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The Dynamics of an Evaporating Meniscus

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With 1 Figure

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Summary — Zusammenfassung

The Dynamics of an Evaporating Meniscus. This work is concerned with evaporation from a meniscus (an air-liquid interface) on the end of a small bore, vertically oriented capillary tube. The equations governing the dynamics of the meniscus are derived by considering an evaporation-driven fluid flow. They lead to a nonlinear differential-integral equation which is linearized for small meniscus motions about steady state equilibrium.

The linearized equation is shown to contain a single dimensionless parameter K. The motion resulting from a step increase in evaporation rate is shown to be monotone for $K < K_0 \sim 47$, while for $K > K_0$ a damped oscillatory motion results. A closed form approximate solution is presented, valid for small K.

This work was motivated by the study of transpiration in plants.

Die Dynamik eines verdunstenden Meniskus. Diese Arbeit beschäftigt sich mit dem Verdunsten eines Meniskus (Trennfläche Luft—Flüssigkeit) am Ende einer engen, vertikal gerichteten Kapillarröhre. Die Gleichungen, die die Dynamik des Meniskus beschreiben, werden durch Betrachtung der durch die Verdunstung hervorgerufenen Flüssigkeitsströmung abgeleitet. Diese führen zu einer nichtlinearen Differential-Integral-Gleichung, die für kleine Meniskusbewegungen um eine stationäre Lage linearisiert wird.

Die linearisierte Gleichung enthält einen einzigen dimensionslosen Parameter K. Die Bewegung hervorgerufen durch eine schrittweise Zunahme der Verdunstungsgeschwindigkeit ist monoton für $K < K_0 \sim 47$, wogegen für $K > K_0$ eine gedämpfte Schwingung auftritt. Für kleine K wird eine geschlossene Näherungslösung angegeben.

Die vorliegende Arbeit wurde durch die Untersuchung der Verdunstung in Pflanzen angeregt.

Introduction

When a small bore glass tube is inserted vertically into a container of water, the water rises in the tube due to capillary forces. If the tube has circular cross-section of sufficiently small radius, then the meniscus at the air-liquid interface has a nearly spherical shape and may be described by its radius of curvature ξ. The pressure difference across the meniscus is well known to equal $2\sigma/\xi$, where σ = surface tension coefficient = 73 dyne/cm for an air-water interface at 20 °C ([1], pp. 230—237).

In a tube of sufficient length the meniscus rises until the pressure difference across the meniscus equals the gravitational hydrostatic pressure deficit in the liquid at the interface $= \varrho g h$, where ϱ = liquid density, g = acceleration of gravity, and h = height of the liquid column. Thus $2\sigma/\xi = \varrho g h$.

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The height of capillary rise h is determined by the minimum value of ξ attainable for the particular materials involved. If R is the tube radius, then $\xi \geqq R$. For water in capillaries made of clean smooth glass, a value of $\xi = R$ is possible. For most materials, however, a greater minimum value of ξ occurs, causing the liquid column to achieve a smaller height h. The minimum value of ξ is often characterized by a contact angle $=$ arccos (R/ξ), which depends upon the materials involved ([2]; [3], p. 50).

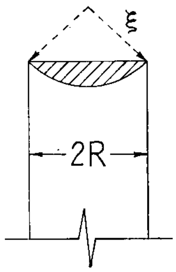

Fig. 1. Meniscus Geometry. The capillary tube has radius R and the spherical meniscus has radius of curvature ξ. The volume V is shown shaded

If the tube length L is smaller than the height h to which the liquid would normally rise, then the meniscus forms at the top of the tube and ξ takes on a value such that $2\sigma/\xi = \varrho g L$. (It is assumed in this work that the meniscus remains affixed to the top of the tube.)

Now if the meniscus is a site of evaporation, then in the steady state the rate at which liquid is transformed into vapor equals the rate at which new liquid is brought up the tube from the liquid reservoir. In steady state the additional pressure drop across the meniscus required to overcome viscous friction in the fluid is given by Poiseuille's law ([1], p. 57),

$$\Delta p = \frac{8L\eta}{\pi R^4 \varrho} Q_1, \tag{1}$$

where $\eta =$ viscosity of liquid, $Q_1 =$ mass flow rate of evaporation.

Assuming that the meniscus remains nearly spherical in shape, this additional pressure drop requires a decrease in ξ such that

$$\frac{2\sigma}{\xi} = \frac{8L\eta}{\pi R^4 \varrho} Q_1 + \varrho g L. \tag{2}$$

If the evaporation rate Q_1 is suddenly changed (for example by a sudden change in the relative humidity of the ambient air in the vicinity of the meniscus), then ξ becomes a function of time t. It is the purpose of this work to consider how ξ varies with t.

In a capillary tube of relatively large radius, the meniscus shape may depart significantly from a spherical surface. Steady state meniscus profiles under evaporation and gravitational loading have recently been investigated [4]. As stated above, it is assumed in this paper that the tube has sufficiently small radius for the meniscus to remain nearly spherical.

This problem is of particular relevance to the phenomenon of transpiration in plants [3], [5]. The sites of evaporation in plants are the mesophyll cell walls in leaves. These cell walls have interstices with a representative "radius" of about 50 Angstroms. A continuous fluid column (of variable cross-section and with branchings) reaches from the soil water at the roots, up the stem and into the evaporation sites at the leaves. (In this paper the fluid column is modeled as having constant cross-section without branchings.) A decrease in relative humidity outside the leaves causes an increase in transpiration rate and a corresponding decrease in the radii of curvature ξ associated with the menisci in the cell wall interstices.

There follows a brief description of some related aspects of the transpiration process.

If the supply of water available at the roots becomes reduced, then greater pressure differences across the cell wall menisci are required for transpiration. Under such conditions of "water stress" the radii of curvature of menisci in some leaf mesophyll cell wall interstices may be reduced to their minimum attainable values. Any additional water stress will cause these menisci to retract away from the cell wall surface, i.e., to move to positions in the capillary interior. This effect, called "incipient drying", decreases the effective evaporation from such menisci (since the evaporating water vapor must now diffuse through an additional distance), and has therefore been conjectured to regulate transpiration in plants [6]. Incipient drying is related to the process of dynamic capillary rise [7], and will not be considered here.

The problem of transpiration is further complicated by the presence of stomata (minute openings) in the leaf surface. Water vapor, after evaporating from the mesophyll cell walls, diffuses through twisting pathways in the leaf's interior and finally emerges from the leaf via the stomata. By opening and closing in response to pressure changes in leaf cells [8], the stomata provide a mechanism for the regulation of transpiration. With regard to the present work, the effects of stomatal movements are assumed to be included in the (prescribed) evaporation rate $Q_1(t)$.

The Governing Equations

The basic equation is derived from conservation of mass: the rate of increase of mass of liquid in the tube equals the rate at which new liquid is brought up the tube by fluid flow minus the rate at which liquid is transformed into vapor by evaporation.

$$\frac{dM}{dt} = Q_2 - Q_1, \tag{3}$$

where M = mass of liquid in the tube,
Q_2 = mass flow rate of liquid into tube ("discharge"),
Q_1 = mass flow rate of evaporation from tube.

Let V represent the volume (of air) between the spherical meniscus and the end of the circular cylindrical tube (Fig. 1). The volume of liquid in the tube equals total tube volume minus volume of air in the tube = constant $-$ V. Therefore

$$\frac{dM}{dt} = -\varrho\frac{dV}{dt}. \tag{4}$$

The volume V is a "spherical segment of one base" [9]. Its volume is

$$V = \frac{\pi}{6}a(3R^2 + a), \tag{5}$$

where a = altitude of segment $= \xi - (\xi^2 - R^2)^{1/2}$,
R = tube radius.

Substituting,

$$V = \frac{2\pi}{3}\left[\xi^3 - \left(\xi^2 + \frac{R^2}{2}\right)(\xi^2 - R^2)^{1/2}\right]. \tag{6}$$

Treating ξ as the dependent variable, $\xi = \xi(t)$,

$$\varrho\frac{dV}{d\xi}\frac{d\xi}{dt} = Q_1 - Q_2 \tag{7}$$

where

$$\frac{dV}{d\xi} = -2\pi R^2\beta, \tag{8}$$

$$\beta = \beta\left(\frac{\xi}{R}\right) = \frac{\xi}{R}\left[\left(\frac{\xi}{R}\right)^2 - \frac{1}{2}\right]\left[\left(\frac{\xi}{R}\right)^2 - 1\right]^{-1/2} - \left(\frac{\xi}{R}\right)^2. \tag{9}$$

The evaporation rate $Q_1(t)$ is here assumed to be a prescibed function of time. The discharge Q_2 is found as a function of ξ and t as follows.

Consider the axisymmetric flow of an incompressible viscous fluid in a circular cylindrical tube. Assuming no radial velocity component, the flow may be described relative to cylindrical polar coordinates r, θ, z, by $w = w(r, z, t)$ = fluid velocity in z direction and $p = p(r, z, t)$ = fluid pressure. Orienting the z axis vertically upwards, the Navier-Stokes equations and the equation of continuity reduce to ([1], p. 51)

$$\frac{\partial w}{\partial z} = \frac{\partial p}{\partial r} = 0, \tag{10}$$

$$\frac{\partial p}{\partial z} = -\varrho\frac{\partial w}{\partial t} + \eta\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \varrho g. \tag{11}$$

From Eqs. (10), $w = w(r, t)$ and $p = p(z, t)$. Thus the LHS of Eq. (11) is a function of z and t only, while the RHS is a function of r and t only. Hence both sides must depend only on t and may be equated to an arbitrary function of t, say $A(t)$. Then

$$p(z, t) = zA(t) + B(t), \tag{12}$$

where $B(t)$ is also arbitrary. $A(t)$ and $B(t)$ are evaluated by using the following boundary conditions.

At the free surface of the fluid reservoir located at the bottom of the tube, $z = 0$, the pressure is taken to be atmospheric, $p = 0$.

$$z = 0, \qquad p = 0 \quad \text{gives} \quad B(t) = 0. \tag{13}$$

In the fluid at the meniscus, the pressure is taken to be below atmospheric by the pressure difference across the meniscus, i.e., by $2\sigma/\xi$.

$$z = L, \qquad p = -2\sigma/\xi \quad \text{gives} \quad A(t) = -\frac{2\sigma}{L\xi(t)}. \tag{14}$$

Here it is assumed that the tube length $L \gg R$ so that little error is introduced by applying the meniscus boundary condition at the end of the tube, $z = L$, instead of along the spherical meniscus surface.

For convenience, the independent variables t, r are now replaced by the dimensionless variables τ, y:

$$\tau = \frac{\eta}{\varrho R^2} t, \qquad y = \frac{r}{R}. \tag{15}$$

Eq. (11) on $w(\tau, y)$ becomes

$$\frac{\partial w}{\partial \tau} - \frac{\partial^2 w}{\partial y^2} - \frac{1}{y}\frac{\partial w}{\partial y} = F(\tau), \tag{16}$$

where

$$F(\tau) = F\big(\xi(\tau)\big) = \frac{2\sigma R^2}{L\eta\xi(\tau)} - \frac{\varrho R^2 g}{\eta}. \tag{17}$$

Eq. (16) is accompanied by the boundary conditions

$$y = 1 \quad (r = R), \qquad w = 0, \tag{18}$$

$$y = 0 \quad (r = 0), \qquad \frac{\partial w}{\partial y} = 0 \tag{19}$$

and by the initial condition

$$\tau = 0 \quad (t = 0), \qquad w = f(y), \tag{20}$$

where $f(y)$ is as yet unspecified.

Eq. (16) has the complementary solution $\big(F(\tau) \equiv 0\big)$

$$w = \sum A_n J_0(\Omega_n y)\, e^{-\Omega_n^2 \tau}, \tag{21}$$

where $\sum$ represents $\sum_{n=1}^{\infty}$ unless otherwise noted,

A_n is an arbitrary constant,
J_0 is the zeroth order Bessel function,
Ω_n is the nth positive root of J_0, i.e.,

$$J_0(\Omega_n) = 0. \tag{22}$$

To obtain a solution to the nonhomogeneous Eq. (16), expand the RHS in a series of the eigenfunctions of the homogeneous equation,

$$F(\tau) = F(\tau) \sum B_n J_0(\Omega_n y), \tag{23}$$

where

$$B_n = \frac{2}{\Omega_n J_1(\Omega_n)} \tag{24}$$

(see, e.g., [10], p. 57). Then substituting

$$w = \sum g_n(\tau)\, J_0(\Omega_n y) \tag{25}$$

into Eq. (16) gives

$$\frac{dg_n}{d\tau} + \Omega_n^2 g_n = B_n F(\tau), \tag{26}$$

which has the general solution

$$g_n(\tau) = A_n e^{-\Omega_n^2 \tau} + B_n e^{-\Omega_n^2 \tau} \int_0^\tau e^{\Omega_n^2 \tau'} F(\tau')\, d\tau', \tag{27}$$

where A_n is chosen to satisfy the initial condition (20),

$$w(y, 0) = f(y) = \sum A_n J_0(\Omega_n y), \tag{28}$$

$$A_n = \frac{\int_0^1 y f(y)\, J_0(\Omega_n y)\, dy}{\frac{1}{2} J_1^2(\Omega_n)} \tag{29}$$

(see, e.g., [10], p. 52).

Now the discharge Q_2 may be computed from Eqs. (25), (27),

$$Q_2 = 2\pi R^2 \varrho \int_0^1 y w(y, \tau)\, dy, \tag{30}$$

$$Q_2 = E(\tau) + 4\pi R^2 \varrho \int_0^\tau \Phi(\tau - \tau')\, F(\tau')\, d\tau', \tag{31}$$

where

$$E(\tau) = 2\pi R^2 \varrho \sum A_n e^{-\Omega_n^2 \tau} \int_0^1 y J_0(\Omega_n y)\, dy, \tag{32}$$

$$E(\tau) = 2\pi R^2 \varrho \sum \frac{J_1(\Omega_n)}{\Omega_n} A_n e^{-\Omega_n^2 \tau}, \tag{33}$$

$$\Phi(\tau - \tau') = \sum \Omega_n^{-2} e^{-\Omega_n^2 (\tau - \tau')}. \tag{34}$$

In evaluating the integral in Eq. (32), Bessel's equation has been used. Note that $E(\tau)$ depends upon the initial condition (cf. Eqs. (28), (29)) while $\Phi(\tau - \tau')$ does not.

Finally the equation governing the meniscus motion $\xi(\tau)$, Eq. (7), becomes (using $d\tau/dt = \eta/\varrho R^2$)

$$\frac{\eta}{R^2}\frac{dV}{d\xi}(\xi)\frac{d\xi}{d\tau} = Q_1(\tau) - E(\tau) - 4\pi R^2 \varrho \int_0^\tau \Phi(\tau - \tau') F(\xi(\tau')) d\tau', \tag{35}$$

where $dV/d\xi$ and F are given by Eqs. (8) and (17) respectively.

Steady State Solution

The nonlinear differential-integral Eq. (35) must possess the steady state solution discussed in the introduction, Eq. (2), if $Q_1(\tau) =$ constant.

At steady state $\xi(\tau) =$ constant and $F = F(\xi(\tau))$ may be removed from under the integral sign in Eq. (35), which becomes

$$Q_1 = E(\tau) + 4\pi R^2 \varrho F \int_0^\tau \Phi(\tau - \tau') d\tau'. \tag{36}$$

From Eq. (34),

$$\int_0^\tau \Phi(\tau - \tau') d\tau' = \sum \Omega_n^{-4}(1 - e^{-\Omega_n^2 \tau}). \tag{37}$$

In order that Eq. (36) be identically satisfied for all τ, $E(\tau)$ must be chosen to correspond to the following initial condition

$$w(y, 0) = f(y) = \frac{F}{4} \cdot (1 - y^2). \tag{38}$$

(This velocity distribution is that of steady state Poiseuille flow.)

Then Eq. (29) gives (using Bessel's differential equation and integrating by parts)

$$A_n = \frac{2F}{\Omega_n^3 J_1(\Omega_n)}. \tag{39}$$

Substitution into Eq. (33) gives

$$E(\tau) = 4\pi R^2 \varrho F \sum \Omega_n^{-4} e^{-\Omega_n^2 \tau}. \tag{40}$$

Using Eqs. (37) and (40), Eq. (36) becomes

$$Q_1 = 4\pi R^2 \varrho F \sum \Omega_n^{-4} \tag{41}$$

or

$$Q_1 = \frac{\pi}{8} R^2 \varrho F(\xi) \tag{42}$$

since $\sum \Omega_n^{-4} = 1/32$, ([10], p. 57).

Eq. (42), together with the equation defining $F(\xi)$, Eq. (17), may be rearranged to give Eq. (2).

Linearization

Let $Q_1(\tau) = Q_1{}^0 =$ constant correspond to the steady state solution $\xi(\tau) = \xi_0$ = constant. Then (cf. Eq. (36))

$$Q_1{}^0 = E(\tau) + 4\pi R^2 \varrho F(\xi_0) \int\limits_0^\tau \Phi(\tau - \tau')\, d\tau', \qquad (43)$$

where $E(\tau)$ is chosen as in Eq. (40). Also (cf. Eq. (42))

$$Q_1{}^0 = \frac{\pi}{8} R^2 \varrho F(\xi_0). \qquad (44)$$

Suppose that at $\tau = 0^-$ the system is in steady state corresponding to $Q_1(\tau) = Q_1{}^0$. Then let $Q_1(\tau)$ suddenly jump at $\tau = 0$,

$$Q_1(\tau) = Q_1{}^0 + \varepsilon Q_1{}^*, \qquad \tau > 0, \qquad (45)$$

where ε is a small parameter and where $Q_1{}^*$ is a constant.

The resulting meniscus motion $\xi(\tau)$ is the subject of the remainder of this paper. It is assumed that the meniscus motion and the fluid motion are initially unperturbed, i.e.,

$$\xi(0) = \xi_0 \qquad (46)$$

and $E(\tau)$ is chosen as in Eq. (40), to identically satisfy Eq. (43).

Substituting

$$\xi(\tau) = \xi_0 + \varepsilon \xi^*(\tau) \qquad (47)$$

into Eq. (35) and linearizing for small ε, obtain

$$\frac{\eta}{R^2} \frac{dV}{d\xi}(\xi_0) \frac{d\xi^*}{d\tau} = Q_1{}^* - 4\pi R^2 \varrho \frac{dF}{d\xi}(\xi_0) \int\limits_0^\tau \Phi(\tau - \tau')\, \xi^*(\tau')\, d\tau', \qquad (48)$$

where Eq. (43) has been used and where $F(\xi)$ has been expanded in a Taylor series about $\xi = \xi_0$.

Before considering Eq. (48), it is desirable to find the new steady state value ξ_s,

$$\xi_s = \xi_0 + \varepsilon \xi_s{}^*. \qquad (49)$$

At steady state (cf. Eq. (44))

$$Q_1{}^0 + \varepsilon Q_1{}^* = \frac{\pi}{8} R^2 \varrho F(\xi_0 + \varepsilon \xi_s{}^*). \qquad (50)$$

To first order in ε, using Eq. (44),

$$Q_1{}^* = \frac{\pi}{8} R^2 \varrho \frac{dF}{d\xi}(\xi_0)\, \xi_s{}^*. \qquad (51)$$

For convenience, the dependent variable $\xi^*(\tau)$ is replaced by the dimensionless variable $x(\tau)$:

$$x(\tau) = \xi^*(\tau)/\xi_s^*, \tag{52}$$

where ξ_s^* is given by Eq. (51). Thus $x(\tau)$ has the steady state value of unity. Moreover the initial condition (46) becomes

$$x(0) = 0. \tag{53}$$

Eq. (48) becomes

$$\frac{dx}{d\tau} = \frac{K}{32} - K \int_0^\tau \Phi(\tau - \tau')\, x(\tau')\, d\tau', \tag{54}$$

where

$$K = \frac{4\pi R^4 \varrho \dfrac{dF}{d\xi}(\xi_0)}{\eta \dfrac{dV}{d\xi}(\xi_0)} = \frac{4R^4 \varrho\sigma}{L\eta^2 \xi_0^2 \beta} \tag{55}$$

(here Eqs. (8) and (17) have been used).

K is a dimensionless parameter which characterizes the behavior of the linear Eq. (54). For example, with $\sigma = 73$ dyne/cm, $\varrho = 1$ g/cm^3, $\eta = .01$ g/cm-sec, $L = 10$ cm, $R = 5$ µm, $\xi_0 = 2R$, computation gives $\beta = 0.041$ (Eq. (9)) and $K = 0.44$.

Solution of the Linearized Equation

Eq. (54) contains a convolution integral. Taking the Laplace transform of Eq. (54) and utilizing the transform's convolution property [11], obtain

$$sL(x) = \frac{K}{32s} - KL(\Phi)\, L(x), \tag{56}$$

where

$$L(x) = \int_0^\infty e^{-s\tau} x(\tau)\, d\tau, \tag{57}$$

$$\begin{aligned} L(\Phi) &= \int_0^\infty e^{-s\tau} \Phi(\tau)\, d\tau \\ &= \int_0^\infty e^{-s\tau} \left(\sum \Omega_n^{-2} e^{-\Omega_n^2 \tau}\right) d\tau \\ &= \sum \Omega_n^{-2} (\Omega_n^2 + s)^{-1}, \end{aligned} \tag{58}$$

and where Eq. (53) has been used. Solving for $L(x)$,

$$L(x) = \frac{K}{32sG(s)} \tag{59}$$

where

$$G(s) = s + K \sum \Omega_n^{-2}(\Omega_n^2 + s)^{-1}. \tag{60}$$

In order to accomplish the Laplace transform inversion, the singularities of $L(x)$ on the complex s-plane must be considered. Since all the singularities are poles (no branch points) [12],

$$x(\tau) = \sum \text{residue}\left(e^{s\tau} L(x)\right), \tag{61}$$

where the sum is taken over all singularities of $L(x)$.

Consider first the simple pole at $s = 0$:

$$\text{res}\left(e^{s\tau} L(x)\right)_{s=0} = \frac{K}{32 G(0)} = \frac{1}{32 \sum \Omega_n^{-4}} = 1 \tag{62}$$

since $\sum \Omega_n^{-4} = 1/32$ [10]. This residue contributes the steady state term to $x(\tau)$ (cf. Eq. (52)).

The other singularities of $L(x)$ occur at the roots of

$$G(s) = 0. \tag{63}$$

Setting $s = u + iv$, this produces two real equations:

$$u + K \sum \Omega_n^{-2}(\Omega_n^2 + u)\,[(\Omega_n^2 + u)^2 + v^2]^{-1} = 0, \tag{64}$$

$$v\left\{1 - K \sum \Omega_n^{-2}[(\Omega_n^2 + u)^2 + v^2]^{-1}\right\} = 0. \tag{65}$$

Numerical evaluation reveals that for $K < K_0 \sim 47$ the only solution to these equations occurs for $v = 0$, $u < 0$. Such roots lead to contributions to $x(\tau)$ which exponentially decay in a monotone fashion. For $K > K_0$ there also exist complex roots, the residues of which correspond to damped oscillatory motions. The Appendix contains a closed form approximation for K_0.

In what follows it is assumed that $K \ll 1$. For $v = 0$, Eq. (64) becomes

$$u + K \sum \Omega_n^{-2}(\Omega_n^2 + u)^{-1} = 0. \tag{66}$$

For small K there exist simple roots near $u = 0$, $-\Omega_1^2$, $-\Omega_2^2, \ldots$ Expressions for these roots will be obtained in the form of power series expansions in K.

Firstly consider the root s_0 near $u = 0$. Substituting

$$s_0 = \alpha_1 K + \alpha_2 K^2 + \alpha_3 K^3 + 0(K^4) \tag{67}$$

into Eq. (66), collecting terms and equating to zero coefficients of K^n ($n = 1, 2, 3$) gives

$$\begin{aligned} \alpha_1 &= -H_4 \\ \alpha_2 &= -H_4 H_6 \\ \alpha_3 &= -H_4^2 H_8 - H_4 H_6^2 \end{aligned} \tag{68}$$

where

$$H_4 = \sum \Omega_n^{-4} = 1/32$$
$$H_6 = \sum \Omega_n^{-6} = 1/192 \tag{69}$$
$$H_8 = \sum \Omega_n^{-8} = 11/12288.$$

(These last results are derivable from the infinite product expression for J_0. See [10], p. 57.)

For the root s_m near $-\Omega_m^2$ ($m = 1, 2, 3, \ldots$), one similarly finds

$$s_m = -\Omega_m^2 + \Omega_m^{-4}K + 0(K^2). \tag{70}$$

(Although s_0 has been found to $0(K^4)$ while s_m to $0(K^2)$, these expressions result in the same order contributions to $x(\tau)$, see below.)

Since all the singularities s_m ($m = 0, 1, 2, \ldots$) (resulting from $G(s) = 0$) are simple poles, the contribution to $x(\tau)$ from s_m is [12]

$$\text{res}\left(e^{s\tau}L(x)\right)_{s=s_m} = \frac{Ke^{s_m\tau}}{32s_m \dfrac{dG}{ds}(s_m)}$$

$$\text{res}\left(e^{s\tau}L(x)\right)_{s=s_m} = \frac{Ke^{s_m\tau}}{32s_m[1 - K\sum \Omega_n^{-2}(\Omega_n^2 + s_m)^{-2}]}. \tag{71}$$

Again consider the $m = 0$ case first. Substituting s_0 from Eq. (67) into Eq. (71) and expanding in powers of K gives

$$\text{res}\left(e^{s\tau}L(x)\right)_{s=s_0} = -e^{s_0\tau}\left(1 + H_4H_8K^2 + 0(K^3)\right). \tag{72}$$

Similarly the residue at $s = s_m$ is found by substituting Eq. (70) into Eq. (71),

$$\text{res}\left(e^{s\tau}L(x)\right)_{s=s_m} = e^{s_m\tau}\left(K^2\Omega_m^{-8}/32 + 0(K^3)\right). \tag{73}$$

Finally, substituting Eqs. (62), (72), (73) into Eq. (61) gives

$$x(\tau) = 1 - e^{s_0\tau} + \frac{K^2}{32}\left[-(\sum \Omega_m^{-8})\, e^{s_0\tau} + \sum \Omega_m^{-8}e^{s_m\tau}\right] + 0(K^3). \tag{74}$$

Here s_0 and s_m are given by Eqs. (67) and (70). Note that s_0 and $s_m < 0$ so that $x(\tau) \to 1$ as $\tau \to \infty$, as expected (cf. Eq. (52)). Note also that $x(0) = 0$ (cf. Eq. (53)).

Appendix. Approximation of K_0

For $K > K_0$ there exist complex roots to Eq. (63), $G(s) = 0$, while for $K < K_0$ all such roots lie on the negative u axis.

In order for there to exist complex roots, the second factor in Eq. (65) must vanish:

$$1 - K\sum \Omega_n^{-2}[(\Omega_n^2 + u)^2 + v^2]^{-1} = 0. \tag{A.1}$$

By continuity, the value $K = K_0$ will correspond to a real root, i.e., to a simultaneous solution of Eq. (A.1) and Eq. (64) such that $v = 0$:

$$1 - K_0 \sum \Omega_n^{-2}(\Omega_n^2 + u)^{-2} = 0, \tag{A.2}$$

$$u + K_0 \sum \Omega_n^{-2}(\Omega_n^2 + u)^{-1} = 0. \tag{A.3}$$

As these series converge rapidly, a close approximation for K_0 may be obtained by taking only one term,

$$1 - K_0 \Omega_1^{-2}(\Omega_1^2 + u)^{-2} = 0, \tag{A.4}$$

$$u + K_0 \Omega_1^{-2}(\Omega_1^2 + u)^{-1} = 0. \tag{A.5}$$

Eliminating K_0 gives $u = -\Omega_1^2/2$ and therefore

$$K_0 \sim \Omega_1^6/4 \sim 48.4 \tag{A.6}$$

since $\Omega_1 \sim 2.405$ ([10], p. 50).

Numerical evaluation of Eqs. (A.2) and (A.3) yields a value of $K_0 \sim 47$.

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