The Stability of Bifurcating Periodic Solutions in a Two-Degree-of-Freedom Nonlinear System

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Dynamical systems which possess bifurcating *equilibrium solutions* have received much attention in recent iterature [1, 2].[3] In particular the stability involved in such bifurcations has been studied and may be illustrated by the following example [1, 3]: The plane of a simple pendulum is rotated about a vertical axis with constant angular velocity ω relative to a Newtonian frame (Fig. 1).

An elementary phase plane analysis [3] reveals that if $\omega < \omega_0 = \sqrt{g/L}$, the only equilibrium points are $\theta = 0, \pi$. However, if $\omega > \omega_0$ then two additional equilibrium solutions exist, having bifurcated out of the $\theta = 0$ solution when $\omega = \omega_0$. Regarding the stability of these equilibria, the $\theta = 0$ solution is a center (stable) for $\omega < \omega_0$ and a saddle (unstable) for $\omega > \omega_0$. The pair of bifurcating equilibria are centers (stable); see Fig. 2.

In this paper we shall consider the stability of bifurcating *periodic motions* (nonlinear normal modes) in a conservative two-degree-of-freedom system. (For an introduction to nonlinear normal modes, see [4]. Rosenberg [8] first studied two-degree-of-freedom systems possessing more than two nonlinear normal modes, his "superabundant modes.")

The system consists of two unit masses which are constrained to move along a straight line and which are restrained by two anchor springs and a coupling spring. The positions of the masses are given by generalized coordinates x and y, both of which are taken to be zero when the springs are unstretched. Let F be the tension in a spring and let d be its elongation beyond its unstretched length. Then for the identical anchor springs, $F = d + Kd^3$, while for the coupling spring, $F = d^3$. It is assumed in what follows that the parameter $K > 0$, whereupon the system is admissible (in the sense of reference [4]) for all energies.

In a previous work [5] it has been shown that this system admits 2 similar (i.e., straight line) nonlinear normal modes if $K \le 4$, but 4 similar normal modes if $K > 4$. We briefly summarize these results as follows: The equations of motion for the system are

$$\ddot{x} = -\frac{\partial V}{\partial x}, \quad \ddot{y} = -\frac{\partial V}{\partial y} \tag{1}$$

where dots represent differentiation with respect to time t, and where

$$V = \frac{1}{2}x^2 + \frac{1}{4}Kx^4 + \frac{1}{4}(x-y)^4 + \frac{1}{2}y^2 + \frac{1}{4}Ky^4 \tag{2}$$

In order that these equations permit a solution of the form $y = Cx$, it is necessary and sufficient [5] that

$$C^4 + (K-2)C^3 - (K-2)C - 1 = 0 \tag{3}$$

Here equation (3) is obtained by substituting $y = Cx$ into equations (1). The roots of equation (3) are

$$C = 1, \quad -1, \quad 1 - \frac{K}{2} \pm \frac{\sqrt{K(K-4)}}{2} \tag{4}$$

When $K \le 4$ there are only 2 similar normal modes, $y = \pm x$. An additional pair of similar normal modes bifurcates out of the $y = -x$ mode when $K > 4$; see Fig. 3 (cf. Fig. 2.)

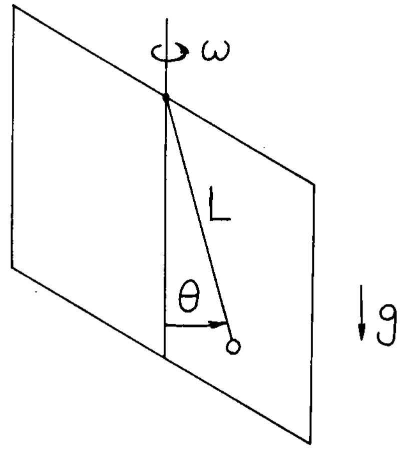

Fig. 1 Plane of a simple pendulum is rotated about a vertical axis with constant angular velocity ω

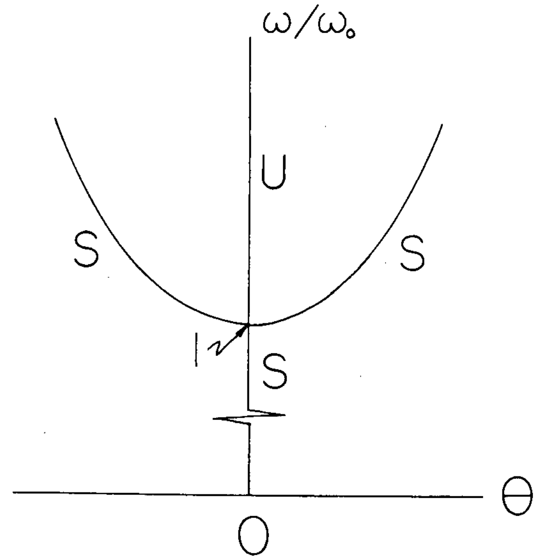

Fig. 2 Stability of bifurcating equilibrium solutions for the system of Fig. 1. S = stable, U = unstable

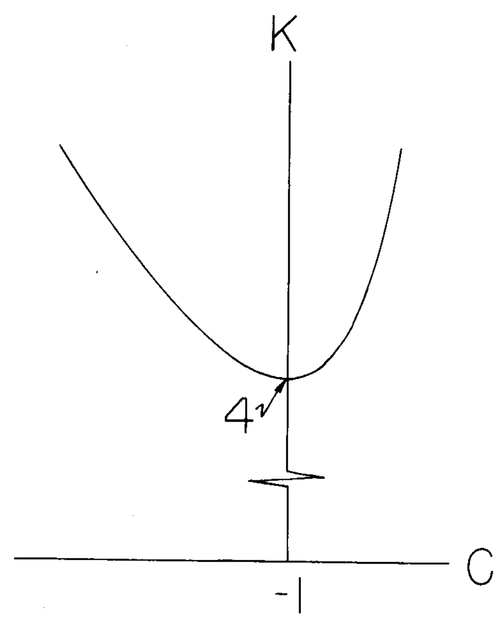

Fig. 3 Bifurcating similar nonlinear normal modes; $C = y(t)/x(t)$

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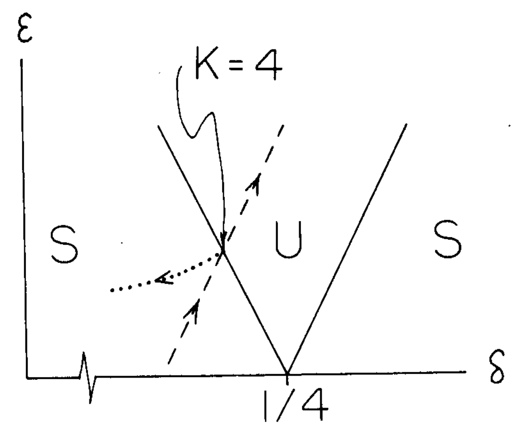

Fig. 4 **Stability chart for the Mathieu equation. Solid lines represent transition curves separating regions of stability (S) from regions of instability (U). (Those shown have the equations $\delta = 1/4 \pm \epsilon/2 + O(\epsilon^2)$.) Dashed line represents stability of the $y = -x$ mode, equation (25). Dotted line represents stability of the pair of bifurcating normal modes, equation (26). Arrows show direction of increasing K.**

We shall investigate the stability of the $y = -x$ mode and the two modes which bifurcate out of it.

For convenience, we introduce new dependent variables u, v which are related to x, y by a rotation ([4, p. 225]).

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(1 + C^2)^{1/2}} \begin{bmatrix} -C & -1 \\ 1 & -C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \qquad (5)$$

Here the u and v axes are, respectively, orthogonal to, and coincident with the nonlinear normal mode $y = Cx$ whose stability is to be considered. Substituting equation (5) into equations (1), obtain

$$\ddot{u} + u + A_1 u^3 + A_2 u^2 v + A_3 u v^2 = 0 \qquad (6)$$

$$\ddot{v} + v + A_4 v^3 + A_3 v u^2 + \frac{A_2}{3} u^3 = 0 \qquad (7)$$

where

$$(1 + C^2)^2 A_1 = K(1 + C^4) + (1 + C)^4 \qquad (8)$$

$$(1 + C^2)^2 A_2 = -3[C^4 + (2 - K)C^3 + (K - 2)C - 1] \qquad (9)$$

$$(1 + C^2)^2 A_3 = 3[C^4 + 2(K - 1)C^2 + 1] \qquad (10)$$

$$(1 + C^2)^2 A_4 = K(1 + C^4) + (1 - C)^4 \qquad (11)$$

and where equation (3) has been used.

Equations (6) and (7) admit the solution $u(t) \equiv 0$, $v(t) = v^*(t)$ where v^* satisfies the homogeneous Duffing equation

$$\ddot{v}^* + v^* + A_4 v^{*3} = 0 \qquad (12)$$

For small amplitudes of vibration $\beta \ll 1$, equation (12) has the following approximate perturbation solution [6]:

$$v^*(t) = \beta \cos \left(1 + \frac{3}{8} A_4 \beta^2\right) t \qquad (13)$$

To investigate stability, set

$$u(t) = 0 + \xi(t) \qquad (14)$$

$$v(t) = v^*(t) + \eta(t) \qquad (15)$$

and substitute into equations (6) and (7). After linearizing the resulting equations on $\xi(t)$ and $\eta(t)$, obtain

$$\ddot{\xi} + \xi + A_3 v^*(t)^2 \xi = 0 \qquad (16)$$

$$\ddot{\eta} + \eta + 3A_4 v^*(t)^2 \eta = 0 \qquad (17)$$

Now replace t by τ as independent variable

$$\tau = 2\left(1 + \frac{3}{8} A_4 \beta^2\right) t \qquad (18)$$

Equations (16) and (17) become Mathieu equations

$$\xi'' + (\delta_1 + \epsilon_1 \cos \tau)\xi = 0 \qquad (19)$$

$$\eta'' + (\delta_2 + \epsilon_2 \cos \tau)\eta = 0 \qquad (20)$$

where primes represent differentiation with respect to τ, and where

$$\delta_1 = \frac{1}{4} + \left(\frac{A_3}{8} - \frac{3}{16} A_4\right) \beta^2 \qquad (21)$$

$$\epsilon_1 = A_3 \beta^2 / 8 \qquad)22)$$

$$\delta_2 = \frac{1}{4} + \frac{3}{16} A_4 \beta^2 \qquad (23)$$

$$\epsilon_2 = 3A_4 \beta^2 / 8 \qquad (24)$$

The stability of these equations may be deduced by comparison with the standard stability chart for the Mathieu equation, obtained by using Floquet theory [6, 7]; see Fig. 4.

As usual, equation (20) lies on the transition curve $\delta = 1/4 + \epsilon/2 + O(\epsilon^2)$ (see [4, p. 227]). This is due to the dependence of frequency upon amplitude in $v^*(t)$, equation (13), typical of nonlinear periodic motions. Although Lyapunov unstable, this effect is well known to give orbital stability [6]. The stability problem is therefore reduced to consideration of equation (19) with equations (21) and (22).

Substitution of equations (10) and (11) into (21) and (22) gives for $C = -1$,

$$\delta_1(K) = \frac{1}{4} + \left(\frac{3}{32} K - \frac{3}{4}\right) \beta^2, \quad \epsilon_1(K) = \frac{3}{16} K \beta^2 \qquad (25)$$

For fixed amplitude β, equation (25) represents a straight line in the δ-ϵ plane with K as parameter. For $K = 4$ this straight line intersects the transition curve $\delta = 1/4 - \epsilon/2 + O(\epsilon^2)$. For $K < 4$ (>4), equation (25) lies in the stable (unstable) region of the δ-ϵ plane; see Fig. 4.

Similarly, for the bifurcating pair of solutions with $2C = 2 - K \pm \sqrt{K(K-4)}$, equations (10), (11), (21), and (22) give

$$\delta_1(K) = \frac{1}{4} + \left(\frac{3K(3 - K)}{16(K - 2)}\right) \beta^2, \quad \epsilon_1(K) = \frac{3K}{8(K - 2)} \beta^2, \quad K \geq 4 \qquad (26)$$

(both bifurcating modes give the same expressions for δ and ϵ.) For fixed amplitude β, equation (26) represents a curve in the δ-ϵ plane with K as parameter. This curve originates at $K = 4$ on the transition curve $\delta = 1/4 - \epsilon/2 + O(\epsilon^2)$. For values of K just larger than 4, equation (26) lies in the stable region of the δ-ϵ plane; see Fig. 4.

In conclusion, it has been shown that upon bifurcation the two new periodic motions enter as stable, while the existing periodic motion changes from stable to unstable. This stability behavior is consistent with the behavior of the bifurcating equilibria in the rotating pendulum example (Fig. 2).

It is to be noted that the linearized stability analysis just used would not be valid for the $C = +1$ mode as equations (16) and (17) become identical in that case, both predicting secular instability with pure imaginary characteristic exponents.

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