COMPTES RENDUS
PROCEEDINGS

CANCAM 73

QUATRIÈME CONGRÈS CANADIEN DE MÉCANIQUE APPLIQUÉE

FOURTH CANADIAN CONGRESS OF APPLIED MECHANICS

28 MAI - 1er JUIN 1973 **MAY 28 - JUNE 1st, 1973**

ÉCOLE POLYTECHNIQUE DE MONTRÉAL

A. BAZERGUI
Éditeur — Editor

A. BIRON
Président du congrès
Chairman of Congress

P.P. FAZIO
Président du comité des communications
Chairman, Papers Committee

NONLINEAR NORMAL MODES IN DYNAMICAL SYSTEMS WITH NON-EUCLIDEAN METRICS

Gopalakrishna Nadig
Graduate Student
Department of Theoretical and Applied Mechanics
Cornell University
Ithaca, N.Y. 14850, U.S.A.

Richard H. Rand
Assistant Professor
Department of Theoretical and Applied Mechanics
Cornell University
Ithaca, N.Y. 14850, U.S.A.

This paper is concerned with the geometrical dynamics of a general class of holonomic, scleronomous, conservative systems with two degrees of freedom, whose generalized co-ordinates cover a two-dimensional Riemannian manifold. Geometrical dynamics is the study of the geometry of the trajectories of a dynamical system in the manifold of configurations, without reference to the motion of system in time. The primary objective of this study is to extend the analysis of recent literature on non-linear normal vibrations ([1],[2]) to systems whose metrical structure is essentially non-Euclidean. Even though the superposition principle is not applicable to nonlinear systems, normal modes are nevertheless of importance inasmuch as resonance occurs in the neighborhood of such motions [1].

Consider a general class of conservative systems whose dynamical properties are defined by the following expressions for the kinetic and potential energies:

$$T = \frac{1}{2}[A(x,y)\dot{x}^2+B(x,y)\dot{y}^2+2C(x,y)\dot{x}\dot{y}], \qquad (1)$$

where $A = A_0+A_1f^2(x,y)$; $B = B_0+B_1f^2(x,y)$; $C = C_1f^2(x,y)$, with $f(x,y) = \lambda x+\mu y$. (A_0, B_0, A_1, B_1, C_1, λ and μ are all constants).

$$V = ax^2+by^2+\alpha x^4+\beta x^3y+\gamma x^2y^2+\sigma xy^3+\tau y^4, \qquad (2)$$

where a, b, ..., τ are constants such that V is positive definite.

The metric associated with the kinetic energy (1),

$$ds^2 = 2Tdt^2 = Adx^2+Bdy^2+2Cdxdy, \qquad (3)$$

is in general non-Euclidean. Previous studies ([1],[2]) have taken A = B = 1, C = 0, giving a Euclidean metric. For general A, B, C, this metric will be Euclidean if and only if the associated Riemann-Christoffel tensor vanishes [5].

Using the energy integral $T+V = h$, one can derive the (time independent) geometrico-dynamical equation for the system in the form [3]

$$\begin{aligned}(h-V)[&2(AB-C^2)y''+(BB_x+CB_y-2BC_y)y'^3 \\ &+(AB_y-2BA_y+3CB_x-2CC_y)y'^2 \\ &-(BA_x-2AB_x+3CA_y-2CC_x)y'-(AA_y+CA_x-2AC_x)] \\ &+(A+By'^2+2Cy')[V_y(A+Cy')-V_x(C+By')] = 0 ,\end{aligned} \qquad (4)$$

where the primes denote differentiation with respect to x. For small energy levels h, ($h \ll 1$), a normal modal solution to (4) is assumed in the form of a power series expansion in the parameter h. To obtain the modal curve which is an analytic continuation of the linear mode $y \equiv 0$, we employ the perturbation scheme

Comptes rendus du quatrième congrès canadien de mécanique appliquée, Montréal, 28 mai - 1er juin 1973.
Proceedings of the Fourth Canadian Congress of Applied Mechanics, Montreal, May 28 - June 1st, 1973.

$$x = h^{1/2}\xi;\ y = h^{1/2}\sum_{k=1}^{\infty} h^k \eta_k(\xi) \quad . \tag{5}$$

Substituting (5) into (4) and equating to zero coefficients of like powers of h, one finds that the equation on η_k, (k=1,2,3,...), is a Tchebycheff differential equation. Solving these successively, the normal modal curve is found to be

$$y(x) = (L+Mh)hx + (N+Ph)x^3 + Qx^5 + 0(h^{7/2}), \tag{6}$$

where the constants, L,M,N,P and Q are known functions of the system parameters [3]. To obtain an equation analogous to (6) for the nonlinear counterpart of the linear normal mode $x \equiv 0$, interchange x and y throughout.

The natural frequency of oscillation corresponding to the normal mode (6) can be found as follows:

Let the modal curve (6) intersect the bounding equipotential curve V = h at the point (X,Y). The elapsed time during the motion of the system from the equilibrium position (0,0) to the extreme position (X,Y) is one-quarter of a period of vibration. Using the modal curve (6) in conjunction with the fact that T + V = h, one can write $\dot{x} = F(h,x)$. Hence the period and the frequency of normal mode vibration are given by

$$T_0 = 4\int_0^X \frac{dx}{F(h,x)}\ ;\ \omega_0 = \frac{2\pi}{T_0} \quad . \tag{7}$$

The validity of the asymptotic formula (6) was checked by numerical integration, as well as by comparison with an example for which an exact solution exists (the Hoop-Pendulum system [3]). Application of the results of the preceding analysis to the fruit-stem system of Ref. [4] gives excellent agreement. It can be easily demonstrated that the analysis of this paper can be utilized to obtain explicit approximate expressions for the nonlinear normal modes of a wide variety of dual-mode mechanical systems. Included in this general class are such two-degree-of-freedom systems as a plane double pendulum, a variety of mass-spring systems, a particle moving on an arbitrary surface under gravity.

The method of this paper fails when the linearized natural frequencies of the system are in the ratio of odd integers, i.e., when

$$(A_0/a)/(B_0/b) = (2n+1)^2, \quad (n=0,1,2,...), \tag{8}$$

because of vanishing denominators in (6).

<u>References</u>

[1] Rosenberg, R.M., Advances in Applied Mechanics, vol. 9, pp. 155-242, Academic Press (1966).

[2] Rand, R.H., Int. Jour. of Nonlinear Mech., vol. 6, pp. 545-547 (1971).

[3] Nadig, G., Ph.D. Thesis, Cornell University (1973).

[4] Rand, R.H. and Cooke, J.R., Jour. of Ag. Engng. Res., vol. 15, pp. 347-363 (1970).

[5] Sokolnikoff, I.S., Tensor Analysis, Wiley (1951).