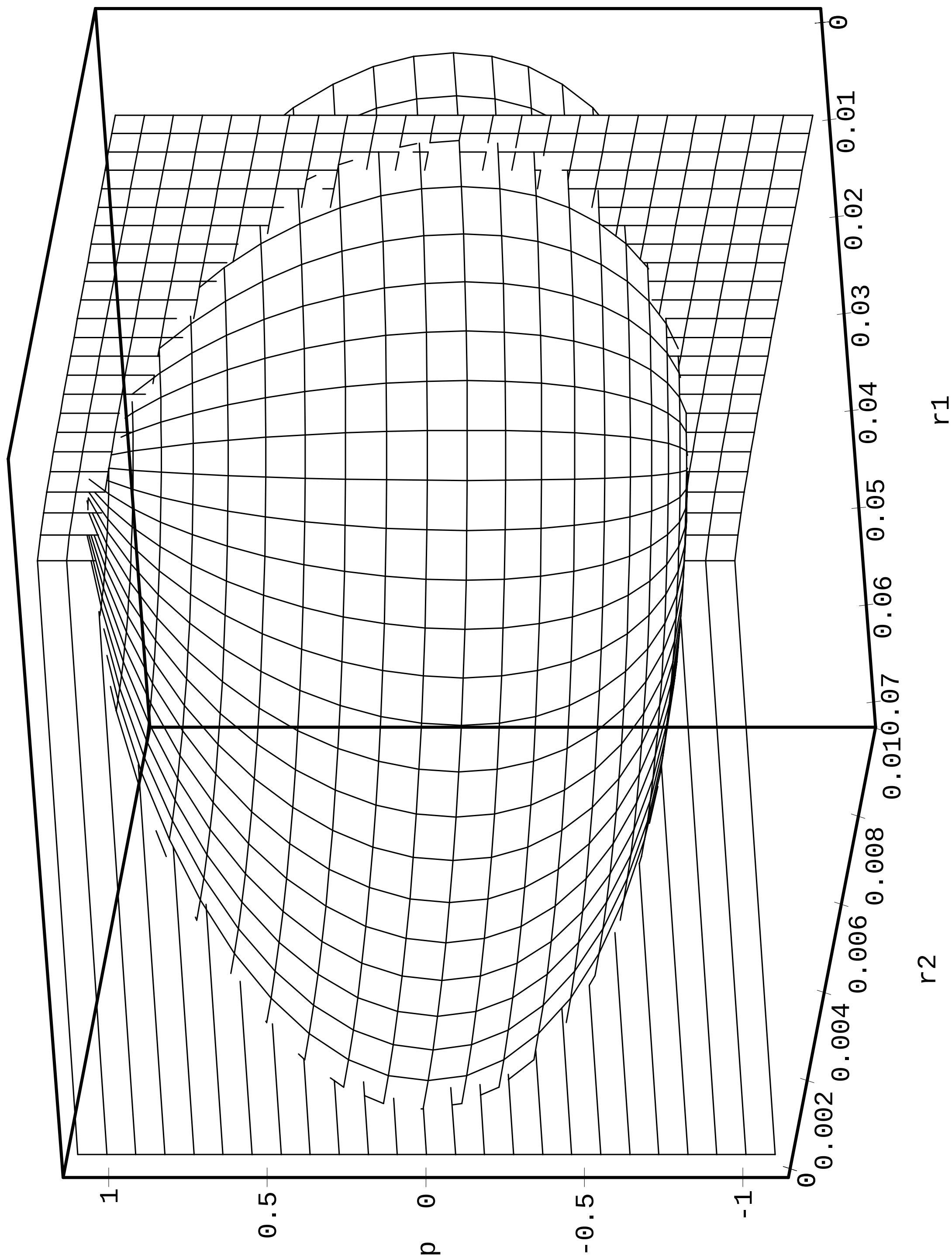

Fig.3. The two first integrals (23),(24).

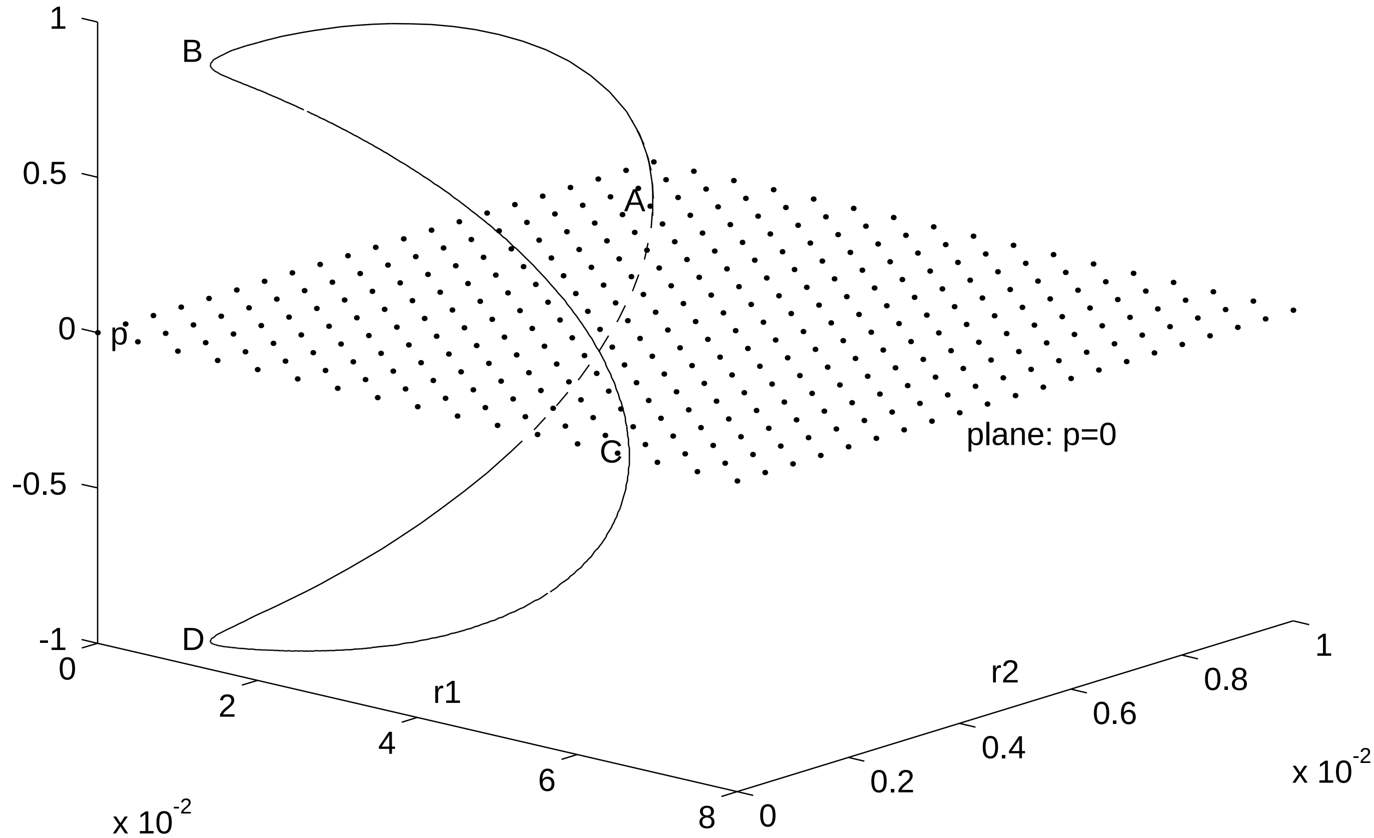

Fig.4. The motion in $r_1 - r_2 - p$ phase space lies on the intersection of the two first integrals (23),(24), cf.Fig.3. The direction of the motion is from A to B to C to D to A. See text for a description of the associated physical motion of the celt.