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LIE TRANSFORMS APPLIED TO A NON-LINEAR PARAMETRIC EXCITATION PROBLEM*

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Abstract We use Lie transforms to approximate the Poincaré map of a weakly non-linear periodic perturbation of the simple harmonic oscillator in order to study the stability of the trivial solution. Resonant frequencies, corresponding to non-removable terms in the differential equation, are identified through $O(\varepsilon^2)$. We show that detuning from resonance stabilizes the trivial solution when the perturbation contains no linear periodic terms. Finally, we study a typical bifurcation between two lowest-order resonant frequencies. A MACSYMA program which performs the Lie transform algorithm to arbitrary order is presented in the Appendix with a sample run.

1. INTRODUCTION

In this paper we present some results concerning the stability of the trivial solution of the equation

$$\ddot{x}+\omega^2 x+\varepsilon f(t,x)=0 \tag{1}$$

where $f(t, x)$ is T-periodic in t, Taylor–Fourier expandable in x and t respectively, and $f(t, x)$ satisfies $f(t, 0)\equiv 0$. The hamiltonian structure of equation (1) permits us to use *Lie transforms* to reduce the non-autonomous hamiltonian induced by equation (1) to an autonomous one by means of a periodic canonical near-identity transformation. The resulting autonomous hamiltonian describes the Poincaré map in a neighborhood of the origin.

Analysis of the Poincaré map gives substantial information concerning the original equation. The presence of a periodic point in the Poincaré map implies the existence of a periodic orbit in the original equation. In particular, a periodic saddle point corresponds to a hyperbolic periodic orbit, and a periodic center corresponds to an elliptic periodic orbit.

We begin by describing the Lie transform algorithm as used in this work. We then present a theorem which defines the $O(\varepsilon)$ and $O(\varepsilon^2)$ resonances for the general case of equation (1), and show that almost all higher order resonances are stable.

Next, we study the properties of the trivial solution of a simple equation of the type (1). We identify the $O(\varepsilon)$ and $O(\varepsilon^2)$ resonances, and characterize the stability of the trivial solution for all non-zero ω. Results of the Lie transform analysis are compared with numerically generated Poincaré maps.

Finally, we study a bifurcation between $O(\varepsilon)$ resonances of cubic and quadratic non-linearities. In this example, a 4π-periodic hyperbolic orbit becomes a 2π-periodic hyperbolic orbit through a sequence of bifurcations.

2. RESULTS

We consider the general equation

$$\ddot{x}+\omega^2 x+\varepsilon\sum_{\alpha} g_\alpha(t)x^{N_\alpha-1}=0 \tag{2}$$

where the $g_\alpha(t)$ are periodic and the N_α are positive integers. This equation was studied

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extensively in [1]. Here we summarize some results and refer the reader to [1] for additional information.

In canonical variables q and p, equation (2) is generated by the hamiltonian

$$h(q, p, t)=\frac{p^2}{2}+\frac{\omega^2 q^2}{2}+\varepsilon\sum_\alpha \frac{1}{N_\alpha} g_\alpha(t) q^{N_\alpha}. \tag{3}$$

The change of variables

$$q=\sqrt{2J/\omega}\sin(\theta+\omega t),$$
$$p=\sqrt{2J\omega}\cos(\theta+\omega t) \tag{4}$$

reduces equation (3) to the $O(\varepsilon)$ hamiltonian

$$H(J, \theta, t)=\varepsilon\sum_\alpha \frac{1}{N_\alpha} g_\alpha(t)\left(\frac{2J}{\omega}\right)^{N_\alpha/2} \sin^{N_\alpha}(\theta+\omega t)\equiv \varepsilon H_1(J, \theta, t). \tag{5}$$

We will apply the Lie transform procedure to this $O(\varepsilon)$ hamiltonian.

Definition 1. *Let*

$$w(t, x, \varepsilon)=w_1(t, x)+\varepsilon w_2(t, x)+\varepsilon^2 w_3(t, x)+\ \ldots$$

be the Lie generating function defining a canonical transformation which reduces the hamiltonian (5) *to an autonomous one. Then* ω *is a resonance at* $O(\varepsilon^n)$ *if it is a pole of* $w_n(t, x)$ *but not of* $w_k(t, x)$, *for* $1\le k\le n-1$.

We denote by Ω_n the set of frequencies which are resonant at $O(\varepsilon^n)$.

An equivalent definition of a resonant frequency may be formulated in terms of the near-identity transformation generated by periodic averaging.

For example, the $O(\varepsilon^n)$ resonance for the linear Mathieu equation

$$\ddot{x}+\omega^2 x+\varepsilon x\cos t=0$$

is $\omega=n/2$, for $n\ge 1$. Resonant frequencies correspond to non-removable terms in the hamiltonian (with respect to Lie transforms) or in the differential equation (with respect to periodic averaging).

In order to show how to generate all $O(\varepsilon)$ and $O(\varepsilon^2)$ resonances for equation (2), we introduce some notation. By assumption, each $g_\alpha(t)$ is periodic and may therefore be expanded in a Fourier series. Let

$$c_\mu^{(\alpha)}=\frac{1}{2\pi}\int_0^{2\pi} g_\alpha(t)e^{-i\mu t}dt$$

be the Fourier coefficients of $g_\alpha(t)$ for integer μ. Let M_α denote the set of frequencies of $g_\alpha(t)$, that is,

$$M_\alpha=\{\mu : c_\mu^{(\alpha)}\ne 0,\ \mu\in Z\}.$$

Then

$$g_\alpha(t)=\sum_{\mu\in M_\alpha} c_\mu^{(\alpha)} e^{i\mu t}.$$

As shown in [1], Ω_1 consists of all ω satisfying

$$\omega=\frac{\mu}{N_\alpha-2\nu}, \qquad \begin{array}{l}\mu\in M_\alpha\\ 0\le\nu\le N_\alpha\end{array}$$

and Ω_2 consists of all ω satisfying

$$\omega=\frac{\mu+\gamma}{N_\alpha+N_\beta-2(\nu+\delta)} \qquad \begin{array}{l}\mu\in M_\alpha\\ \gamma\in M_\beta\\ 0\le\nu\le N_\alpha\\ 0\le\delta\le N_\beta\\ \delta N_\alpha\ne\nu N_\beta\end{array}$$

which are not also included in Ω_1.

It is also shown in [1] that if $\omega \notin \Omega_1 \cup \Omega_2 \cup \{0\}$, then the resulting reduced hamiltonian is of the form

$$K = \varepsilon f_1(J) + \varepsilon^2 f_2(J) + \ldots$$

where f_1 and f_2 contain integer or half-integer powers of J. This implies that the origin of the Poincaré map is a center, and therefore the trivial solution is a stable elliptic orbit.

We show here that if the perturbation is strictly non-linear then detuning creates a hyperbolic periodic orbit which traps the trivial solution, stabilizing it. We then analyze a bifurcation problem between periodic orbits near resonances showing how a 4π-periodic orbit bifurcates into a 2π-periodic orbit.

3. LIE TRANSFORMS

An important characteristic of autonomous hamiltonian systems is that the hamiltonian is constant along solutions of the system of differential equations. If the phase space has dimension two then the solutions are level curves of the hamiltonian. The reader is referred to [3] or [4] for a complete discussion of hamiltonian mechanics.

In this work we use Lie transforms to reduce equation (3) to an autonomous hamiltonian, and then analyze the level curves of this autonomous hamiltonian to determine the behavior of solutions which have initial conditions close to the trivial solution. The implementation of the Lie transform algorithm which is presented here implicitly constructs a canonical change of coordinates which performs the reduction to an autonomous form. It is obvious that no autonomous canonical change of variables can make this reduction. Therefore the hamiltonian with respect to the new coordinates must be determined by means of a generating function or some equivalent method which takes into account the nonautonomous nature of the transformation. The Lie transform method is an efficient perturbation scheme which explicitly generates the functional form of the reduced hamiltonian under an implicitly defined canonical periodic near-identity transformation.

Let x and y denote the old and new coordinates, respectively. Let ε denote the perturbation parameter. Let H denote the hamiltonian with respect to the x coordinates, and let K denote the transformed hamiltonian. We assume that H and K may each be written as power series in ε,

$$H(t, x, \varepsilon) = H_0(t, x) + \varepsilon H_1(t, x) + \varepsilon^2 H_2(t, x) + \ldots$$

and

$$K(t, x, \varepsilon) = K_0(t, x) + \varepsilon K_1(t, x) + \varepsilon^2 K_2(t, x) + \ldots$$

The relation between x and y is defined implicitly in terms of a *Lie generating function* $w(t, x)$ as

$$\frac{\partial y_i}{\partial \varepsilon} = \{x_i, w\} \tag{6}$$

where $\{\ ,\ \}$ is the *Poisson bracket* operator. For two-dimensional phase space, the Poisson bracket operator is

$$\{f, g\} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}.$$

In words, the new coordinate system evolves from the old one by means of a "hamiltonian flow" in the evolution quantity ε. See [3] for a complete discussion of this procedure. It is straightforward to show that the change of variables $x \to y$ defined by equation (6) is canonical. This consists of showing that the fundamental Lagrange brackets are preserved under the transformation.

The reduced hamiltonian K is related to H by

$$\begin{aligned} K_0 &= H_0 \\ K_1 &= H_1 + \{w_1, H_1\} + \frac{\partial w_1}{\partial t} \\ K_2 &= H_2 + \frac{1}{2}\{w_1, K_1 + H_1\} + \frac{1}{2}\frac{\partial w_2}{\partial t} + \{w_2, H_0\}. \\ &\vdots \end{aligned} \tag{7}$$

Although this sequence can be written in closed form to arbitrary order, we need it only through $O(\varepsilon^2)$. See [2] for full details of this topic.

It is important to interpret equation (7) correctly. The right-hand side of each equation is a function of x and t, and the Poisson brackets are computed with respect to the x coordinate system. The resulting function K is evaluated at $K(t, x)$. The x are dummy variables, and may be replaced by y to give the transformed hamiltonian.

The sequence (7) gives the transformed hamiltonian for an arbitrary generating function w. The trick is to choose successive w_i to make the corresponding K_i as simple as possible. This means choosing w_i at the ith step such that

$$\frac{\partial w_i}{\partial t} + \{w_i, H_0\}$$

removes as many terms as possible in the right-hand side of the ith equation in (7). While this operator is linear, it has a non-trivial kernel; therefore some terms may not be removable. In the context of periodic perturbations, this means that w_i cannot be chosen to make K_i autonomous directly. However, after all non-essential terms have been removed to desired order using Lie transforms, a final canonical transformation of the form

$$J \to I$$
$$\theta \to \Phi + \alpha t$$

for some scalar α can always be found which makes it autonomous.

The method may be simplified considerably by the following trick: Apply a canonical transformation which removes the $O(1)$ terms of the hamiltonian so that $H_0 \equiv 0$. Then all Poisson brackets in (7) involving H_0 vanish, and the terms which are removable are precisely the t-dependent ones. The appropriate choice of w_i is to take $-w_i$ as the t-antiderivative of the t-dependent terms. The resulting K is autonomous by construction. This modified Lie transform algorithm has been implemented in MACSYMA since the amount of algebra required to carry the perturbation scheme through even $O(\varepsilon^2)$ is too daunting to compute by hand with any confidence. The program and sample runs are given in the Appendix. For a further discussion on the use of computer algebra in perturbation schemes, see [1, 6, 7].

While the simplification of the algorithm is important from the computer algebra point of view, it is perhaps more important for analytical purposes. This modified method was used to determine the $O(\varepsilon)$ and $O(\varepsilon^2)$ resonances of the general equation given previously.

In principle, this strategy may be used in any system where the $\varepsilon = 0$ problem may be solved exactly. For example, a system of linear oscillators with weak non-linear coupling may be studied using this simplification.

4. DETERMINING THE RESONANT FREQUENCIES

In this section we briefly describe the procedure by which resonances may be found using Lie transforms. For complete details, see [1].

We assume that the equation is of the form

$$\ddot{x} + \omega^2 x + \varepsilon \sum_\alpha g_\alpha(t) x^{N_\alpha - 1} = 0.$$

In canonical variables q and p, this equation gives rise to the hamiltonian

$$h(q, p, t) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \varepsilon \sum_\alpha \frac{1}{N_\alpha} g_\alpha(t) q^{N_\alpha}. \tag{8}$$

Our first step, as described at the end of the previous section, will be to perform a transformation of coordinates to a system in which the hamiltonian contains no $O(1)$ terms. The change of variables

$$\begin{aligned} q &= \sqrt{2J/\omega}\,\sin(\theta + \omega t), \\ p &= \sqrt{2J\omega}\,\cos(\theta + \omega t) \end{aligned} \tag{9}$$

reduces equation (8) to the simplified hamiltonian

$$H(J,\theta,t)=\varepsilon\sum_{\alpha}\frac{1}{N_\alpha}g_\alpha(t)\left(\frac{2J}{\omega}\right)^{N_\alpha/2}\sin^{N_\alpha}(\theta+\omega t)\equiv\varepsilon H_1(J,\theta,t). \tag{10}$$

We now apply the Lie transform procedure to transform equation (10) into an autonomous hamiltonian. Note that no $O(1)$ terms are present. As noted at the end of the previous section, this simplification permits us to compute the Lie generating function at each step by integration of exponentials.

To identify the resonances, we first compute w_1 for arbitrary ω. This gives a function similar in form to H_1 but whose coefficients are rational functions of ω. The poles of these coefficients, which correspond to non-removable terms in $H_1(J,\theta,t)$, are frequencies which are resonant at $O(\varepsilon)$. Having identified the $O(\varepsilon)$ resonances, we may then implicitly compute w_2 to identify possible $O(\varepsilon^2)$ resonances.

We first introduce some notation. By assumption, each $g_\alpha(t)$ is periodic and may therefore be expanded in a Fourier series. Let

$$c_\mu^{(\alpha)}=\frac{1}{2\pi}\int_0^{2\pi}g_\alpha(t)e^{-i\mu t}dt$$

be the Fourier coefficients of $g_\alpha(t)$ for integer μ. Let M_α denote the set of frequencies of $g_\alpha(t)$, that is,

$$M_\alpha=\{\mu: c_\mu^{(\alpha)}\neq 0, \mu\in Z\}.$$

Then

$$g_\alpha(t)=\sum_{\mu\in M_\alpha}c_\mu^{(\alpha)}e^{i\mu t}.$$

Expanding the trigonometric functions with the binomial theorem and inserting the expansion for $g_\alpha(t)$ in equation (10) gives

$$H_1(J,\theta,t)=\sum_\alpha\sum_{\mu\in M_\alpha}\sum_{\nu=0}^{N_\alpha}a_{\mu\nu}^{(\alpha)}e^{i\theta(2\nu-N_\alpha)}e^{it((2\nu-N_\alpha)\omega+\mu)} \tag{11}$$

where

$$a_{\mu\nu}^{(\alpha)}=\frac{1}{N_\alpha}c_\mu^{(\alpha)}\left(\frac{J}{2\omega}\right)^{N_\alpha/2}\binom{N_\alpha}{\nu}(-1)^{N_\alpha/2-\nu}. \tag{12}$$

Proceeding formally, w_1 is just the negative of the t-antiderivative of H_1:

$$w_1=\sum_\alpha\sum_{\mu\in M_\alpha}\sum_{\nu=0}^{N_\alpha}\frac{-a_{\mu\nu}^{(\alpha)}e^{i\theta(2\nu-N_\alpha)}e^{it((2\nu-N_\alpha)\omega+\mu)}}{i((2\nu-N_\alpha)\omega+\mu)}.$$

This choice of w_1 makes K_1 the t-independent part of H_1.

The poles of w_1 are $\omega=0$, which we shall ignore, and

$$\omega=\frac{\mu}{N_\alpha-2\nu},\qquad \begin{matrix}\mu\in M_\alpha\\ 0\le\nu\le N_\alpha.\end{matrix}$$

Let Ω_1 denote the set of $O(\varepsilon)$ resonances. Let Ω_2 denote the set of poles of w_2 which are not in Ω_1. Then Ω_2 the set of $O(\varepsilon^2)$ resonances. The equation defining w_2 is

$$\frac{\partial w_2}{\partial t}=2K_2-\{w_1,K_1\}-\{w_1,H_1\}. \tag{13}$$

It is sufficient to determine the possible exponents introduced in the right-hand side of equation (13) since the poles of w_2 correspond to roots of the exponents. Since K_1 and K_2 are autonomous by construction, the new resonances can come only from the term $\{w_1, H_1\}$. It is clear from the definition of $a_{\mu\nu}^{(\alpha)}$ that

$$\frac{\partial a_{\mu\nu}^{(\alpha)}}{\partial J}=\frac{N_\alpha}{2J}a_{\mu\nu}^{(\alpha)}.$$

Therefore,

$$\{w_1, H_1\} = \sum_{\alpha,\beta} \sum_{\substack{\mu\in M_\alpha \\ \gamma\in M_\beta}} \sum_{v=0}^{N_\alpha} \sum_{\delta=0}^{N_\beta} \frac{a_{\mu v}^{(\alpha)} a_{\gamma\delta}^{(\beta)} (\delta N_\alpha - v N_\beta)}{i((2v - N_\alpha)\omega + \mu)} e^{i\theta(2(v+\delta) - N_\alpha - N_\beta)} e^{it((2(v+\delta) - N_\alpha - N_\beta)\omega + \mu + \gamma)}. \tag{14}$$

If a frequency is a resonance, then it is a root of the t-dependent exponential. It is easily seen that Ω_2 consists of all ω which satisfy

$$\omega = \frac{\mu+\gamma}{N_\alpha + N_\beta - 2(v+\delta)} \qquad \begin{matrix} \mu \in M_\alpha \\ \gamma \in M_\beta \\ 0 \le v \le N_\alpha \\ 0 \le \delta \le N_\beta \\ \delta N_\alpha \ne v N_\beta \end{matrix}$$

but which are not also included in Ω_1.

We conclude this section with some examples which demonstrate how to compute the resonant frequencies.

4.1. *Examples*

Example 1

$$\ddot{x} + \omega^2 x + \varepsilon x \cos(t) = 0.$$

For this example,

$$N_1 = 2, \qquad M_1 = \{-1, 1\}.$$

Ω_1 is generated by the numerators ± 1 and denominators $2 - 2v$ with $v = 0$ or $v = 2$. The only positive resonance is $\omega = 1/2$. For Ω_2, the possible numerators are $1 + 1$ and $-1 - 1$. The denominators are given by $2 + 2 - 2(v + \delta)$ where $v = 0, 1, 2$ and $\delta = 0, 1, 2$ with $v \ne \delta$. Therefore $v + \delta$ can take on the values 1, 2, and 3, and consequently the allowed denominators are ± 2. The only frequency generated is $\omega = 1$. Therefore

$$\Omega_1 = \{\tfrac{1}{2}\},$$
$$\Omega_2 = \{1\}.$$

This agrees with the classical result for the Mathieu equation, which is that the $O(\varepsilon^n)$ resonance is $n/2$.

Example 2

$$\ddot{x} + \omega^2 x + \varepsilon x \cos(t) + \varepsilon x^3 = 0$$

Then

$$N_1 = 2, \qquad M_1 = \{-1, 1\}.$$
$$N_2 = 4, \qquad M_2 = \{0\}.$$

Ω_1 is determined exactly as in the previous example since the set M_2 cannot contribute a non-zero frequency. [The $O(\varepsilon)$ resonances can always be found by considering each term of the perturbation separately]. For Ω_2, the resonance $\omega = 1$ is generated as in the previous example. The "mixing" of the sets M_1 and M_2 introduces the possible numerators $\pm 1 + 0$, with corresponding denominators $2 + 4 - 2(v + \delta)$ where $v = 0, 1, 2$ and $\delta = 0, 1, 2, 3, 4$. The forbidden pairs are $(v, \delta) = (0, 0)$, $(v, \delta) = (1, 2)$, and $(v, \delta) = (2, 4)$. The allowed values of $v + \delta$ are 1, 2, 3, 4, and 5, giving allowed denominators ± 2 and ± 4, so the new resonance is $\omega = 1/4$:

$$\Omega_1 = \{\tfrac{1}{2}\},$$
$$\Omega_2 = \{1, \tfrac{1}{4}\}.$$

Example 3

$$\ddot{x} + \omega^2 x + \varepsilon x^n \cos(t) = 0, \qquad n \text{ odd}.$$

Here

$$N_1 = n+1, \qquad M_1 = \{-1, 1\}.$$

The resonant frequencies are

$$\Omega_1 = \left\{\frac{1}{n+1}, \frac{1}{n-1}, \frac{1}{n-3}, \ldots, \frac{1}{2}\right\},$$

$$\Omega_2 = \left\{\frac{1}{n}, \frac{1}{n-2}, \frac{1}{n-4}, \ldots, 1\right\}.$$

Example 4

$$\ddot{x} + \omega^2 x + \varepsilon x^n \cos(t) = 0, \qquad n \text{ even}.$$

For this example,

$$N_1 = n+1, \qquad M_1 = \{-1, 1\}.$$

The resonant frequencies are

$$\Omega_1 = \left\{\frac{1}{n+1}, \frac{1}{n-1}, \frac{1}{n-3}, \ldots, 1\right\},$$

$$\Omega_2 = \left\{\frac{1}{n}, \frac{1}{n-2}, \frac{1}{n-4}, \ldots, \frac{1}{2}\right\}.$$

Example 5

$$\ddot{x} + \omega^2 x + \varepsilon x(\cos t + \cos 5t) + \varepsilon x^2(1 + \cos 3t + \cos 7t) = 0$$

Then

$$N_1 = 2 \qquad M_1 = \{-1, 1, -5, 5\}$$

$$N_2 = 3 \qquad M_2 = \{0, -3, 3, -7, 7\}.$$

The resonant frequencies are

$$\Omega_1 = \{\tfrac{1}{2}, 1, \tfrac{7}{3}, \tfrac{5}{2}, 3, 7\},$$

$$\Omega_2 = \{\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{3}, \tfrac{3}{2}, \tfrac{5}{3}, \tfrac{7}{4}, 2, \tfrac{8}{3}, \tfrac{7}{2}, 4, 5, 6, 8, 12\}.$$

Example 6

$$\ddot{x} + \omega^2 x + \varepsilon x \cos t + \varepsilon x^2 \cos 5t + \varepsilon x^3 \cos 12t + \varepsilon x^4(1 + \cos 22t) = 0$$

Here

$$N_1 = 2 \qquad M_1 = \{-1, 1\}$$

$$N_2 = 3 \qquad M_2 = \{-5, 5\}$$

$$N_3 = 4 \qquad M_3 = \{-12, 12\}$$

$$N_4 = 5 \qquad M_4 = \{0, -22, 22\}.$$

The $O(\varepsilon)$ resonant frequencies are

$$\Omega_1 = \{\tfrac{1}{2}, \tfrac{5}{3}, 3, \tfrac{22}{5}, 5, 6, \tfrac{22}{3}, 22\}.$$

The $O(\varepsilon^2)$ resonances are

$$\Omega_2 = \{34, 23, 21, 17, \tfrac{27}{2}, 12, \tfrac{34}{3}, 11, 10, \tfrac{17}{2}, \tfrac{23}{3}, 7, \tfrac{34}{5},$$
$$\tfrac{27}{4}, \tfrac{13}{2}, \tfrac{17}{3}, \tfrac{11}{2}, \tfrac{34}{7}, \tfrac{23}{5}, \tfrac{9}{2}, \tfrac{17}{4}, \tfrac{21}{5}, 4, \tfrac{11}{3}, \tfrac{17}{5}, \tfrac{10}{3},$$
$$\tfrac{13}{4}, \tfrac{17}{6}, \tfrac{11}{4}, \tfrac{5}{2}, \tfrac{12}{5}, \tfrac{7}{3}, 2, \tfrac{12}{7}, \tfrac{10}{7}, \tfrac{7}{5}, \tfrac{4}{3}, \tfrac{5}{4}, 1, \tfrac{5}{6}, \tfrac{1}{3}, \tfrac{1}{5}\}.$$

5. THE STABILITY OF THE TRIVIAL SOLUTION NEAR RESONANCE

Having identified the resonant frequencies, we now study the behavior of solutions close to resonance. We first study the major qualitative difference between linear and non-linear parametric excitation.

We consider equations of the form

$$\ddot{x}+\omega^2 x+\varepsilon f(t, x)=0 \tag{15}$$

where $f(t, x)$ is periodic in t and strictly non-linear in x. The case when $f(t, x)$ contains terms which are linear in x with periodic coefficients has been studied previously [5].

Let ω_0 be a resonance, and take ω in equation (15) to be

$$\omega^2=\omega_0^2+\varepsilon\omega_1+\varepsilon^2\omega_2+\ \ldots$$

Then equation (15) becomes

$$\ddot{x}+\omega_0^2 x+\varepsilon f(t, x)+(\varepsilon\omega_1+\varepsilon^2\omega_2+\ \ldots\ )x=0.$$

Detuning from resonance introduces a linear t-independent perturbation. Since detuning at $O(\varepsilon^n)$ introduces a term of the form $\varepsilon^n J$ to H, it also contributes a term to K_n which is independent of θ and *linear* in J. Since a non-linear term of order $O(x^m)$ in $f(t, x)$ contributes terms of order $O(J^{(m+1)/2})$ to K_1 and terms of higher order to subsequent K_i, the stabilizing effect of the detuning will dominate in a sufficiently small neighborhood of the origin. (This analysis requires that ε be held fixed, while J may be taken as small as necessary. For sufficiently small J the linear term dominates.) This implies that a strictly non-linear periodic perturbation cannot cause the trivial solution to be unstable away from resonance.

6. THE EFFECT OF DETUNING FROM RESONANCE

We demonstrate the effect of detuning from resonance on the equation

$$\ddot{x}+\omega^2 x+\varepsilon x^3\cos t=0. \tag{16}$$

The resonances, as shown in a previous section, are

$$\Omega_1=\{\tfrac{1}{2}, \tfrac{1}{4}\},$$

$$\Omega_2=\{1, \tfrac{1}{3}\}.$$

The MACSYMA implementation of the Lie transform algorithm, which is listed in the Appendix, shows that for $\omega^2=\frac{1}{4}+\varepsilon\omega_1+\varepsilon^2\omega_2$ the $O(\varepsilon^2)$ the reduced hamiltonian is

$$\begin{aligned} K= & -\tfrac{3}{2}\varepsilon^2 J^3\cos 4\theta+4\omega_1\varepsilon^2 J^2\cos 2\theta-\varepsilon J^2\cos 2\theta \\ & -\tfrac{4}{3}\varepsilon^2 J^3+\omega_2\varepsilon^2 J-\omega_1^2\varepsilon^2 J+\omega_1\varepsilon J. \end{aligned} \tag{17}$$

The fixed points satisfy

$$\frac{\partial K}{\partial J}=0,$$

$$\frac{\partial K}{\partial \theta}=0.$$

Solving for fixed points gives the $O(1)$ pairs of fixed points

$$J=-\frac{\omega_1}{2}-\frac{7\omega_1^2+8\omega_2}{16}\varepsilon+\frac{7\omega_1^3+8\omega_1\omega_2}{64}\varepsilon^2+\ \ldots$$

$$\theta=\frac{\pi}{2}, \frac{3\pi}{2}$$

for $\omega_1<0$ and

$$J=\frac{\omega_1}{2}+\frac{7\omega_1^2+8\omega_2}{16}\varepsilon-\frac{7\omega_1^3+8\omega_1\omega_2}{64}\varepsilon^2+\ \ldots$$

$$\theta=0, \pi$$

for $\omega_1>0$. [Solutions which are $O(1/\varepsilon)$ also exist, but we ignore them since we are interested in the behavior of the trivial solution a neighborhood of the origin. These fixed points indicate the presence of elliptic periodic orbits contained in the homoclinic loops of the $O(1)$ fixed points.]

We now classify the non-trivial fixed points by studying the hamiltonian in a neighborhood of the fixed points. Below resonance, for the fixed points at $\theta=\pi/2$ and $\theta=3\pi/2$, the hamiltonian is

$$K=\frac{\varepsilon}{16}((24\omega_2\varepsilon+17\omega_1^2\varepsilon+24\omega_1)\cos 2\theta-8\omega_2\varepsilon-19\omega_1^2\varepsilon-8\omega_1)J.$$

Above resonance, for the fixed points $\theta=0$ and $\theta=\pi$, the hamiltonian is

$$K=-\frac{\varepsilon}{16}((24\omega_2\varepsilon+17\omega_1^2\varepsilon+24\omega_1)\cos 2\theta+8\omega_2\varepsilon+19\omega_1^2\varepsilon+8\omega_1)J.$$

Both translated hamiltonians represent saddle points. As $\omega\to\frac{1}{2}^-$ the saddle points move in toward the origin along the lines $\cos 2\theta=-1$. At $\omega=\frac{1}{2}$, the origin is saddle-like. As ω increases from 1/2 the saddle points move out from the origin on the lines $\cos 2\theta=1$. Figure 1 shows Poincaré maps below, at, and above resonance. The Poincaré maps were generated by integrating the second-order equation equation (16). Figure 2 shows the level curves of the reduced hamiltonian (17), plotted on the same scale as the numerically generated Poincaré maps.

7. A BIFURCATION BETWEEN RESONANCES

Finally, we consider a bifurcation between two $O(\varepsilon)$ resonances. We will use $O(\varepsilon)$ Lie transforms to study how the Poincaré map changes as the amplitudes of two perturbations change. The equation to study is

$$\ddot{x}+(1+\varepsilon\omega_1)x+\varepsilon(sx^2\cos(t)+(1-s)x^3\cos(2t))=0$$

for $0\leq s\leq 1$. When $s=0$, the quadratic term is absent and the cubic term is resonant. When $s=1$, the quadratic term is resonant and the cubic term is absent. When $0<s<1$ both terms are resonant. The interaction of the two resonances is of interest.

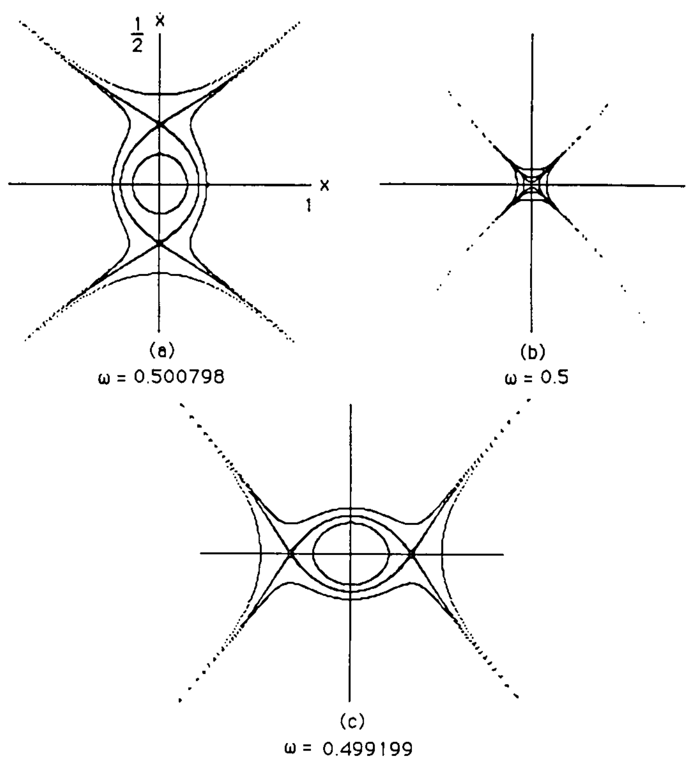

Fig. 1. Numerically integrated Poincaré maps (Σ: $t=0$ mod 2π) of equation (16).

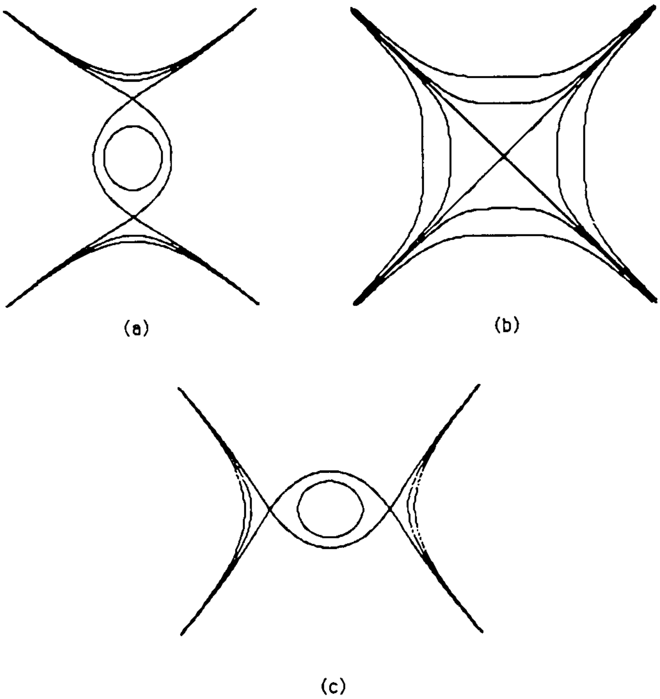

Fig. 2. Level curves of the reduced $O(\varepsilon^2)$ hamiltonian (17) constructed by Lie transforms.

Without loss of generality, set $\omega_1 = 1$. The resulting reduced hamiltonian is

$$K_1 = \frac{1}{2} J + \frac{\sqrt{2}s}{4} J^{3/2} \sin\theta - \frac{1-s}{4} J^2 \cos 2\theta. \tag{18}$$

Fixed points satisfy

$$\frac{\partial K_1}{\partial J} = 0,$$

$$\frac{\partial K_1}{\partial \theta} = 0$$

which give the fixed points

$$\theta = \frac{3\pi}{2}, \qquad \sqrt{J} = \frac{\sqrt{2}}{8(1-s)}\left(3s \pm \sqrt{9s^2 + 32s - 32}\right) \tag{19}$$

and

$$\sin\theta = \frac{\sqrt{2}s}{4(s-1)\sqrt{J}}, \qquad J = \frac{8 - 8s - s^2}{8(1-s)^2}. \tag{20}$$

The requirement that the radicand of equation (19) be non-negative restricts s to the interval $0.8138 \le s \le 1$. The requirement that $|\sin\theta| \le 1$ in equation (20) restricts s to $0 \le s \le 0.828427$.

We now classify the stability of the fixed points. The stability of the critical point is characterized by the sign of the determinant of the Hessian $\mathbf{H}h$ evaluated at the fixed point. A rather lengthy computation (cf. [1]) shows that

$$\det(\mathbf{H}h(\mathbf{x}_0)) = h_{qq}h_{pp} - (h_{qp})^2|_0$$
$$= h_{JJ}h_{\theta\theta} - (h_{J\theta})^2|_0.$$

The critical point is a saddle if

$$h_{JJ}h_{\theta\theta}-(h_{J\theta})^2|_0<0$$

and is a center if

$$h_{JJ}h_{\theta\theta}-(h_{J\theta})^2|_0>0.$$

For the fixed points satisfying $\theta=3\pi/2$,

$$(h_{JJ}h_{\theta\theta}-h_{J\theta}^2)|_0<0$$

gives the stability criterion

$$-\frac{J}{32}(16(1-s)^2J-10\sqrt{2}s(1-s)\sqrt{J}+3s^2)>0$$

where, from equation (19),

$$\sqrt{J}=\frac{\sqrt{2}}{8(1-s)}(3s\pm\sqrt{9s^2+32s-32}).$$

Substituting this into the inequality shows, after some algebra, that the limits of stable s are roots of the polynomial equation

$$s^4+\frac{440}{21}s^3+\frac{463}{9}s^2-\frac{8576}{63}s+\frac{4096}{63}=0.$$

The roots in the interval for which the fixed points exist are found to be $s=0.818337$ (for the $+$ root) and $s=0.88871$ (for the $-$ root). The stabilities of the fixed points for various s are listed in a table below.

For the other fixed points, the stability criterion is

$$-\frac{(1-s)^2J^2}{2}-\frac{(1-s)^2J^2}{2}\sin^2(2\theta)>0$$

where

$$J=\frac{1}{1-s}-\frac{s^2}{8(1-s)^2}$$

and

$$\sqrt{J}\sin\theta=\frac{s\sqrt{2}}{4(s-1)}.$$

Inserting these relations gives the stability criterion

$$\frac{(s^2+4s-4)(s^2+8s-8)}{64(1-s)^2}<0.$$

Since these fixed points exist for $0\le s\le 0.828427$ and the inequality is not satisfied on this interval, these fixed points are always saddles. The behavior for various s is summarized below:

- At $s=0$, saddles exist at $\sqrt{J}=1$, $\theta=0$ and $\sqrt{J}=1$, $\theta=\pi$.
- As s increases, the saddles move into the left side of the plane and toward the horizontal axis.
- At $s=0.8138$, a center appears at $\sqrt{J}=2.31718$, $\theta=3\pi/2$.
- As s increases, the centers separate, remaining on the horizontal axis.
- At $s=0.818337$ the center farthest from the origin on the horizontal axis becomes a saddle.
- At $s=2\sqrt{2}-2$ the two saddles and the inner center coalesce and form a center.
- At $s=0.88871$ the inner center becomes a saddle.
- As $s\to 1$ the inner saddle moves to $\sqrt{J}=\sqrt{8}/3$ and the outer one moves off to infinity.

Figure 3 shows the transitions for various values of s.

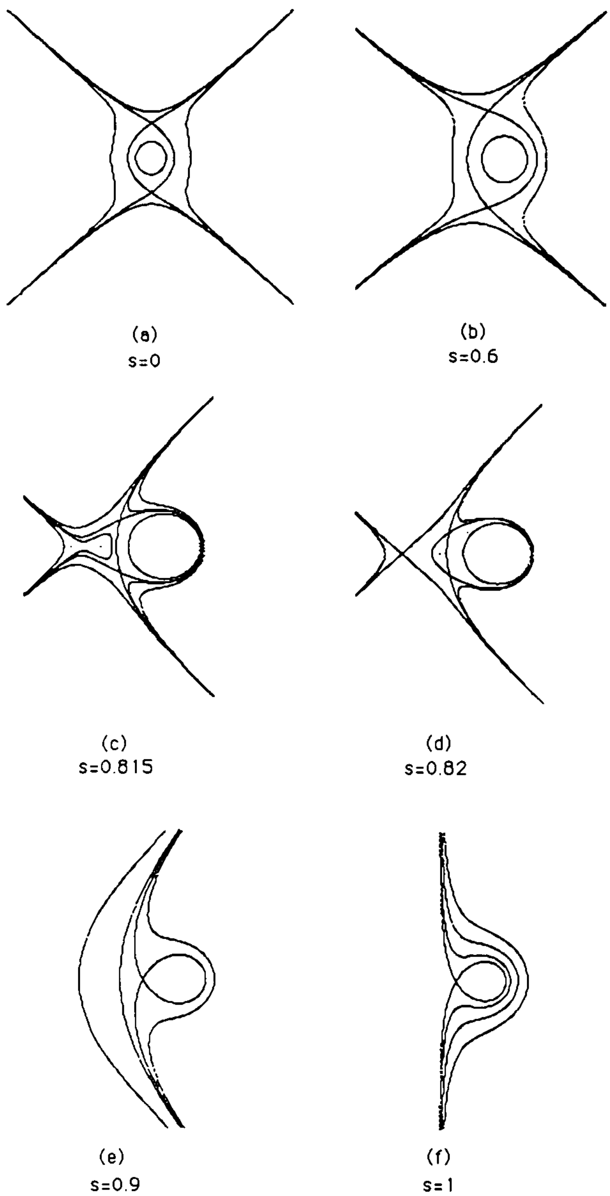

Fig. 3. Level curves of the reduced hamiltonian (18) for indicated values of s.

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APPENDIX

The following MACSYMA program computes the lie transform of a weak perturbation of the simple harmonic oscillator.

```
/* Program to compute the lie transform near a resonance.*/
/*  If detuning is requested, the program will supply it */
/*in the form dw[i]*e^i. */
lie():=block
(
 kill(y,dw,n,j,dotran),
 assume(j>0),
 maperror:false,
 print(timedate()),
 trunc:read("Truncation order:"),
 om:read("Frequency"),
 f:read("Perturbation (use x, e, and t):"),
 detoon:read("Detune from resonance [y/n] ?"),
  if detoon = y then f:f+sum(e^i * dw[i],i,1,trunc)*x,
 dotran:read("Compute the co-ordinate
                             transformation [y/n] ?"),
 print("Equation to work with:"),
 print('diff(x,t,2) + om^2*x + f = 0),

/* Construct the Hamiltonian in complex */
/* slow-flow co-ordinates. */

 hh:map(pseudo_int_x,expand(exponentialize(f))),

/* Do the canonical change of co-ordinates to */
/* slow action-angle variables. */

 hh:ev(hh,x=%e^(%i*om*t)*q/(2*%i*om) - %e^(-%i*om*t)*p),
 hh:ev(hh,q=sqrt(2*j*om)*%e^(%i*th),
                 p=sqrt(2*j*om)/(2*%i*om)*%e^(-%i*th)),

/* Now taylor expand hh to order trunc */
/* and assign h[i] values. */
 tmp: expand(taylor(hh, e, 0, trunc)),
 for i from 0 thru trunc do
 (
  h[i] : coeff(tmp,e,i)
 ),


 /* Initialize the new hamiltonian. */

  k[0] : h[0],

 /* This loop does the transforms. */

  for n from 1 thru trunc do
  (
   print("Loop # ",n, "of ",trunc),
   temp: h[n] + sum(poisson(w[n-m],k[m]), m, 1, n-1)/n,
   temp: expand(temp + sum(m*inverse_evolution
                            (n-m, h[m]), m, 1, n-1)/n),
 /* We don't need w[trunc] unless we are going to */
 /* compute the net transformation. */
   if (dotran = y or n < trunc) then w[n]: getw(n,temp),
 /* Cheat here.  w[n] was chosen to make k[n] the */
 /* t-independent part of temp. */
   k[n]: map(nuke_t,temp)
  ),
 /* The result is in new action-angle variables. */
 /* Tell me what we got. */

  print(""),
  kk:sum(k[i]*e^i,i,0,trunc),
  kk:expand(rat(kk)),

 /* Tell me all about the reduced hamiltonian. */
```

```
 print("The reduced hamiltonian in transformed
                              action-angle variables:"),
 realkk: expand(rat(realpart(kk))),
 print(realkk),

/* if requested, compute the co-ordinate transformation. */
 if dotran = y then
  block
   (

/* use the inverse evolution operator give the relation */

/* between old and new. */

   physical_j:sum(e^i*inverse_evolution(i,j),i,0,trunc),
   physical_th:sum(e^i*inverse_evolution(i,th),i,0,trunc),

   physical_j:expand(realpart(physical_j)),
   physical_th:expand(realpart(physical_th)),

/* Now tell me how big the transformation is: */

   print(""),
   print("The co-ordinate transformation has been
                                          computed."),
   print("length(physical_j)=",length(physical_j)),
   print("length(physical_th)=",length(physical_th))
   )
  else
   print("You told me not to compute the
                         co-ordinate transformation."),

/* Finished.  */

)$

/* Function to look like integration in x. */
/* This function is mapped. */
pseudo_int_x(f):=
(
 f*x/(hipow(f,x) + 1)
)$

/* function to compute poisson brackets in (th,j) space. */
poisson(f,g):=
(
 diff(f,th) * diff(g,j) - diff(f,j) * diff(g,th)
)$

/* function to nuke t-independent stuff. */
/* this function is mapped. */
nuke_no_t(f):=
(
 if freeof(t,f) then
  0
 else
  f
)$

/* function to nuke t-dependent stuff. */
/* this function is mapped. */
nuke_t(f):=
(
 if freeof(t,f) then
  f
 else
  0
)$

/* Function to compute generating function to */
/* nuke t-dependent terms.  This function is not mapped. */
getw(n,f):=
```

```
(
 [tmp],
 tmp:expand(-n*map(nuke_no_t,f)),
/* Factor the exponents in case an unspecified */
/* omega is given.  Note: lambda returns a list. */
 tmp:map(lambda([u],scanmap(factor,[u])),tmp),
 tmp:part(tmp,1),
 map(innegrate,tmp)
)$


/* Function to look like integration of complex */
/* exponential, hence the name. */
/* This function is mapped. */
 innegrate(f):=
(
 [nn,mm,tmp,z],
 matchdeclare([nn,mm],freeof(t)),


/* Define the pattern-matching rules for sines */
/* and cosines.  Note that the rules do not commute,*/
/* and ins must be performed before inc. */


 defrule(ins, sin(nn+mm*t),-cos(nn+mm*z)/mm),
 defrule(inc, cos(nn+mm*t),sin(nn+mm*t)/mm),
 tmp:expand(demoivre(f)),
 tmp:expand(applyb1(tmp,ins)),
 tmp:applyb1(map(nuke_no_t,tmp),inc)+map(nuke_t,tmp),
 tmp:ev(tmp,z=t),
 tmp:expand(exponentialize(tmp))
)$



/* Recursive function to compute kth term of */
/* inverse of evolution operator acting on h. */
inverse_evolution(k,h) :=
(
 if k = 0 then h
 else
 sum(poisson(w[k-m],inverse_evolution(m,h)), m, 0, k-1)/k
)$


/* Recursive function to compute kth term of evolution */
/* operator acting on h. Note that this function is not */
/* used by the program.  */
evolution(k,h) :=
(
 if k = 0 then h
 else
 -sum(evolution(m,poisson(w[k-m],h)),m,0,k-1)/k
)$
```

The following examples were run with the MACSYMA option "SHOW-TIME:ALL" on a VAX 8500.

```
(c4) lie()$
Wed Jun 10 15:59:17 1987


Truncation order:
1;
Frequency
1/2;
Perturbation (use x, e, and t):
e*x^3*cos(t);
Detune from resonance [y/n] ?
y;
```

```
Compute the co-ordinate transformation [y/n] ?
y;

Equation to work with:

 2
d x             3             x
--- + e cos(t) x  + dw  e x + - = 0
  2                   1       4
dt

Loop #  1 of  1



The reduced hamiltonian in transformed action-angle
                                              variables:

              2
dw  e j - e j  cos(2 th)
  1

The co-ordinate transformation has been computed.

length(physical_j)= 5

length(physical_th)= 6

Totaltime= 52500 msec.  GCtime= 20116 msec.
```

Next, a run with a symbolic frequency to show how the resonant frequencies may be computed:

```
(c7) lie()$
Wed Jun 10 16:01:32 1987

Truncation order:
1;
Frequency
omega;
Perturbation (use x, e, and t):
e*x^3*cos(t);
Detune from resonance [y/n] ?
n;
Compute the co-ordinate transformation [y/n] ?
y;

Equation to work with:

 2
d x             3        2
--- + e cos(t) x  + omega  x = 0
  2
dt

Loop #  1 of  1



The reduced hamiltonian in transformed action-angle
                                              variables:

0



The co-ordinate transformation has been computed.

length(physical_j)= 5

length(physical_th)= 6

Totaltime= 72200 msec.  GCtime= 26500 msec.
```

```
(c8) factor(denom(rat(ev(w[1],t=0))));
Totaltime= 3250 msec.  GCtime= 1266 msec.

                 2
(d8) 32 omega  (2 omega - 1) (2 omega + 1)
                    (4 omega - 1) (4 omega + 1)
```

The poles of the generating function at $O(\epsilon)$ are $\omega = 1/2$ and $\omega = 1/4$.