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An Application of the Poincaré Map to the Stability of Nonlinear Normal Modes

The stability of periodic motions (nonlinear normal modes) in a nonlinear two-degree-of-freedom Hamiltonian system is studied by deriving an approximation for the Poincaré map via the Birkhoff-Gustavson canonical transformation. This method is presented as an alternative to the usual linearized stability analysis based on Floquet theory. An example is given for which the Floquet theory approach fails to predict stability but for which the Poincaré map approach succeeds.

Introduction

This paper is concerned with certain periodic motions, called nonlinear normal modes (NNM's), which have been shown to occur in a wide class of coupled oscillators [24, 18, 10]. In particular we shall be interested in the stability of NNM's. As usual in discussing the stability of periodic motions in a nonlinear system, we shall be concerned only with orbital (and not Liapunov) stability [4].

The usual procedure for determining stability of a periodic motion in a nonlinear system involves perturbing the solution whose stability is to be studied, and then linearizing the perturbed equations. The resulting linear system of equations has periodic coefficients and lies in the realm of Floquet theory; e.g., for small amplitudes of vibration the stability problem is often reduced to consideration of the Mathieu equation [28, 22, 16]. The question of the stability of a given NNM will often depend upon the values of the system parameters (e.g., spring constants, geometry, etc.) The parameter space is typically decomposed into stable and unstable regions separated by a codimension 1 "transition" surface. (e.g., in the case of the Mathieu equation $\ddot{x} + (\delta + \epsilon \cos t)x = 0$ the $\delta - \epsilon$ parameter plane has regions of stability separated from regions of instability by transition curves [12].) In a particular problem, analytical expressions may be obtained for the transition surfaces by perturbation methods [26, 21].

It is our purpose in this paper to provide an alternate procedure for determining the stability of NNM's in autonomous two-degree-of-

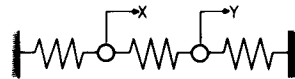


Fig. 1 The system S

freedom Hamiltonian systems. The procedure is based on using the Birkhoff-Gustavson (B-G) canonical transformation to obtain an approximation for the Poincaré map. We believe that this method offers some advantage over the usual Floquet theory approach just described. Specifically, the procedure does not require linearization in the neighborhood of any particular motion and hence it yields a global picture of the flow. Thus it not only permits conclusions to be drawn about the stability of NNM's, but it also provides insight into the dynamical structure of the system.

Moreover, there exist systems for which the Floquet theory approach simply fails to predict stability for a given NNM! We will describe such a system in this paper. The failure of Floquet theory in this case is related to the essentially *nonlinear* nature of the stability problem. We will show that the method based on the B-G transformation and the Poincaré map does, however, predict stability for this problem.

The System

Consider a nonlinear autonomous two-degree-of-freedom Hamiltonian system S which consists of two unit masses constrained to move along a straight line and restrained by two anchor springs and a coupling spring, Fig. 1. The positions of the masses are given by generalized coordinates x and y , both of which are taken to be zero when the springs are unstretched. We assume that the restoring force F for the identical anchor springs is given by $F = d + ka^3$, while for the

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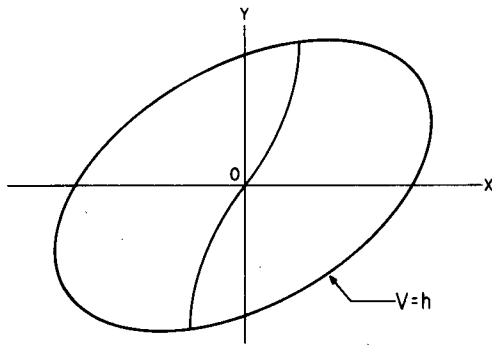


Fig. 2 A nonlinear normal mode (NNM)

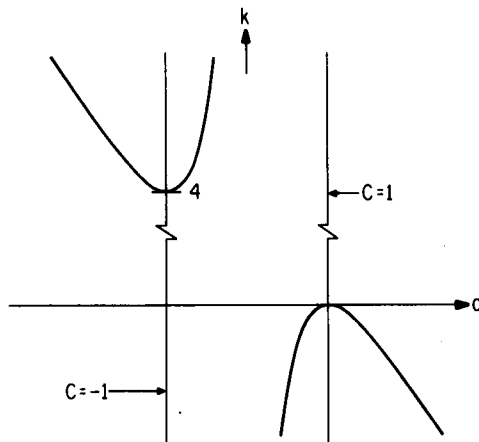


Fig. 3 Bifurcation of SNM's ($y = Cx$) for system S

coupling spring, $F = d^3$. Here d is the spring deformation and k is a parameter.

The equations of motion for the system S are

$$\ddot{x} = -\frac{\partial V}{\partial x}, \quad \ddot{y} = -\frac{\partial V}{\partial y} \quad (1)$$

where the potential energy $V(x, y)$ is given by

$$V = \frac{1}{2}(x^2 + y^2) + \frac{k}{4}(x^4 + y^4) + \frac{1}{4}(x - y)^4. \quad (2)$$

The system S possesses a first integral corresponding to the conservation of energy,

$$H(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x, y) = h \quad (3)$$

where $H(x, y, \dot{x}, \dot{y})$ is the Hamiltonian.

In a previous work [23] it has been shown that the system S possesses bifurcating NNM's. In a manner similar to Rosenberg [24] we take NNM's to be periodic motions which pass through the origin 0 and which have precisely two rest points, Fig. 2. NNM's which project onto the xy -plane as straight lines have been called similar normal modes (SNM's) [24].

The system S possesses SNM's, $y = Cx$, where

$$C = 1, -1, 1 - \frac{k}{2} \pm \frac{\sqrt{k(k-4)}}{2}. \quad (4)$$

When $0 \leq k \leq 4$ there are only two SNM's, $y = \pm x$. An additional pair of SNM's bifurcates out of the $y = -x$ mode when $k > 4$, and out of the $y = x$ mode when $k < 0$, Fig. 3.

The stability of the $y = -x$ out-of-phase mode and of the two SNM's which bifurcate out of it has been investigated in a previous

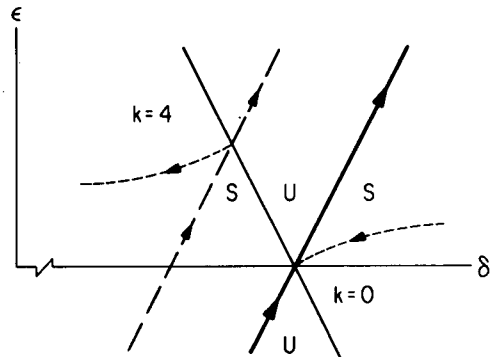


Fig. 4 Stability results from Floquet theory. The stability chart is for the Mathieu equation $\ddot{x} + (\delta + \epsilon \cos t)x = 0$ (S = stable, U = unstable). Solid lines represent transition curves, $\delta = 1/4 \pm \epsilon/2 + O(\epsilon^2)$. The location of the SNM's for varying k is displayed as follows: The dashed line corresponds to the out-of-phase mode $y = -x$; the dark solid line corresponds to the in-phase mode $y = x$ (which lies on the transition curve); the dotted lines correspond to the pairs of bifurcating modes. Arrows indicate direction of increasing k .

work [16]. Results were obtained using a linearized stability analysis and Floquet theory. The stability analysis, valid for small energies h , was shown to finally reduce to the study of a single Mathieu equation. This Mathieu equation governs perturbations which are orthogonal to the SNM under investigation, and therefore the stability being considered is orbital stability [15].

It was shown [16] that upon bifurcation at $k = 4$ the two new periodic motions enter as stable, while the existing $y = -x$ periodic motion changes from stable to unstable. See Fig. 4 which shows the location of the SNM's for varying k on a Mathieu equation stability chart.

The same analysis can also be used to investigate the stability of the modes which bifurcate out of the $y = x$ in-phase mode when $k = 0$. It has been shown that these modes enter as stable [15], Fig. 4.

However, this same kind of linear stability analysis fails to predict stability for the $y = x$ in-phase mode [16]. In this case the SNM lies along a transition curve in the associated Mathieu equation stability chart, Fig. 4.

In what follows we present an alternate method of stability analysis which will be shown to yield stability information for the $y = x$ mode of system S .

The Poincaré Map

In this section we will discuss a powerful technique for analyzing two-degree-of-freedom problems, the Poincaré map. Its purpose is to provide a two-dimensional description of a flow which is occurring in a four-dimensional phase space.

Let us consider an autonomous two-degree-of-freedom Hamiltonian system. Although the phase space (x, y, \dot{x}, \dot{y}) is four-dimensional, the first integral $H = h$, equation (3), restricts the motion to a three-dimensional surface. If another independent first integral exists then the energy manifold, $H = h$, is fibered by invariant two-dimensional tori. These may be pictured in three-dimensional space as a family of concentric tori, Fig. 5.

Now consider the two-dimensional surface Σ which results from "slicing" the three-dimensional energy manifold $H = h$ with a plane, say $x = 0$, Fig. 6. Σ , which looks locally like a two-dimensional plane, will in general intersect a particular motion infinitely often. A motion beginning on Σ returns to Σ after making a circuit around the torus. This produces a mapping of Σ onto itself. This map is known as the Poincaré map [20, 9, 2, 1], and the two-dimensional surface $\Sigma = \{x = 0\} \cap \{H = h\}$ is called a surface of section. We will choose y, \dot{y} as coordinates on Σ and will project Σ down onto the $y - \dot{y}$ -plane for convenience.

In order to insure that the Poincaré map gives a realistic picture of the flow in the neighborhood of a given motion, we require that the motion intersect the surface of section Σ transversally (nontangen-

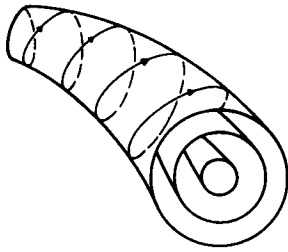


Fig. 5 In a two-degree-of-freedom Hamiltonian system, the existence of a first integral which is independent of the Hamiltonian implies that the three-dimensional energy manifold will be fibered into invariant tori; a given motion remains on a single torus for all time

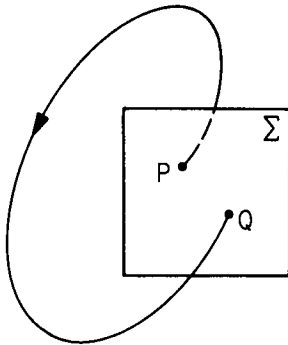


Fig. 6 A motion starting at P on the surface of section $\Sigma = \{x = 0\} \cap \{H = h\}$ reintersects Σ at Q producing a Poincaré map

tially) [5]. A motion will fail to intersect Σ transversally whenever the normal to the "slicing" surface (i.e., $x = 0$) is perpendicular to the tangent of the motion. The normal to the surface $x = 0$ is $(1, 0, 0, 0)$ and the tangent to the motion (x, y, \dot{x}, \dot{y}) is $(\dot{x}, \dot{y}, \ddot{x}, \ddot{y})$. Thus transversality is violated whenever $(\dot{x}, \dot{y}, \ddot{x}, \ddot{y}) \cdot (1, 0, 0, 0) = 0$ or $\dot{x} = 0$. We impose the additional restriction that $\dot{x} > 0$ when $x = 0$ in order to insure that the motion always pierces Σ from the "same side." Therefore successive intersections of a motion with the y, \dot{y} surface of section Σ lie inside the region $\dot{x}(y, \dot{y}, h) \geq 0$. This region is bounded by the curve $\dot{x}(y, \dot{y}, h) = 0$, obtained by solving the equation $H(0, y, \dot{x}, \dot{y}) = h$ equation (3), for \dot{x} .

Periodic motions which pierce the surface of section Σ once per cycle appear as fixed points of the Poincaré map. More complicated periodic orbits may appear as n -cycles of the map, i.e., as fixed points of the map composed with itself n times.

Suppose a periodic motion which corresponds to a fixed point of the Poincaré map is stable. Then in phase space the closed curve which corresponds to the periodic motion will be enclosed by tori which contain neighboring motions. The Poincaré map will reveal the fixed point to be surrounded by concentric closed curves. Thus stable periodic motions become centers in the $y - \dot{y}$ -plane. Similarly unstable periodic motions appear as saddle points in $y - \dot{y}$ -plane. It is important to recognize that the dynamic on the $y - \dot{y}$ -plane is that of a map and not of a flow, and although we speak of centers and saddles these must not be confused with the singular points associated with the flows of phase plane analysis (cf. [14]).

The Poincaré map can be found analytically as follows: Let $f(x, y, \dot{x}, \dot{y}) = c$ be a first integral which is independent of the energy integral $H = h$. The intersection of the $f = c$ surface with the surface of section Σ represents the invariant curves of the Poincaré map. For fixed energy h these invariant curves may be written

$$\tilde{f}(y, \dot{y}) = f(0, y, \dot{x}(y, \dot{y}, h), \dot{y}) = c. \quad (5)$$

Note that if no such independent first integral $f = c$ exists then the energy manifold will not, in general, be fibered by invariant tori, and the motion may even be ergodic.

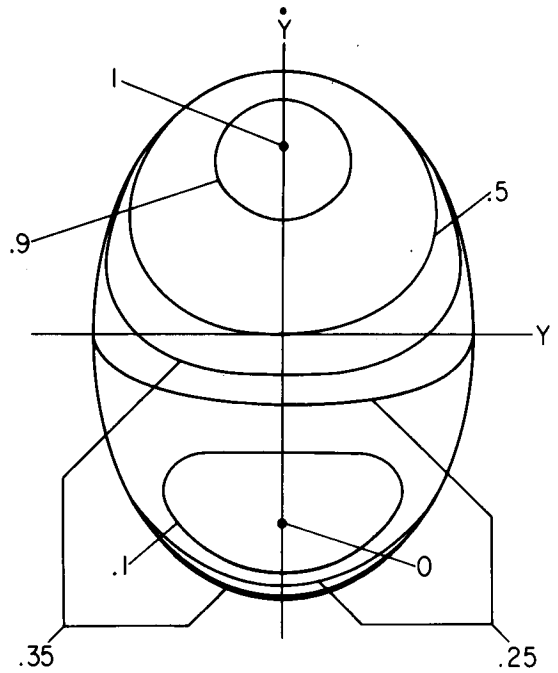


Fig. 7 The Poincaré map for the linear system L for $h = 1$; each invariant curve is associated with a value of c

To familiarize the reader with the Poincaré map we will now consider a well-known integrable two-degree-of-freedom linear system L . The system L consists of two unit masses which are constrained to move along a straight line and which are restrained by two linear anchor springs and a linear coupling spring. All three springs have a spring constant of unity.

Choosing generalized coordinates x, y as in Fig. 1, the potential energy for L becomes

$$V(x, y) = x^2 + y^2 - xy. \quad (6)$$

The general solution to the equations of motion (1) consists, of course, of a superposition of the two linear normal modes: here the $x = y$ in-phase mode has frequency 1 and the $x = -y$ out-of-phase mode has frequency $\sqrt{3}$ [6, p. 370].

In addition to the conservation of energy for the entire system, $H = h$, equation (3), the system L possesses an independent first integral corresponding to conservation of energy in either mode alone. Taking the $x = y$ in-phase mode, conservation of energy gives

$$\frac{1}{2} \dot{u}^2 + \frac{1}{2} u^2 = c \quad (7)$$

where $u = (1/\sqrt{2})(x + y)$. That is,

$$f(x, y, \dot{x}, \dot{y}) = \frac{1}{4} (\dot{x} + \dot{y})^2 + \frac{1}{4} (x + y)^2 = c. \quad (8)$$

To generate the Poincaré map, set $x = 0$ in equations (3) and (6) and solve for $\dot{x}(y, \dot{y}, h)$:

$$\dot{x} = (2h - 2y^2 - \dot{y}^2)^{1/2}. \quad (9)$$

Then set $x = 0$ in equation (8) and use equation (9) to obtain

$$\tilde{f}(y, \dot{y}) = \frac{1}{4} [(2h - 2y^2 - \dot{y}^2)^{1/2} + \dot{y}]^2 + \frac{1}{4} y^2 = c. \quad (10)$$

For fixed h , say $h = 1$, equation (10) represents a one-parameter family of invariant curves with c as parameter, Fig. 7. These curves fill the interior of the region in the $y - \dot{y}$ plane bounded by the ellipse

$$2y^2 + \dot{y}^2 = 2h \quad (11)$$

obtained by setting $\dot{x} = 0$ in equation (9).

From Fig. 7 we see that the system L exhibits two stable periodic

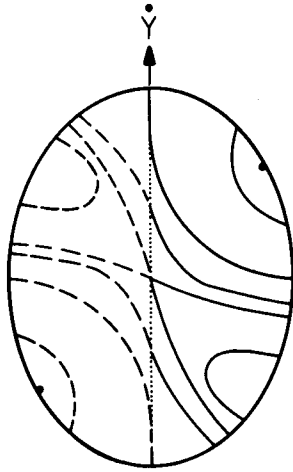


Fig. 8 The surface of section Σ for system L is topologically equivalent to a 2-sphere. Here we see a "side" view of Σ (the y -axis is directed into the paper). Solid lines are invariant curves which lie on the $\dot{x} > 0$ ("front") side of Σ . Dashed lines lie on the $\dot{x} < 0$ ("rear") side of Σ . Dotted lines represents $\dot{x} = 0$.

motions, $c = 0$ ($x = -y$) and $c = 1$ ($x = y$). There are no other periodic motions since the natural frequencies are incommensurable. For $0 < c < 0.25$ the invariant curves are closed and surround the out-of-phase mode $c = 0$, while for $0.5 < c < 1$, they surround the in-phase mode $c = 1$. However for $0.25 < c < 0.5$ the invariant curves are composed of two disconnected pieces. Nevertheless the motions which correspond to these disconnected curves are not qualitatively different from the other nonperiodic motions of the system. Their disconnected appearance is due to the topology of the energy manifold $H = h$, equation (3). We offer the following description of this situation.

The energy manifold $H = h$ for the system L is topologically equivalent to a 3-sphere. By slicing it with the 3-plane $x = 0$ we produce a surface of section Σ which is topologically equivalent to a 2-sphere. The Poincaré map of Fig. 7 is a projection of this surface onto the $y - \dot{y}$ -plane. The front half of Σ corresponds to $\dot{x} > 0$ while the rear half corresponds to $\dot{x} < 0$. Since we have taken $\dot{x} > 0$ in equation (9) we see only the front half of Σ in Fig. 7. The bounding curve $\dot{x} = 0$ separates the front half of Σ from the rear half. As mentioned previously, any motion which intersects Σ at $\dot{x} = 0$ does so nontransversally. Thus every disconnected invariant curve in Fig. 7 has a motion associated with it which intersects Σ nontransversally.

In fact any such disconnected invariant curve in Fig. 7 forms a closed connected curve on Σ . Such a curve includes points on both the front and rear halves of Σ . Due to the symmetry of the system L , the invariant curves on Σ are symmetric with respect to the origin. Therefore the rear half of Σ ($\dot{x} < 0$) looks just like the front half ($\dot{x} > 0$) turned upside-down, Fig. 8.

The stability of a periodic motion follows directly from the nature of the invariant curves in the neighborhood of a fixed point of the Poincaré map. It has been shown that stability results which are obtained from a linear stability analysis and Floquet theory are equivalent to results obtained by expanding the Poincaré map about a fixed point and neglecting cubic and higher-order terms [15, 8, 13, 19]. In cases where Floquet theory fails to predict stability (e.g., system S), the Poincaré map will still give stability results if higher-order terms are retained in expansions about a fixed point. We will return to this point when we find the Poincaré map for system S . First, however, we will use the B-G transformation to determine an approximate first integral for the system S .

The Birkhoff-Gustavson Transformation

In this section we will use an approximate method developed by Birkhoff [3] and extended by Gustavson [7]. The idea of the method

is to choose new dependent variables via a generating function in order to simplify the form of the Hamiltonian. Although the method can be continued to all orders of accuracy, we shall neglect terms of $O(6)$, i.e., terms of the form $x^\alpha y^\beta \dot{x}^\gamma \dot{y}^\delta$ where $\alpha, \beta, \gamma, \delta$ are non-negative integers and $\alpha + \beta + \gamma + \delta \geq 6$. As a result, our conclusions will only be valid for small energies h . In order to simplify the notation in this section, we will write

$$x = x_1, \quad y = x_2, \quad \dot{x} = \dot{x}_1, \quad \dot{y} = \dot{x}_2. \quad (12)$$

The Hamiltonian for system S , equation (3), may be written

$$H = H(x_j, \dot{x}_j) = H^{(2)} + H^{(4)} \quad (13)$$

where $H^{(n)}$ is a homogeneous polynomial of degree n . We first transform variables from (x_j, \dot{x}_j) to (ξ_j, η_j) by a near-identity canonical transformation based on the generating function $x_1\eta_1 + x_2\eta_2 + W^{(4)}(x_i, \eta_i)$;

$$\xi_j = x_j + \frac{\partial W^{(4)}}{\partial \eta_j}, \quad \dot{x}_j = \eta_j + \frac{\partial W^{(4)}}{\partial x_j}, \quad j = 1, 2. \quad (14)$$

Substituting (14) into (13) we find, neglecting terms of $O(6)$.

$$\begin{aligned} H^{(2)}(x_j, \dot{x}_j) &= \frac{1}{2} \sum_{j=1}^2 x_j^2 + \dot{x}_j^2 \\ &= \frac{1}{2} \sum_{j=1}^2 \xi_j^2 + \eta_j^2 + DW^{(4)}(\xi_j, \eta_j) \end{aligned} \quad (15)$$

where

$$D \equiv \sum_{j=1}^2 \eta_j \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \eta_j} \quad (16)$$

and

$$H^{(4)}(x_j, \dot{x}_j) = H^{(4)}(\xi_j, \eta_j). \quad (17)$$

Note that the replacement of x_j by ξ_j in the argument of $W^{(4)}$ is valid to $O(6)$.

Before judiciously selecting $W^{(4)}$, we perform a second canonical transformation which will simplify the operator D . We transform from (ξ_j, η_j) to (q_j, p_j) :

$$\xi_j = (q_j + ip_j)/\sqrt{2}, \quad \eta_j = (iq_j + p_j)/\sqrt{2} \quad (18)$$

where $i = \sqrt{-1}$. The transformed Hamiltonian $K(q_j, p_j)$ becomes

$$K(q_j, p_j) = K^{(2)}(q_j, p_j) + EW^{(4)}(q_j, p_j) + K^{(4)}(q_j, p_j) \quad (19)$$

where

$$K^{(2)}(q_j, p_j) = H^{(2)}(\xi_j, \eta_j) = i \sum_{j=1}^2 p_j q_j$$

$$EW^{(4)}(q_j, p_j) = DW^{(4)}(\xi_j, \eta_j)$$

$$= i \sum_{j=1}^2 \left(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} \right)$$

and where $K^{(4)}(q_j, p_j)$ is $H^{(4)}(\xi_j, \eta_j)$ transformed to q, p variables.

We now choose $W^{(4)}$ so that the transformed Hamiltonian K has a simple form which will yield a first integral. A typical term of $W^{(4)}$ is $C_{\alpha\beta\gamma\delta} q_1^\alpha q_2^\beta p_1^\gamma p_2^\delta$, where $\alpha + \beta + \gamma + \delta = 4$. It will be more convenient to work with $W^{(4)}$ as a column vector, which we may do in view of the isomorphism between the space of homogeneous polynomials of degree four and R^{35} , see Table 1.

From equation (19) we would like to choose $W^{(4)}$ so that

$$EW^{(4)} = -K^{(4)}. \quad (20)$$

Here E is a 35×35 diagonal matrix with typical diagonal term $i(\alpha + \beta - \gamma - \delta)$, $W^{(4)}$ is a 35-column vector whose elements are to be determined, and $K^{(4)}$ is a 35-column vector with known elements.

It if were possible to choose $W^{(4)}$ to satisfy equation (20) then equation (19) would be immediately integrable. However, the matrix E is singular. There are nine diagonal terms of E which vanish. They correspond to the following values of $\alpha, \beta, \gamma, \delta$:

Table 1 Isomorphism between the space of homogeneous polynomials of degree four and R^{35} . A basis vector in the former space may be written $q_1^\alpha q_2^\beta p_1^\gamma p_2^\delta$ where $\alpha + \beta + \gamma + \delta = 4$, and is represented in the table by $(\alpha\beta\gamma\delta)$. The corresponding basis vector in R^{35} is $\bar{e}_n = (0, 0, \dots, 0, 1, 0, \dots, 0)$, in which the n th element is unity and all others are zero. \bar{e}_n is represented in the table simply by n .

n	$(\alpha\beta\gamma\delta)$	n	$(\alpha\beta\gamma\delta)$	n	$(\alpha\beta\gamma\delta)$
1	(0004)	13	(0301)	25	(1300)
2	(0013)	14	(0310)	26	(2002)
3	(0022)	15	(0400)	27	(2011)
4	(0031)	16	(1003)	28	(2020)
5	(0040)	17	(1012)	29	(2101)
6	(0103)	18	(1021)	30	(2110)
7	(0112)	19	(1030)	31	(2200)
8	(0121)	20	(1102)	32	(3001)
9	(0130)	21	(1111)	33	(3010)
10	(0202)	22	(1120)	34	(3100)
11	(0211)	23	(1201)	35	(4000)
12	(0220)	24	(1210)		

$$(\alpha\beta\gamma\delta) = (0202), (0211), (0220), (1102), (1111), (1120), (2002), (2011), (2020). \quad (21)$$

Thus equation (20) has no solution for a general $K^{(4)}$.

Nevertheless we may choose $W^{(4)}$ to simplify the transformed Hamiltonian K , equation (19), as much as possible. We proceed as follows:

Since E is a diagonal matrix, its null space and range are complementary spaces. Let P be a projection operator onto the range of E and let Q be a projection operator onto the null space of E . Then we may write

$$K^{(4)} = PK^{(4)} + QK^{(4)}. \quad (22)$$

Now instead of equation (20) we may choose $W^{(4)}$ so that

$$EW^{(4)} = -PK^{(4)}. \quad (23)$$

The general solution to equation (23) is not unique, since any vector in the null space of E may be appended to a given solution. However if we require $W^{(4)}$ to lie in the range of E then a unique solution emerges. (The transformation which maps the right-hand side of equation (23) into $W^{(4)}$ is called the *pseudo-inverse* of E , [25]). It is to be noted that any first integral which is independent of the Hamiltonian H is never unique since, e.g., functional combinations of H and itself may be added to it. The nonuniqueness of $W^{(4)}$ reflects this fact.

If $W^{(4)}$ is chosen so as to satisfy equation (23), then equation (19) becomes

$$K = K^{(2)} + QK^{(4)}. \quad (24)$$

By direct computation for system S we obtain the following expression for the projection of $K^{(4)}$ onto the null space of E (i.e., the space spanned by the basis vectors of equation (21)),

$$QK^{(4)} = c_1 q_2^2 p_2^2 + c_2 q_2^2 p_1 p_2 + c_3 q_2^2 p_1^2 + c_4 q_1 q_2 p_2^2 + c_5 q_1 q_2 p_1 p_2 + c_6 q_1 q_2 p_1^2 + c_7 q_1^2 p_2^2 + c_8 q_1^2 p_1 p_2 + c_9 q_1^2 p_1^2 \quad (25)$$

where

$$c_1 = c_9 = -3(k+1)/8$$

$$c_2 = c_4 = c_6 = c_8 = 3/4$$

$$c_3 = c_7 = -3/8$$

$$c_5 = -1/2.$$

To determine a first integral corresponding to the approximate Hamiltonian K , we note that

$$EK^{(2)} = 0 \quad (26)$$

and therefore, using equation (24),

$$EK = EK^{(2)} + EQK^{(4)} = 0. \quad (27)$$

Next we note that

$$\begin{aligned} \dot{K}^{(2)}(q_j, p_j) &= i \sum_{j=1}^2 (p_j \dot{q}_j + \dot{p}_j q_j) \\ &= i \sum_{j=1}^2 \left(-\frac{\partial K}{\partial q_j} q_j + \frac{\partial K}{\partial p_j} p_j \right) \\ &= -EK. \end{aligned} \quad (28)$$

From equations (27) and (28) we see that $\dot{K}^{(2)} = 0$ and therefore $K^{(2)} = \text{constant}$ is a first integral. Moreover, since $K = \text{constant}$ is also a first integral, we may with greater convenience choose the difference $K - K^{(2)} = QK^{(4)} = \text{constant}$ as a first integral corresponding to the system based on the approximate Hamiltonian K . As far as the original system S is concerned, $QK^{(4)} = \text{constant}$ is an approximate first integral valid when terms of $O(6)$ are neglected.

Using the inverse of the canonical transformation (18) and again neglecting terms of $O(6)$, we may transform the expression (25) for $QK^{(4)}$ back to the original variables. In this way we find the approximate first integral to be

$$\begin{aligned} f(x, y, \dot{x}, \dot{y}) &= (1+k)[(x^2 + \dot{x}^2)^2 + (y^2 + \dot{y}^2)^2] \\ &\quad + 4(x^2 + \dot{x}^2)(y^2 + \dot{y}^2) - 4(x^2 + \dot{x}^2 + y^2 + \dot{y}^2)(xy + \dot{x}\dot{y}) \\ &\quad + 2(x^2 - \dot{x}^2)(y^2 - \dot{y}^2) + 8xy\dot{x}\dot{y} \\ &= c. \end{aligned} \quad (29)$$

A simple computation shows that $df/dt = 0$ if terms of $O(6)$ are neglected.

As a check, we note that system S is integrable when $k = 0$. Setting $u = (1/\sqrt{2})(x + y)$ and $v = (1/\sqrt{2})(-x + y)$ uncouples the two equations of motion (1) in this case. It turns out that for $k = 0$

$$\begin{aligned} F(x, y, \dot{x}, \dot{y}) &= x^2 + \dot{x}^2 + y^2 + \dot{y}^2 - 2(xy + \dot{x}\dot{y}) + (x - y)^4 \\ &= \text{constant} \end{aligned} \quad (30)$$

is an *exact* first integral independent of the energy integral (3). If we set $k = 0$ in our approximate first integral (29), we find that

$$f = F^2 + O(6) \quad (31)$$

which confirms that f is a first integral to $O(6)$.

Note that we did not have to find $W^{(4)}$ in order to obtain the approximate first integral (29). This situation usually occurs only for the low-order 1:1 or 3:1 resonance cases, i.e., for systems in which the linearized normal modal frequencies are in the ratio 1:1 or 3:1. (System S has a 1:1 resonance.) In all other cases the B-G method proceeds in a slightly different manner. There follows a brief outline of the B-G procedure for such cases.

In systems without 1:1 or 3:1 resonances, the transformed Hamiltonian K of equation (19) has been shown by Birkhoff [3] to be a function only of the product terms $\pi_1 = p_1 q_1$ and $\pi_2 = p_2 q_2$. This result implies that π_1 and π_2 are two independent first integrals, since

$$\dot{p}_j = -\frac{\partial K}{\partial q_j} = -p_j \frac{\partial K}{\partial \pi_j} \quad (32)$$

and

$$\dot{q}_j = \frac{\partial K}{\partial p_j} = q_j \frac{\partial K}{\partial \pi_j} \quad (33)$$

and therefore

$$\dot{\pi}_j = \dot{p}_j q_j + p_j \dot{q}_j = 0. \quad (34)$$

Thus, for such systems π_1 and π_2 are chosen as the two independent first integrals. They represent the energies in each of the approximately uncoupled oscillators. In this case, however, $W^{(4)}(\xi_j, \eta_j)$ is needed in order to transform back to the original variables x_j, \dot{x}_j to

0(6). We note that since $\dot{K}^{(2)} = 0$, we could again choose $QK^{(4)} = K - K^{(2)}$ as an approximate first integral to 0(6). Here, however $QK^{(4)}$ could be zero, whereas it is generally not zero in the 1:1 and 3:1 resonance cases.

Application

In this section we will use the approximate first integral (29) to obtain an approximate Poincaré map (valid for small energies h) for the system S .

To generate the Poincaré map, set $x = 0$ in equations (2) and (3) and solve for $\dot{x}(y, \dot{y}, h)$:

$$\dot{x} = \left[2h - y^2 - \left(\frac{k+1}{2} \right) y^4 - \dot{y}^2 \right]^{1/2} > 0. \quad (35)$$

Then set $x = 0$ in equation (29) to obtain

$$\begin{aligned} \bar{f}(y, \dot{y}) &= (1+k)[\dot{x}^4 + (y^2 + \dot{y}^2)^2] + 2\dot{x}^2(y^2 + 3\dot{y}^2) \\ &\quad - 4\dot{x}\dot{y}(x^2 + y^2 + \dot{y}^2) \\ &= c \end{aligned} \quad (36)$$

in which $\dot{x} = \dot{x}(y, \dot{y}, h)$ is given by equation (35). For fixed h, k , equation (36) represents a one-parameter family of invariant curves with c as parameter. These curves fill the interior of the region of the $y - \dot{y}$ -plane bounded by the curve

$$y^2 + \left(\frac{k+1}{2} \right) y^4 + \dot{y}^2 = 2h \quad (37)$$

obtained by setting $\dot{x} = 0$ in equation (35).

NNM's appear as fixed points of the Poincaré map which lie on the \dot{y} -axis ($y = 0$). This follows from the requirement that a NNM pass through the origin ($x = 0, y = 0$). At any fixed point, the level curves $\bar{f}(y, \dot{y}) = c$ possess a singularity. Thus, for a NNM, we require

$$\frac{\partial \bar{f}}{\partial y}(0, \dot{y}) = 0, \quad \frac{\partial \bar{f}}{\partial \dot{y}}(0, \dot{y}) = 0 \quad (38)$$

From equations (35) and (36) we see that $\bar{f}(y, \dot{y}) = \bar{f}(-y, \dot{y})$ so that the first condition of (38) is satisfied identically. The second condition of (38) gives, after some algebra,

$$\dot{y} = \pm \sqrt{h}, \quad y^2 = h \left(1 \pm \frac{\sqrt{k(k-4)}}{|k-2|} \right). \quad (39)$$

These values of \dot{y} , together with the condition $y = 0$, give the position on the Poincaré map of all the NNM's which occur in system S . The in-phase mode $y = x$ corresponds to $\dot{y} = \sqrt{h}$, the out-of-phase mode $y = -x$ corresponds to $\dot{y} = -\sqrt{h}$, and the bifurcating modes of equation (4) correspond to the last two values of \dot{y} in equation (39). This analysis proves that system S possesses no other NNM's than the SNM's referred to in equation (4) (subject to the assumption that h is small).

In order to investigate the stability of these NNM's, we will examine the nature of the invariant curves of the Poincaré map in the neighborhood of the corresponding singular points.

First we move the origin to the fixed points by substituting

$$w = \dot{y} - \dot{y}_0 \quad (40)$$

in the expression for \bar{f} , equation (36), where \dot{y}_0 is one of the NNM values of equation (39). Then we expand \bar{f} in a Taylor series about $y = 0, w = 0$, and neglect cubic and higher-order terms.

For the out-of-phase mode $y = -x$ this procedure gives

$$(k-4)w^2 - y^2 = \frac{c - c_0}{8h} \quad (41)$$

where $c_0 = 2h^2(k+8)$ is the value of c at the fixed point. For $k < 4$, equation (41) is a family of ellipses while for $k > 4$ it is a family of hyperbolas. Thus the out-of-phase mode is stable for $k < 4$ and unstable for $k > 4$, a result which is in agreement with the Floquet theory approach, cf. Fig. 4.

For the in-phase mode $y = x$, however, the same procedure gives

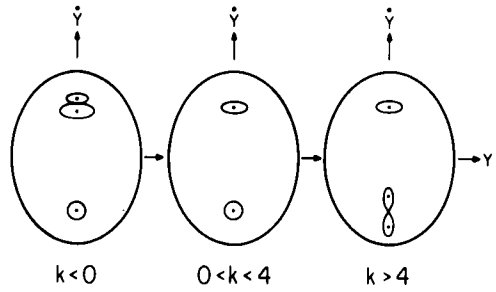


Fig. 9 Sketches of the Poincaré map for system S for different ranges of the bifurcation parameter k ; the upper (lower) fixed points correspond to the in-phase (out-of-phase) mode and to the modes which bifurcate from it

$$kw^2 = \frac{c - c_0}{8h} \quad (42)$$

where now $c_0 = 2h^2k$. Thus the invariant curves in the neighborhood of the fixed point appear to be parallel lines and therefore equation (42) fails to predict stability. This result agrees with Floquet theory which also failed to predict stability for this mode.

In the case of the Poincaré map, however, we may consider the effect of nonlinear perturbations about the periodic motion in the following manner. As noted previously, equations (35) and (36) show that $\bar{f}(y, \dot{y})$ is an even function of y . We therefore set

$$z = y^2 \quad (43)$$

and substitute (40) and (43) into (35) and (36). Expanding \bar{f} in a Taylor series about $z = 0, w = 0$ and neglecting higher-order terms, we obtain

$$8hkw^2 + 8\sqrt{h}kwz + [2k + 1 - hk(1+k)]z^2 = c - c_0 \quad (44)$$

The nature of this conic section depends upon the sign of the discriminant of (44) which equals

$$-32hk + O(h^2) \quad (45)$$

To $O(h^2)$, when $k < 0$ the discriminant is positive, the conics (44) are hyperbolas and the singularity in $w - z$ coordinates is a saddle. The two separatrices of this saddle for $z \geq 0$ map into two separatrices in $w - y$ coordinates and hence in $w - y$ coordinates the singularity is a (nonlinear) saddle. For $k > 0$ the discriminant is negative and the conics (44) in $w - z$ coordinates are ellipses. In $w - y$ coordinates this singularity becomes a (nonlinear) center. Therefore the Poincaré map predicts that the in-phase mode is unstable for $k < 0$ and stable for $k > 0$.

The stability of the bifurcating NNM's may be investigated in a similar fashion. Sketches of the Poincaré map for various values of k are given in Fig. 9.

Conclusion

We have investigated the dynamical structure of a pair of nonlinear coupled oscillators S by generating an approximation for the Poincaré map via the B-G canonical transformation. In particular we investigated the existence and stability of NNM's and found that the system S admits only 2 or 4 NNM's, depending upon the value of the parameter k . The bifurcating modes enter as stable while the mode from which they bifurcated changes from stable to unstable upon bifurcation.

It is to be noted that all these results are valid only for small h . This is not only because the B-G method is a perturbation method valid for small h , but also because KAM (Kolmogorov-Arnold-Moser) theory tells us that the invariant tori generally do not fill the energy manifold $H = h$ in systems which are not integrable [17, 1]. The set of motions which lie on invariant tori typically have decreasing measure as h is increased. For large enough h the Poincaré map (ob-

tainable by numerical integration) loses all semblance of order and appears to be filled with "noise," i.e., with motions which seem to be ergodic.

It is also to be noted that although we used the B-G method to generate the approximate Poincaré map, it could have been obtained by other equivalent methods. We mention Whittaker's adelpic integral [27] and the method of Lie transforms [11] and refer the reader to [15] where all three methods are compared and are shown to be equivalent to lowest-order terms.

Finally, we note that for negative k system S admits additional equilibrium points besides the origin $x = y = 0$ [15]. In order that the energy manifold $H = h$ be compact (i.e., closed and bounded) for negative k , it is necessary to restrict the energy h to be smaller than a limiting value dependent on k . If the energy manifold is not compact, the bounding curve for the Poincaré map will generally possess branches which reach to infinity. In such a case the system S will not be physically realistic.

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References

- 1 Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Springer, 1978.
- 2 Arnold, V. I., and Avez, A., *Ergodic Problems of Classical Mechanics*, Benjamin, 1968.
- 3 Birkhoff, G. D., *Dynamical Systems*, AMS Colloquium 9, 1927.
- 4 Cesari, L., *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, 3rd ed., Springer, 1971.
- 5 Chillingworth, D. R. J., *Differential Topology With a View to Applications*, Pitman, 1976.
- 6 Greenberg, M. D., *Foundations of Applied Mathematics*, Prentice Hall, 1978.
- 7 Gustavson, F. G., "On Constructing Formal Integrals of a Hamiltonian System Near an Equilibrium Point," *Astronomical Journal*, Vol. 71, 1963, pp. 670-686.
- 8 Hartman, P., *Ordinary Differential Equations*, Wiley, 1964.
- 9 Hirsch, M. W., and Smale, S., *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press, 1974.

- 10 Johnson, T. L., and Rand, R. H., "On the Existence and Bifurcation of Minimal Normal Modes," *International Journal of Non-Linear Mechanics*, Vol. 14, 1979, pp. 1-12.
- 11 Kamel, A. A., "Perturbation Theory Based on Lie Transforms," NASA Contractor Report 1622, 1970.
- 12 Kauderer, H., *Nichtlineare Mechanik*, Springer, 1958.
- 13 Lefschetz, S., *Differential Equations: Geometric Theory*, 2nd ed., Dover, 1977.
- 14 Minorsky, N., *Nonlinear Oscillations*, Van Nostrand, 1962.
- 15 Month, L. A., "On Approximate First Integrals of Hamiltonian Systems With an Application to Nonlinear Normal Modes in a Two-Degree-of-Freedom Nonlinear Oscillator," PhD Thesis, Cornell University, 1979.
- 16 Month, L. A., and Rand, R. H., "The Stability of Bifurcating Periodic Solutions in a Two-Degree-of-Freedom Nonlinear System," *ASME JOURNAL OF APPLIED MECHANICS*, Vol. 44, 1977, pp. 782-784.
- 17 Moser, J., "Stable and Random Motions in Dynamical Systems," *Annals of Mathematical Studies*, Vol. 77, Princeton University Press, 1973.
- 18 Pak, C. H., and Rosenberg, R. M., "On the Existence of Normal Mode Vibrations in Nonlinear Systems," *Quarterly of Applied Mathematics*, Vol. 26, 1968, pp. 403-416.
- 19 Pecelli, G., and Thomas, E. S., "Normal Modes, Uncoupling, and Stability for a Class of Nonlinear Oscillators," SUNY Albany Report No. 461, 1978.
- 20 Poincaré, H., "New Methods of Celestial Mechanics," Vol. 3, NASA Technical Translation TT F-452, 1967.
- 21 Rand, R. H., "On the Stability of Hill's Equation With Four Independent Parameters," *ASME JOURNAL OF APPLIED MECHANICS*, Vol. 36, 1969, pp. 885-886.
- 22 Rand, R. H., and Tseng, S. F., "On the Stability of a Differential Equation With Application to the Vibrations of a Particle in the Plane," *ASME JOURNAL OF APPLIED MECHANICS*, Vol. 36, 1969, pp. 311-313.
- 23 Rand, R., and Vito, R., "Nonlinear Vibrations of Two-Degree-of-Freedom Systems With Repeated Linearized Natural Frequencies," *ASME JOURNAL OF APPLIED MECHANICS*, Vol. 39, 1972, pp. 296-297.
- 24 Rosenberg, R. M., "On Nonlinear Vibrations of Systems With Many Degrees of Freedom," *Advances in Applied Mechanics*, Academic Press, 1966, pp. 155-242.
- 25 Stackgold, I., "Branching of Solutions of Nonlinear Equations," *SIAM Review*, Vol. 13, 1971, pp. 289-332.
- 26 Stoker, J. J., *Nonlinear Vibrations*, Interscience, 1950.
- 27 Whittaker, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, 4th ed., Cambridge University Press, 1937.
- 28 Yang, T. L., and Rosenberg, R. M., "On the Vibrations of a Particle in the Plane," *International Journal of Nonlinear Mechanics*, Vol. 2, 1967, pp. 1-25.