

Dynamics of three coupled limit cycle oscillators with vastly different frequencies

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Summary. An autonomous system of three coupled nonlinear oscillators with widely separated frequencies is investigated using the method of direct partition of motion (DPM). Approximate expressions for two of the limit cycle oscillations are presented in terms of Jacobi elliptic functions whose amplitude and frequency depend on the coupling parameters. Critical values for the coupling parameters were found, above which the oscillations of one of the limit cycles are quenched.

Introduction

The effects of high frequency excitation on nonlinear mechanical systems have been extensively studied and reviewed in recent years [1, 4, 8, 9]. These effects include apparent changes in system properties such as the number of equilibrium points, stability of equilibrium points, natural frequencies, stiffness, bifurcation paths [9], and existence of limit cycles [2]. Such problems can be analyzed using standard perturbation methods such as the method of multiple timescales or the method of averaging [9]. However, the method of direct partition of motion (DPM) developed by Blekhman [1] serves to facilitate the study of such problems. Unlike the averaging method or the method of multiple timescales, DPM offers no systematic way to obtain higher order terms in an asymptotic expansion of the solution, and instead is limited to the leading order dynamics of the system. In return for this limitation, one gains efficiency in terms of the required mathematical manipulations. Particularly, DPM is most useful when the main interest is in the leading order slow motion of the system that is subject to the fast excitation. A common feature of all the aforementioned works is that the fast excitation is due to an external source, that is, all the systems considered are nonautonomous. We assert that similar non-trivial effects could occur even if the fast excitation is internal to the system, instead of coming from an external source. An example of such a case would be a nonlinear oscillator coupled to a much faster oscillator. Systems of coupled nonlinear oscillators with widely separated frequencies have been investigated in the literature [10, 5]. Often, the method of averaging is used to study the dynamics, while here, we extend the standard DPM procedure to study an autonomous system of three coupled nonlinear oscillators with widely separated frequencies. When uncoupled, each of the oscillators possesses a limit cycle solution with a frequency $\omega_1 = O(1)$, $\omega_2 = O(1/\varepsilon)$ and $\omega_3 = O(1/\varepsilon^2)$ respectively, where $\varepsilon \ll 1$. We find that the coupling between such oscillators causes a change in the amplitude and frequency of the limit cycles of oscillators 1 and 2, and if the coupling between the oscillators is strong enough then the stable limit cycle of one of these two oscillators disappears. The limit cycle of the fastest oscillator, to leading order, is unchanged by the coupling.

Dynamics of the three coupled limit cycle oscillators

We will consider three van der Pol type limit cycle oscillators x , y and z , which are governed by the following equations:

$$\begin{aligned} \frac{d^2x}{dt_1^2} + x + (a_1 + b_1x^2) \frac{dx}{dt_1} &= \gamma_1 (1 + g_1x^2) \frac{dy}{dt_1} \\ \frac{d^2y}{dt_2^2} + y + (a_2 + b_2y^2) \frac{dy}{dt_2} &= (1 + g_2y^2) \left[\gamma_1 \frac{dx}{dt_2} + \gamma_2 \frac{dz}{dt_2} \right] \\ \frac{d^2z}{dt_3^2} + z + (a_3 + b_3z^2) \frac{dz}{dt_3} &= \gamma_2 (1 + g_3z^2) \frac{dy}{dt_3} \end{aligned} \quad (1)$$

where $t_1 = \omega_1 t$, $t_2 = \frac{\omega_2}{\varepsilon} t$, $t_3 = \frac{\omega_3}{\varepsilon^2} t$, $\varepsilon \ll 1$

Here, ω_1 , ω_2 and ω_3 are $O(1)$ quantities. We are interested in values of a_i and b_i ($i=1,2,3$) for which each of the equations above for x , y and z possesses a stable limit cycle solution that is an $O(\varepsilon)$ perturbation off of a simple harmonic motion that occurs on the time scales t_1 , t_2 and t_3 respectively. We will investigate the case of nearest neighbor nonlinear coupling. This particular form of coupling is inspired by the work of Bourkha and Belhaq [2] in which the point of suspension of a self-excited pendulum is subjected to a horizontal parametric forcing. Without loss of generality, from now on, we will assume $\omega_1 = 1$.

The main idea of The Method of Direct Partition of Motion (DPM) is that the solution is partitioned into a slow motion and a fast motion. Accordingly, we will look for a solution partitioned in the following manner:

$$x = X(t_1) + \varepsilon \xi(t_1, t_2, t_3), \quad y = Y(t_2) + \varepsilon \eta(t_1, t_2, t_3), \quad z = Z(t_3) + \varepsilon \zeta(t_1, t_2, t_3) \quad (2)$$

We will use DPM to investigate the dynamics of the leading order motions X , Y and Z . The key assumptions to be enforced when applying DPM [1, 4, 8] are (a) ξ is periodic and has a zero average over the t_2 and t_3 time scales and (b) η is periodic and has a zero average over the t_3 timescale. Using these assumptions and following the standard DPM procedure [1] we solve for approximate formal expressions for the leading order motions X , Y and Z . Z takes the form of a harmonic oscillation of frequency ω_3 with an amplitude that depends on a_3 and b_3 . X and Y are each found to be governed by a van der pol-duffing type equations. The limit cycles of X and Y are then solved for by finding a root of a Melnikov integral[7], and these limit cycles when they exist, take the form of a Jacobi elliptic function [3]. The amplitude and frequency of the Y oscillation is found to depend on a_2 , b_2 , a_3 , b_3 , g_2 and γ_2 , while that of X depends on the following parameters: a_i and b_i for $i=1,2,3$, as well as g_i and γ_i for $i=1,2$.

Results

Since there are many parameters affecting the dynamics, we limit our investigation to the effect of varying the coupling strengths γ_1 and γ_2 while holding the other parameters fixed. It is found that as γ_2 is increased, the period of oscillation of the y oscillator increases, and for γ_2 equal to a critical value $\gamma_{2,cr}$, the limit cycle suddenly disappears. That is, for $\gamma_2 \geq \gamma_{2,cr}$, the Melnikov integral associated with the Y equation has no real roots. Similarly, holding all other parameters fixed, as γ_1 is increased the period of the x oscillations increases and then the limit cycle of x suddenly disappears for γ_1 equal to a critical value $\gamma_{1,cr}$. That is, for $\gamma_1 \geq \gamma_{1,cr}$, the Melnikov integral associated with the X equation has no real roots. While the Y motion is independent of γ_1 , the X motion depends on both γ_2 and γ_1 , consequently, the value of $\gamma_{1,cr}$ varies as γ_2 is varied. These formal approximate findings were checked against solutions from numerical integration and the two solutions were found to agree well. For details see [6]

References

- [1] I.I. Blekhman (2000) *Vibrational mechanics-nonlinear dynamic effects, general approach, application*, Singapore: World Scientific
- [2] R Bourkha, M Belhaq (2007) Effect of fast harmonic excitation on a self-excited motion in van der Pol oscillator, *Chaos, Solitons & Fractals*, Vol. 34, 2, p. 621
- [3] P. Byrd, M. Friedman (1971) *Handbook of elliptic integrals for engineers and scientists*, 2nd ed., Berlin: Springer-Verlag
- [4] J.S. Jensen (1999) *Non-Trivial Effects of Fast Harmonic Excitation, PhD dissertation*, DCAMM Report, S83, Dept. Solid Mechanics, Technical University of Denmark
- [5] A.H. Nayfeh, C.M. Chin (1995) Nonlinear Interactions in a parametrically excited system with widely spaced frequencies, *Nonlinear Dynamics* 7: 195-216
- [6] H. Sheeitli, R.H. Rand (2011) Dynamics of three coupled limit cycle oscillators with vastly different frequencies, *Nonlinear Dynamics*, 64: 131-145.
- [7] R.H. Rand (2005) *Lecture Notes in Nonlinear Vibrations*, version 52, <http://audiophile.tam.cornell.edu/randdocs>
- [8] J.J. Thomsen (2003) *Vibrations and Stability, Advanced Theory, Analysis and Tools*, Germany: Springer-Verlag Berlin Heidelberg
- [9] J.J. Thomsen (2005) Slow high-frequency effects in mechanics: problems, solutions, potentials, *Int. J. Bifurcation Chaos* 15: 2799-2818
- [10] J.M. Tuwankotta, F. Verhulst (2003) Hamiltonian systems with widely separated frequencies, *Nonlinearity* 16: 689706