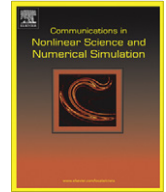




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Short communication

## Straight-line backbone curve

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## ABSTRACT

In this work we propose a class of problems in nonlinear vibrations related to avoiding undesirable hysteresis and jump phenomena and offer sample conservative systems for which the backbone curve is a straight vertical line.

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## 1. Introduction

The dynamics of a conservative system like the unforced, undamped Duffing oscillator,

$$\ddot{x} + x \pm x^3 = 0 \quad (1)$$

involves a relationship between the amplitude of vibration  $R$  of a typical periodic motion and its period or frequency  $\omega$ . This relation, when plotted as a curve in the  $\omega$ – $R$  plane, is called a backbone curve (see Fig. 1). Note that the backbone curve for Eq. (1) is bent to the right for the upper sign (hardening Duffing oscillator) and to the left for the lower sign (softening Duffing oscillator). This behavior is well-known to cause hysteresis and jump phenomena in the forced equation (see, for example, [1,2] or [3]).

There may be some situations where the jump phenomenon is undesirable. This leads us to the question of designing a differential equation which is similar to the Duffing equation (1) in that it is conservative, but for which the backbone curve is a straight vertical line in the  $\omega$ – $R$  plane, as shown in Fig. 1.

In what follows we offer an example of such a system and discuss some related oscillators.

## 2. Derivation

We propose to focus our investigation on equations of the form:

$$\ddot{x} + x + x\dot{x}^2 + f(x) = 0 \quad (2)$$

where  $f(x)$  is odd and strictly nonlinear in  $x$ :

$$f(x) = a_3x^3 + a_5x^5 + a_7x^7 + \dots \quad (3)$$

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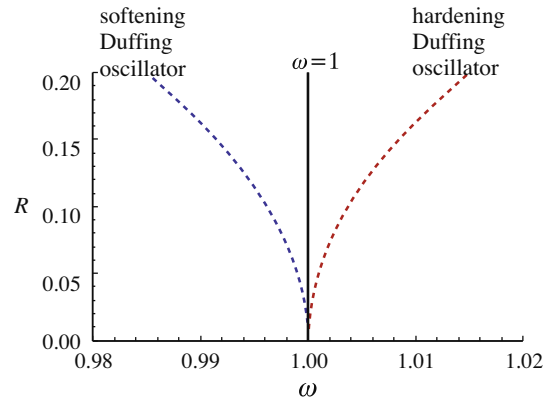


Fig. 1. Numerically obtained backbone curves of the Duffing oscillator (1) (dotted lines) and a straight-line backbone curve (solid line).

We ask how can we define the coefficients  $a_i$  in  $f(x)$  such that Eq. (2) will exhibit a straight-line backbone curve, and thus when forced and possibly damped, will not exhibit jumps.

To begin with, we note that Eq. (2) is conservative and may be derived using Lagrange’s equation. We take the Lagrangian to be of the form

$$L = \exp(x^2) \left( \frac{1}{2} \dot{x}^2 - g(x) \right) \tag{4}$$

where  $g(x)$  is to be determined. Lagrange’s equation becomes:

$$\ddot{x} + x\dot{x}^2 + \frac{dg(x)}{dx} + 2xg(x) = 0 \tag{5}$$

Comparing Eqs. (2), (3), (5) we see that  $g(x)$  must satisfy the equation:

$$\frac{dg(x)}{dx} + 2xg(x) = x + a_3x^3 + a_5x^5 + a_7x^7 + \dots \tag{6}$$

This may be solved for  $g(x)$  by taking  $g(x)$  in the form of a power series with even powered terms:

$$g(x) = b_2x^2 + b_4x^4 + b_6x^6 + \dots \tag{7}$$

Substitution of (7) into (6) leads to expressions for the  $b_i$  coefficients, the first few of which are:

$$b_2 = \frac{1}{2}, \quad b_4 = \frac{a_3 - 1}{4}, \quad b_6 = \frac{2a_5 - a_3 + 1}{12}, \dots \tag{8}$$

Thus the Lagrangian (4) produces the differential equation of motion (2) which exhibits the first integral

$$\exp(x^2) \left( \frac{1}{2} \dot{x}^2 + g(x) \right) = \text{constant} \tag{9}$$

### 3. Straight-line backbone curve

We set  $x = \sqrt{\epsilon}\tilde{x}$  in Eqs. (2) and (3) and drop the tilde for convenience, giving:

$$\ddot{x} + x + \epsilon x\dot{x}^2 + \epsilon a_3x^3 + \epsilon^2 a_5x^5 + \epsilon^3 a_7x^7 + \dots = 0 \tag{10}$$

In order to obtain an approximate solution to Eq. (10), we expand  $x$  in a power series in  $\epsilon$ :

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots \tag{11}$$

Note that since we are after a straight-line backbone curve, there is no need to expand frequency in a power series in  $\epsilon$  as is usual in Lindstedt’s method [3]. Substituting (11) into (10) and collecting terms we get a sequence of equations of which the first few are given by:

$$\ddot{x}_0 + x_0 = 0 \tag{12}$$

$$\ddot{x}_1 + x_1 = -x_0\dot{x}_0^2 - a_3x_0^3 \tag{13}$$

$$\ddot{x}_2 + x_2 = -2x_0\dot{x}_0\dot{x}_1 - \dot{x}_0^2x_1 - 3a_3x_0^2x_1 - a_5x_0^5 \tag{14}$$

We take the solution to Eq. (12) to be

$$x_0 = R \cos t \tag{15}$$

whereupon (13) becomes, after some trig reduction:

$$\ddot{x}_1 + x_1 = \frac{a_3 - 1}{4} R^3 \cos 3t + \frac{3a_3 + 1}{4} R^3 \cos t \tag{16}$$

We take  $a_3 = -1/3$  to remove resonance terms, and obtain

$$\ddot{x}_1 + x_1 = -\frac{1}{3} R^3 \cos 3t \tag{17}$$

which gives the particular solution:

$$x_1 = -\frac{1}{24} R^3 \cos 3t \tag{18}$$

Substituting (15, 18) into (14) gives

$$\ddot{x}_2 + x_2 = \frac{15a_5 - 1}{24} R^5 \cos t + \text{NRT} \tag{19}$$

where NRT stands for non-resonant terms. For no resonance, we choose  $a_5 = 1/15$ . Proceeding in this way, we obtain the following values for the coefficients  $a_i$  in Eq. (3):

$$a_3 = -1/3 \tag{20}$$

$$a_5 = 1/15 = 1/(3 * 5) \tag{21}$$

$$a_7 = -1/105 = -1/(3 * 5 * 7) \tag{22}$$

$$a_9 = 1/945 = 1/(3^3 * 5 * 7) \tag{23}$$

$$a_{11} = -1/10395 = -1/(3^3 * 5 * 7 * 11) \tag{24}$$

$$a_{13} = 1/135135 = 1/(3^3 * 5 * 7 * 11 * 13) \tag{25}$$

$$a_{15} = -1/2027025 = -1/(3^4 * 5^2 * 7 * 11 * 13) \tag{26}$$

$$a_{17} = 1/34459425 = 1/(3^4 * 5^2 * 7 * 11 * 13 * 17) \tag{27}$$

Now it is a remarkable fact that the typical term in the foregoing list of coefficients may be written in the following compact form:

$$a_{2n+1} = \frac{(-1)^n}{(2n+1)!!} \tag{28}$$

Thus the straight-line backbone differential equations (2) and (3) may be written:

$$\ddot{x} + x + x\dot{x}^2 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!!} = 0 \tag{29}$$

or absorbing the  $x$  term into the sum, cf. Eqs. (5) and (6),

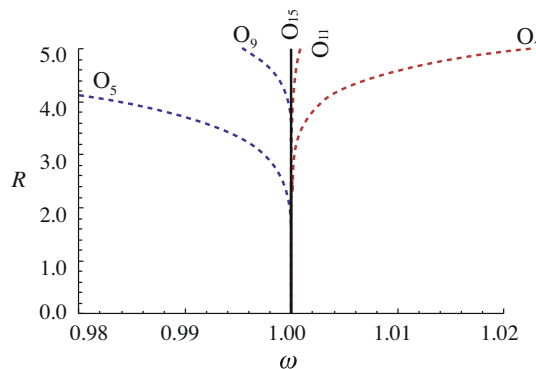


Fig. 2. Numerically obtained backbone curves of the oscillator (30) for a different number of odd-powered terms (the highest power is indicated in the subscript).

$$\ddot{x} + x\dot{x}^2 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!!} = 0 \quad (30)$$

In order to check this analytical result, Eq. (30) was solved numerically and the frequency was extracted from the time response for different values of the initial amplitude  $R$ . This was done for a different number of odd-powered polynomial terms in the sum from Eq. (30). The backbone curves of the corresponding oscillators ( $O_j$ ) are shown in Fig. 2, where the subscript  $j$  denotes the highest power included into the sum. Thus, the oscillator  $O_5$  has the backbone of the softening type. Additional terms change the way how the backbone curve is bent, alternating it between hardening and softening, making it be straighter for higher amplitudes. The oscillator  $O_{15}$  has a backbone curve that is straight on the region of  $R$  considered. The backbone curve of the oscillators with higher powers of nonlinearity remains straight on that region, too.

#### 4. Closed form solution

It is another remarkable fact that the sum in Eq. (30) can be written in the following closed form:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!!} = \sqrt{\frac{\pi}{2}} e^{-\frac{x^2}{2}} \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) \quad (31)$$

where  $\operatorname{erfi}$  denotes the “imaginary error function” defined as [4]

$$\operatorname{erfi}(z) = -i \operatorname{erf}(iz) \quad (32)$$

where  $\operatorname{erf}$  represents the error function [4,5],

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \quad (33)$$

We note that  $\operatorname{erfi}(z)$  satisfies the equation [4]:

$$\frac{d}{dz} \operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \exp z^2 \quad (34)$$

Thus starting from a local perturbation analysis we have been able to obtain an expression for the straight-line backbone differential equation which is valid for all  $x$ , namely

$$\ddot{x} + x\dot{x}^2 + \sqrt{\frac{\pi}{2}} e^{-\frac{x^2}{2}} \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) = 0 \quad (35)$$

The terms  $a_i$  obtained by use of the perturbation series (11) were derived with the requirement that the frequency of oscillations be unity, this being equivalent to requiring that the period of oscillations be  $2\pi$ . Because of the local nature of such a procedure, results may be expected to be valid only for small amplitudes of vibration. Since we were able to sum the series in closed form, and since it is easy to show using the ratio test that the resulting series converges for all  $x$ , it should be possible to show directly the straight-line backbone property, i.e., that all motions of Eq. (35) have period  $2\pi$ , regardless of amplitude.

From Eq. (9) with the initial condition  $x(0) = R, \dot{x}(0) = 0$ , we obtain

$$\exp(x^2) \left( \frac{1}{2} \dot{x}^2 + g(x) \right) = \exp(R^2) g(R) \quad (36)$$

We may use this equation to compute the period of oscillation once we know  $g(x)$ . From Eqs. (5), (6) and (35), we have

$$\frac{dg(x)}{dx} + 2xg(x) = \sqrt{\frac{\pi}{2}} e^{-\frac{x^2}{2}} \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) \quad (37)$$

Multiplying by  $\exp(x^2)$ , an integrating factor, we obtain

$$\frac{d}{dx} \left( e^{x^2} g(x) \right) = \sqrt{\frac{\pi}{2}} e^{\frac{x^2}{2}} \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) \quad (38)$$

Now from Eq. (34) we have

$$\frac{d}{dx} \operatorname{erfi}^2\left(\frac{x}{\sqrt{2}}\right) = 2\sqrt{\frac{2}{\pi}} e^{\frac{x^2}{2}} \operatorname{erfi}\left(\frac{x}{\sqrt{2}}\right) \quad (39)$$

Comparison of Eqs. (38) and (39) gives that we may take  $g(x)$  as

$$g(x) = \frac{\pi}{4} e^{-x^2} \operatorname{erfi}^2\left(\frac{x}{\sqrt{2}}\right) \quad (40)$$

Having found  $g(x)$ , we may now return to Eq. (36), which may be solved for  $\dot{x}$ , leading to the following expression for period of oscillation  $T$ :

$$T = 4 \int_0^R \frac{dx}{|\dot{x}|} = 4 \int_0^R \frac{dx}{\sqrt{-2g(x) + 2 \exp(R^2 - x^2)g(R)}} \tag{41}$$

Substitution of Eq. (40) in (41) gives

$$T = 4 \sqrt{\frac{2}{\pi}} \int_0^R \frac{\exp(x^2/2) dx}{\sqrt{\operatorname{erfi}^2(R/\sqrt{2}) - \operatorname{erfi}^2(x/\sqrt{2})}} \tag{42}$$

Careful differentiation gives the result that

$$\frac{d}{dx} \arctan \frac{\operatorname{erfi}(x/\sqrt{2})}{\sqrt{\operatorname{erfi}^2(R/\sqrt{2}) - \operatorname{erfi}^2(x/\sqrt{2})}} = \sqrt{\frac{2}{\pi}} \frac{\exp(x^2/2) dx}{\sqrt{\operatorname{erfi}^2(R/\sqrt{2}) - \operatorname{erfi}^2(x/\sqrt{2})}}, \tag{43}$$

whereupon we have

$$T = 4 \arctan \frac{\operatorname{erfi}(x/\sqrt{2})}{\sqrt{\operatorname{erfi}^2(R/\sqrt{2}) - \operatorname{erfi}^2(x/\sqrt{2})}} \Big|_0^R = 4 \frac{\pi}{2} = 2\pi. \tag{44}$$

So, the period of oscillations is  $2\pi$ .

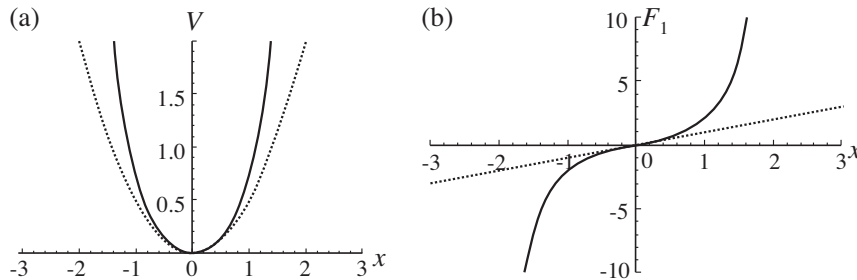
### 5. Restoring force

There are two ways of approaching the concept of restoring force for the foregoing oscillator. On the one hand we may consider the Lagrangian (4) in which the potential energy is given by  $V = \exp(x^2)g(x)$ , where  $g(x)$  is defined by Eq. (40), which gives

$$V = \frac{\pi}{4} \operatorname{erfi}^2\left(\frac{x}{\sqrt{2}}\right) \tag{45}$$

The potential energy  $V$  is plotted as a solid line in Fig. 3(a). In this case the force that corresponds to it is  $F_1 = -\partial V/\partial x$ . Omitting the minus sign, we can recognize that the restoring force–displacement law is given by the right-hand side of Eq. (38), which is shown as a solid line in Fig. 3(b). For comparison, these same figures contain plots of the potential energy  $V = x^2/2$  and restoring force  $x$  of an harmonic oscillator which has the same period  $2\pi$ . Note that both the nonlinear oscillator under consideration and the corresponding harmonic oscillator have a single-well potential. An example of a mechanical system that mimics this behavior is a particle whose mass changes exponentially with its position, and which moves along a fixed smooth horizontal surface, being connected to the spring whose potential energy and the force–displacement law correspond to those given in Fig. 3.

Another approach to treating the restoring force is to take the nonlinear oscillator as being given in the form (35). In this case we may identify the restoring force as  $F_2$ , its value being given by the right hand side of (31). This force–displacement law is plotted in Fig. 4. It is seen that  $F_2$  has a limited codomain, with a maximum at  $x^* \approx 1.307$ , for which  $F_2(x^*) = 1/x^* \approx 0.765$ . As  $x$  further increases,  $F_2$  approaches zero monotonically.



**Fig. 3.** (a) Potential energy  $V$  defined by Eq. (45) (solid line); (b) The corresponding force–displacement law  $F_1 - x$ , given by the right-hand side of Eq. (38) (solid line). The potential energy and force–displacement law of an harmonic oscillator with period  $2\pi$  are plotted as dotted lines.

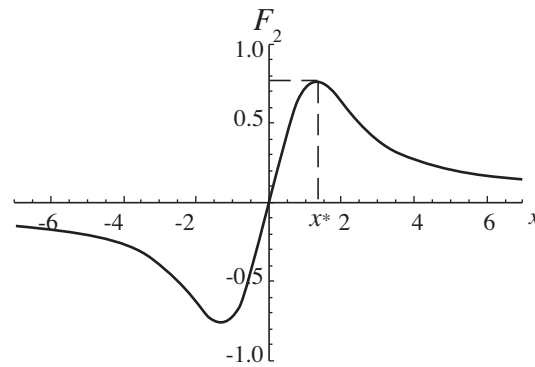


Fig. 4.  $F_2$  given by the right-hand side of Eq. (31) plotted as a function of  $x$ .

## 6. Related oscillators

The oscillator given by Eqs. (2) and (3) may be generalized by including even powered terms in the expansion for  $f(x)$ :

$$\ddot{x} + x + a_0 x \dot{x}^2 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots = 0 \quad (46)$$

In this case we may simplify the  $a_0$  coefficient by stretching  $x$ . We set  $x = \mu y$  and obtain

$$\ddot{y} + y + a_0 \mu^2 y \dot{y}^2 + a_2 \mu y^2 + a_3 \mu^2 y^3 + \dots = 0 \quad (47)$$

Thus by choosing  $\mu^2 = 1/|a_0|$  we may make the coefficient of the  $y \dot{y}^2$  term equal to 1 or to  $-1$ , i.e. to  $\text{sgn } a_0$ .

Returning to (46), we prepare it for perturbation treatment by setting  $x = \epsilon \tilde{x}$ :

$$\ddot{\tilde{x}} + \tilde{x} + a_0 \epsilon^2 \tilde{x} \dot{\tilde{x}}^2 + a_2 \epsilon \tilde{x}^2 + a_3 \epsilon^2 \tilde{x}^3 + a_4 \epsilon^3 \tilde{x}^4 + \dots = 0 \quad (48)$$

where we have dropped the tilde for convenience. Proceeding as before, requiring the solution to have frequency  $\omega = 1$ , collecting terms, solving equations and removing resonances, we obtain:

$$9a_3 = 10a_2^2 - 3a_0 \quad (49)$$

$$135a_5 = 9a_0^2 + 63a_2^2 a_0 + 378a_2 a_4 - 280a_4^2 \quad (50)$$

$$14175a_7 = -135a_0^3 + 192a_2^2 a_0^2 + 11934a_2 a_4 a_0 - 40880a_2^4 a_0 + 48600a_2 a_6 + 20412a_4^2 - 186480a_2^3 a_4 + 148400a_2^6 \quad (51)$$

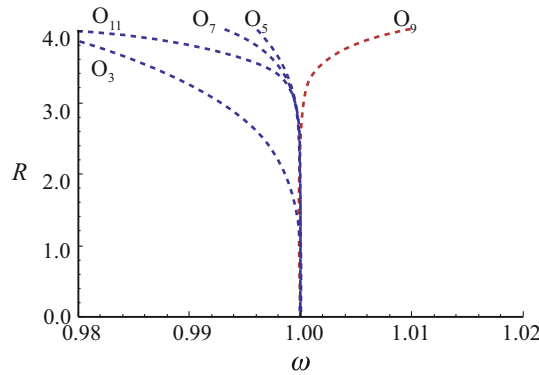
$$\begin{aligned} 1148175a_9 = & 1215a_0^4 + 6993a_2^2 a_0^3 + 73386a_2 a_4 a_0^2 - 1216755a_2^4 a_0^2 + 886950a_2 a_6 a_0 + 500823a_4^2 a_0 \\ & - 15790140a_2^3 a_4 a_0 + 29912400a_2^6 a_0 + 4677750a_2 a_8 + 3608550a_4 a_6 - 24057000a_2^3 a_6 \\ & - 29189160a_2^2 a_4^2 + 131392800a_2^5 a_4 - 93940000a_2^8 \end{aligned} \quad (52)$$

$$\begin{aligned} 442047375a_{11} = & -42525a_0^5 - 435024a_2^2 a_0^4 + 2671542a_2 a_4 a_0^3 - 112405293a_2^4 a_0^3 + 38321100a_2 a_6 a_0^2 \\ & + 17272926a_2^2 a_0^2 - 2156954184a_2^3 a_4 a_0^2 + 8833359150a_2^6 a_0^2 + 314940150a_2 a_8 a_0 \\ & + 388520550a_4 a_6 a_0 - 7348433400a_2^3 a_6 a_0 - 10286259984a_2^2 a_4^2 a_0 + 97229424600a_2^5 a_4 a_0 \\ & - 133948183600a_2^8 a_0 + 2089678500a_2 a_{10} + 1532430900a_4 a_8 - 14189175000a_2^3 a_8 \\ & + 703667250a_6^2 - 31086455400a_2^2 a_4 a_6 + 97459362000a_2^5 a_6 - 8459018568a_2^2 a_4^3 \\ & + 188659270800a_2^4 a_4^2 - 586687701600a_2^7 a_4 + 379443064000a_2^{10} \end{aligned} \quad (53)$$

We now discuss a number of special cases of Eq. (46).

### 6.1. Eq. (46) with $a_0 = 0$

In this case the formulas (49)–(53) show that the coefficients may be chosen so that the resulting equation has a straight-line backbone curve. The first few of them are



**Fig. 5.** Numerically obtained backbone curves of the oscillator (48) with  $a_0=0$  and  $\epsilon = 0.1$  for the coefficients of the even powers equal to unity and the coefficients of the odd powers defined by Eqs. (49)–(56).

$$a_3 = \frac{10}{9} a_2^2 \tag{54}$$

$$a_5 = \frac{378a_2a_4 - 280a_2^4}{135} \tag{55}$$

$$a_7 = \frac{48600a_2a_6 + 20412a_4^2 - 186480a_2^3a_4 + 148400a_2^6}{14175} \tag{56}$$

The corresponding backbone curves obtained numerically from the differential equations of motion are shown in Fig. 5 for  $\epsilon = 0.1$  and for the coefficients of the even powers equal to unity. The oscillator  $O_3$  with a quadratic and cubic nonlinear term corresponds to the so-called Helmholtz–Duffing oscillator [2]. The relationship between  $a_3$  and  $a_2$  in Eq. (54) agrees with the well-know result yielding the frequency independent of amplitude, which is in the literature obtained for the calculation procedure when terms of  $O(\epsilon^3)$  are neglected (see, for example, [6, pp. 55 or pp. 198]). As seen from Fig. 5, an increase in the number of terms in the truncation ceases to produce an increase in the straightness of the backbone curve for values of amplitude  $R$  which are larger than about 2.5. We conjecture that this effect is due to divergence of the associated perturbation series.

6.2. Eq. (46) with  $a_1 = a_3 = a_5 = \dots = 0$

When all the odd-powered terms are absent, the formulas (49)–(53) determine the following coefficients of the even-powered terms such that the backbone curve is a straight line:

$$a_2 = \pm 3 \sqrt{\frac{a_0}{30}} \tag{57}$$

$$a_4 = \mp \frac{1}{14} \sqrt{\frac{a_0^3}{30}} \tag{58}$$

$$a_6 = \mp \frac{2209}{18900} \sqrt{\frac{a_0^5}{30}} \tag{59}$$

$$a_8 = \pm \frac{91673}{1157625} \sqrt{\frac{a_0^7}{30}} \tag{60}$$

$$a_{10} = \pm \frac{2014958909}{35006580000} \sqrt{\frac{a_0^9}{30}} \tag{61}$$

Note that there are two sets of these coefficients (with the upper and lower sign), i.e. two types of oscillators with even-powered terms that have a straight-line backbone curve. Note however that solutions exist only for  $a_0 > 0$ . By solving Eq. (48) with even powered-terms for the coefficients (57)–(61) with the upper signs and for  $a_0 = 1, \epsilon = 0.1$ , the backbone curves presented in Fig. 6 are obtained.

6.3. Eq. (46) with  $a_2 = a_4 = a_6 = \dots = 0$

This case has been the subject of the earlier sections of this paper based on Eq. (10) where we took  $a_0 = 1$ . However now we note that the form of the closed form solution is dependent on the sign of  $a_0$ . When  $a_0 > 0$ , the coefficients  $a_{2n+1}$  are defined by Eq. (28) and have an alternating sign. When  $a_0 < 0$ , they are all positive and defined by

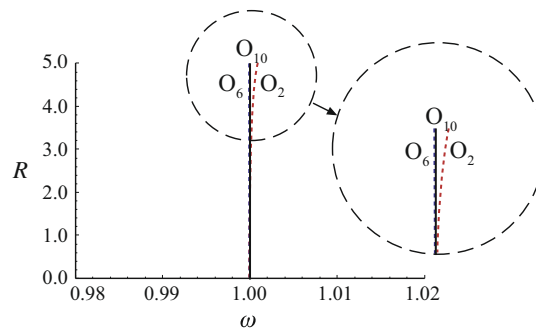


Fig. 6. Numerically obtained backbone curves of the oscillator (48), (57)–(61) with  $a_0 = 1$  and  $\epsilon = 0.1$  for a different number of even-powered terms.

$$a_{2n+1} = \frac{|a_0|^n}{(2n+1)!!} \quad (62)$$

In order to show how additional terms change the shape of the corresponding backbone curve, Eq. (46) with no even-powered-terms was first rescaled to correspond to Eq. (10) and then solved numerically for  $a_0 = -1$ ,  $\epsilon = 0.1$  and for  $a_{2n+1}$  defined by Eq. (62). On the basis of these numerical results, an animation is created, which is given as [Supplementary Material](#). The backbone curves are of the hardening type, but every additional even-powered term unbends it more and more, producing the desirable effect of a straight-line backbone curve on the region of  $R$  considered.

In the case of Eq. (46) with no even-powered terms but with  $a_0 < 0$ , the sum can be expressed in the following closed form

$$\sum_{n=0}^{\infty} \frac{|a_0|^n x^{2n+1}}{(2n+1)!!} = \sqrt{\frac{\pi}{2|a_0|}} e^{\frac{a_0 x^2}{2}} \operatorname{erf}\left(\sqrt{\frac{|a_0|}{2}} x\right) \quad (63)$$

so that Eq. (46) becomes

$$\ddot{x} - |a_0|x\dot{x}^2 + \sqrt{\frac{\pi}{2|a_0|}} e^{\frac{|a_0|x^2}{2}} \operatorname{erf}\left(\sqrt{\frac{|a_0|}{2}} x\right) = 0 \quad (64)$$

## 7. Conclusions

In this work we have set out to find an oscillator of the form of Eq. (2) whose period is insensitive to amplitude  $R$ . In the process of doing so we have found several classes of such oscillators and have been able to determine the closed form expression of the associated series in some of them. It would be interesting to obtain similar closed form results for the other cases mentioned in the previous section, but we have been unable to do so.

## Acknowledgement

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## Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at <http://dx.doi.org/10.1016/j.cnsns.2012.11.031>.

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