

set (p. 129). On a cylinder it has, in general, two exterior regions (see the figure), and its boundary is not uniquely determined (p. 138), in contrast to the planar case where the unique hull has a single exterior face. Delaunay triangulations—duals of Voronoi diagrams—have proven extremely useful in the plane, but the dual of a Voronoi diagram on a cylinder, for example, is not necessarily a triangulation (p. 143). Therefore the authors pursue one of the more useful properties enjoyed by planar Delaunay triangulations—they maximize the minimum angle over all triangulations—and show that such an optimal triangulation on the cylinder can be computed in $O(n \log n)$ time, just as in the plane.

Delaunay triangulations can be constructed in the plane by “flipping” diagonals of convex quadrilateral to increase the angles, because the *flip graph* is connected. The authors prove that the flip graph can be disconnected on a cylinder (p. 154), but only because of the nonuniqueness of the boundary of a triangulation. With that fixed, the flip graph is connected (p. 157). Triangulations of the torus (following the maximal-set-of-segments definition) include a face that is not a triangle if some quadrant is devoid of points (p. 160). But this turns out to be the only impediment to connectivity: the graph of triangulations is either empty or connected. These connectivity results are, however, special in the sense that every surface admits a metric in which, for some polygons, the triangulation graph is disconnected (p. 170).

The next frontiers, not broached in this book, include generalization to arbitrary 2-manifolds (an area of active current research), and non-Euclidean 3-manifolds, as yet relatively unexplored.

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Analytical Mechanics. By A. I. Lurie. Springer-Verlag, New York, 2002. \$139.00. 864 pp., hardcover. ISBN 3-540-42982-4.

This book is a translation of a book originally printed in Russian in 1961. It offers a treatment of the dynamics of mechanical systems. The topics treated may be classified as follows:

(a) deriving the equations of motion (9 chapters, 504 pages),

(b) obtaining exact solutions by Hamilton–Jacobi theory (1 chapter, 62 pages),

(c) obtaining approximate solutions by perturbation theory (1 chapter, 84 pages),

(d) variational principles (1 chapter, 112 pages).

There are also 67 pages of appendices treating matrix algebra and tensor calculus.

Topic (a), deriving the equations of motion, constitutes the major portion of this book and, in my opinion, is the most interesting topic covered. The treatment involves a very careful examination of the assumptions and techniques by which one may obtain the equations of motion of mechanical dynamical systems, that is, systems involving particles and rigid bodies, with special emphasis on nonholonomic constraints. For example, the concept of a virtual displacement (denoted by the variation symbol δ) is contrasted with ordinary differentiation (denoted by the differential d), a topic that is often found confusing by students. It is shown that the operations of differentiation and variation commute when applied to generalized coordinates q_i .

In the case of an n -degree-of-freedom system with m nonholonomic constraints, the author contrasts the usual approach of using Lagrange’s equations (including Lagrange multipliers) with another scheme that is “simpler in form and more symmetrical than Lagrange’s equations for many classes of dynamical problems.” These equations

are called the “Euler–Lagrange equations” and, according to Lurie, were first derived by Boltzmann in 1902 and Hamel in 1904, and were named by Hamel. Since in English the “Euler–Lagrange equations” commonly refer to the conditions for a stationary value of a functional in the calculus of variations, to avoid confusion I shall refer to the equations presented in Lurie’s book as the BHL equations (for Boltzmann–Hamel–Lurie).

A brief description of these BHL equations of motion will allow the reader to compare them with other more familiar methods for deriving equations of motion. The key idea is to replace the generalized velocities \dot{q}_i by affine combinations of them, called quasi-velocities ω_s :

$$(1) \quad \omega_s = \sum_{i=1}^n a_{si} \dot{q}_i + a_{s,n+1} \quad (s = 1, \dots, n).$$

If there are m nonholonomic constraints, then each of m of the quasi-velocities is chosen to equal a nonholonomic constraint equation, so that the constraints become simply $\omega_s = 0$ for $s = 1, 2, \dots, m$.

Once the quasi-velocities have been chosen, the procedure for obtaining the BHL equations of motion begins with the computation of what Lurie refers to as “Boltzmann three-index symbols,” γ_{ism}^r . Although these are defined in terms of the combination coefficients a_{si} of (1), we are shown by Lurie how to compute them more easily from the equation

$$(2) \quad (\delta\pi_s)^* - \delta\omega_s = \sum_{i=1}^n \sum_{m=1}^n \gamma_{ism}^s \omega_i \delta\pi_m,$$

where π_s is called a quasi-coordinate, and its variation is given by

$$(3) \quad \delta\pi_s = \sum_{i=1}^n a_{si} \delta q_i.$$

The bullet \bullet in (2) represents differentiation with respect to time t (Lurie’s notation). Thus to obtain the three-index symbols we take the time derivative of (3) and subtract from it the variation of (1). By collecting the resulting terms we may identify the coefficients γ_{ism}^r . Incidentally, the fact that the right-hand side of (2) does not in general

vanish shows that the operations of differentiation and variation do *not* commute when applied to quasi-coordinates. It may be said that the BHL equations are based upon this property.

Once the three-index symbols have been found, we must next compute the kinetic and potential energies T, V of the system. This step is common to Lagrange’s equations. Once these have been expressed in terms of the quasi-velocities, we may obtain the BHL equations directly from the formula

$$(4) \quad \frac{d}{dt} \frac{\partial T}{\partial \omega_s} + \sum_{r=1}^n \sum_{i=1}^n \gamma_{irs}^r \frac{\partial T}{\partial \omega_r} \omega_i - \frac{\partial T}{\partial \pi_s} = - \frac{\partial V}{\partial \pi_s} \quad (s = m + 1, \dots, n).$$

Equations (4) are $n - m$ first-order ODEs, which together with the n equations (1) (which may be solved for the \dot{q}_i) give a total of $2n - m$ first-order ODEs.

A strength of the book is the large number of paradigm examples that are presented in detail. These both illustrate the BHL method and provide a convenient reference for the resulting equations of motion. The examples include the following systems:

- A general-shaped body rolling on a plane, including the special cases of a sphere and a ring.
- Cardan’s suspension (model of a gyrocompass), including effects due to the Earth’s rotation.
- A “two-axle trolley,” being a vehicle with four wheels such that the front axle is permitted to rotate about a vertical axis relative to the rear axle. This system involves $n = 8$ generalized coordinates related by $m = 6$ nonholonomic constraints.

In addition to the BHL equations of motion, the book also treats Lagrange’s equations and Appell’s equations (often called the Gibbs–Appell equations in English). The latter, first presented in 1899, utilizes the “energy of accelerations” S :

$$(5) \quad S = \frac{1}{2} \sum_{i=1}^N m_i (\ddot{x}_i^2 + \ddot{y}_i^2 + \ddot{z}_i^2),$$

where the sum is over N particles. Appell's equations are obtained by equating $\frac{\partial S}{\partial \dot{q}_i}$ to an appropriate generalized force.

A serious criticism of the book is that it is dated. Having been written in 1961, it is missing numerous relevant topics that have been invented since then. For example, Lurie's book would have been a good place to compare the methods of Lagrange, BHL, and Appell with Kane's equations. These were first presented in 1961 in a paper by T. R. Kane, and, like the other methods, offer a scheme for dealing with systems having nonholonomic constraints. (See [1] for a thorough treatment of Kane's equations, or [2] for a brief summary.)

Other topics that are missing include the following:

- Chaos, and especially KAM theory. These topics provide an important explanation of why the Hamilton–Jacobi PDE, which is discussed extensively in Chapter 10 of Lurie's book, cannot be solved by separation of variables in most problems. The reason is that if separation of variables works, then the problem is integrable and the phase space is foliated into invariant tori. Such a situation is nongeneric and cannot support chaos. (See [3] for a treatment of KAM theory.)
- Lie transforms, being a perturbation method that utilizes the Hamiltonian nature of the problem and that is superior to the methods presented in Lurie's Chapter 11. (See [2] for an introduction to Lie transforms.)
- Exterior calculus and differential forms. These algebraic devices offer a convenient condition for a transformation to be canonical, that is, to preserve the Hamiltonian structure of the problem. (See [4] for an introduction to differential forms.)

A surprising feature of Lurie's book is the frequent occurrence of tensor calculus, a topic that is usually absent from comparable texts, such as [5]. These occur, for example, in the explicit form of Lagrange's equations:

$$(6) \quad \ddot{q}_s + \sum_{k=1}^n \sum_{m=1}^n \left\{ \begin{matrix} s \\ km \end{matrix} \right\} \dot{q}_k \dot{q}_m = F,$$

where $\left\{ \begin{matrix} s \\ km \end{matrix} \right\}$ is Christoffel's symbol of the second kind based on a metric derived from the quadratic generalized velocity terms in the kinetic energy, and where F is a known function of the generalized forces. An appendix provides a nice summary of tensor calculus.

The final chapter of the book deals with variational principles. It includes a discussion of both Hamilton's principle and the principle of least action, and, by treating the second variation, addresses the question of whether these principles give a minimum. Applications to distributed systems include the vibrations of a hanging chain and of a rotating elastic rod.

In summary, Lurie's *Analytical Mechanics* would be a useful reference book for a person who is interested in engineering dynamics. It could also serve as a textbook for a graduate course in engineering dynamics, especially if it were supplemented with material covering topics that have become important since the book was written.

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