



DYNAMICS OF A NONLINEAR PARAMETRICALLY-EXCITED PDE: 2-TERM TRUNCATION

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Introduction

Parametrically-excited ode's have found numerous applications in mechanics (e.g. dynamic buckling, stability of motion, and water waves in a vertically forced channel.) Recent research work involving a line of coupled pendula with vertical forcing [1],[2] has led to the investigation of the dynamics of a parametrically-excited *pde*. Although the theory of parametrically-excited ode's is well known [3],[4], the comparable treatment of pde's has received little attention. In this paper we study a two-term truncation of a parametrically-excited pde, and we show by using averaging that the truncated system exhibits a rich diversity of dynamical behavior. The pde itself is expected to be far more complicated than the two-term approximation studied here.

The subject of this paper is the pde:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + \epsilon \beta \frac{\partial u}{\partial t} + [\delta + \epsilon \gamma \cos t] u = \epsilon \alpha u^3 \quad (1)$$

with the boundary conditions

$$\frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \pi \quad (2)$$

This system is a generalization of Mathieu's equation,

$$\frac{d^2 u}{dt^2} + [\delta + \epsilon \cos t] u = 0 \quad (3)$$

to which nonlinear ($\epsilon \alpha u^3$), damping ($\epsilon \beta \partial u / \partial t$), and diffusive ($c^2 \partial^2 u / \partial x^2$) terms have been added. Equations (1),(2) represent a paradigm system for parametrically-excited pde's. They have been shown to model a line of coupled pendula with vertical forcing [1].

The present paper is an extension of the paper [2] by Rand et al. which presented a preliminary investigation of (1),(2). By using numerical integration on a 3-term truncation,

$$u = f_0(t) + f_1(t) \cos x + f_2(t) \cos 2x \quad (4)$$

Rand et al. [2] showed that for given values of the parameters $\alpha, \beta, \gamma, \epsilon, \delta, c$, the asymptotic behavior of (1),(2) depended upon the initial conditions. The following types of stable steady states were observed: (i) the trivial solution $u = 0$, (ii) one of the modes $\{1, \cos x, \cos 2x\}$, varying periodically in time, (iii) combinations of these spatial modes, called "multimodes", varying periodically or quasiperiodically in time, and (iv) unbounded growth. In an attempt to explain this behavior, perturbation theory valid for small ϵ was used in [2] to show that each of the modes $\{1, \cos x, \cos 2x\}$ was predicted to be a stable periodic steady state for particular ranges of parameter values. These ranges overlapped, and so for some parameter values the coexistence of different steady states was explained. No explanation was given, however, for the appearance in the numerical integrations of stable motions which involved more than one mode (multimodes). The paper [2] concluded with the optimistic comment "we hope for a perturbation analysis which yields the nature and bifurcation of the multi-mode steady states which we have observed numerically." It is the purpose of the present paper to offer such an analysis in the simpler case of a 2-term truncation:

$$u = f_0(t) + f_1(t) \cos x \quad (5)$$

Analysis

Substituting (5) into (1),(2), simplifying and collecting terms gives the ode's:

$$\ddot{f}_0 + \omega_0^2 f_0 + \epsilon(\beta \dot{f}_0 + \gamma f_0 \cos t) = \epsilon\alpha \left[f_0^3 + \frac{3}{2} f_0 f_1^2 \right] \quad (6)$$

$$\ddot{f}_1 + \omega_1^2 f_1 + \epsilon(\beta \dot{f}_1 + \gamma f_1 \cos t) = \epsilon\alpha \left[3f_0^2 f_1 + \frac{3}{4} f_1^3 \right] \quad (7)$$

where $\omega_0^2 = \delta$ and $\omega_1^2 = \delta + c^2$.

Eqs.(6),(7) represent a system of two coupled damped nonlinear Mathieu eqs. We are interested in the dynamics of (6),(7) for small values of ϵ . As is well-known [3],[4],[5], the major resonance in the uncoupled undamped linear Mathieu eq. (3) occurs when the forcing frequency ($= 1$) is twice the natural frequency ($= \omega_i$). In the uncoupled damped nonlinear Mathieu eq., the instability associated with this 2 : 1 resonance is balanced by the loss of energy due to damping resulting in a stable periodic motion [4],[6]. In order to investigate the nature of the dynamics when *both* oscillators (6),(7) are in 2 : 1 resonance with the driver $\cos t$ (as well as in 1 : 1 resonance with each other), we set

$$\delta = \frac{1}{4} + \delta_1 \epsilon, \quad c = c_1 \sqrt{\epsilon} \quad (8)$$

where δ_1 is a detuning coefficient and where the wave speed c has been scaled in order to make $\omega_1 = \frac{1}{2} + O(\epsilon)$.

An approximate solution to the resulting eqs. may be obtained for small ϵ by using the method of averaging [5],[7]. We set

$$f_i = a_i(t) \cos \frac{t}{2} + b_i(t) \sin \frac{t}{2}, \quad i = 0, 1 \tag{9}$$

and obtain eqs. on $a_i(t), b_i(t)$, which are simplified by averaging to give:

$$\begin{aligned} 8 \frac{da_0}{dt} &= -\epsilon (b_0 [4 \gamma - 8 \delta_1] \\ &\quad + \alpha [9 b_0 b_1^2 + 6 a_0 a_1 b_1 + 6 b_0^3 + 3 a_1^2 b_0 + 6 a_0^2 b_0]) \end{aligned} \tag{10}$$

$$\begin{aligned} 8 \frac{db_0}{dt} &= -\epsilon (a_0 [4 \gamma + 8 \delta_1] \\ &\quad - \alpha [3 a_0 b_1^2 + 6 a_1 b_0 b_1 + 6 a_0 b_0^2 + 9 a_0 a_1^2 + 6 a_0^3]) \end{aligned} \tag{11}$$

$$\begin{aligned} 16 \frac{da_1}{dt} &= -\epsilon (b_1 [8 \gamma - 16 \delta_1 - 16 c_1^2] \\ &\quad + \alpha [9 b_1^3 + 36 b_0^2 b_1 + 9 a_1^2 b_1 + 12 a_0^2 b_1 + 24 a_0 a_1 b_0]) \end{aligned} \tag{12}$$

$$\begin{aligned} 16 \frac{db_1}{dt} &= -\epsilon (a_1 [8 \gamma + 16 \delta_1 + 16 c_1^2] \\ &\quad - \alpha [9 a_1 b_1^2 + 24 a_0 b_0 b_1 + 12 a_1 b_0^2 + 9 a_1^3 + 36 a_0^2 a_1]) \end{aligned} \tag{13}$$

where we have set $\beta = 0$ in order to simplify the subsequent analysis.

Equilibria of eqs.(10)-(13) represent periodic motions of eqs.(6),(7) and are found as:

$$u = 0 \text{ mode : } \{ a_0 = 0, b_0 = 0, a_1 = 0, b_1 = 0 \} \tag{14}$$

$$a_0 \text{ mode : } \left\{ a_0^2 = \frac{2 (\gamma + 2 \delta_1)}{3 \alpha}, b_0 = 0, a_1 = 0, b_1 = 0 \right\} \tag{15}$$

$$b_0 \text{ mode : } \left\{ a_0 = 0, b_0^2 = -\frac{2 (\gamma - 2 \delta_1)}{3 \alpha}, a_1 = 0, b_1 = 0 \right\} \tag{16}$$

$$a_1 \text{ mode : } \left\{ a_0 = 0, b_0 = 0, a_1^2 = \frac{8 (\gamma + 2 \delta_1 + 2 c_1^2)}{9 \alpha}, b_1 = 0 \right\} \tag{17}$$

$$b_1 \text{ mode : } \left\{ a_0 = 0, b_0 = 0, a_1 = 0, b_1^2 = -\frac{8 (\gamma - 2 \delta_1 - 2 c_1^2)}{9 \alpha} \right\} \tag{18}$$

$$a_0 a_1 \text{ mode: } \left\{ a_0^2 = \frac{2(\gamma + 2\delta_1 + 4c_1^2)}{15\alpha}, b_0 = 0, a_1^2 = \frac{16(\gamma + 2\delta_1 - c_1^2)}{45\alpha}, b_1 = 0 \right\} \quad (19)$$

$$a_0 b_1 \text{ mode: } \left\{ a_0^2 = \frac{2(5\gamma + 2\delta_1 - 4c_1^2)}{3\alpha}, b_0 = 0, b_1^2 = -\frac{16(\gamma - c_1^2)}{3\alpha}, a_1 = 0 \right\} \quad (20)$$

$$b_0 a_1 \text{ mode: } \left\{ a_0 = 0, b_0^2 = \frac{2(-5\gamma + 2\delta_1 - 4c_1^2)}{3\alpha}, a_1^2 = \frac{16(\gamma + c_1^2)}{3\alpha}, b_1 = 0 \right\} \quad (21)$$

$$b_0 b_1 \text{ mode: } \left\{ a_0 = 0, b_0^2 = -\frac{2(\gamma - 2\delta_1 - 4c_1^2)}{15\alpha}, a_1 = 0, b_1^2 = -\frac{16(\gamma - 2\delta_1 + c_1^2)}{45\alpha} \right\} \quad (22)$$

From the requirement that a_i, b_i be real, we find that these modes exist in the following regions, respectively (assuming $\alpha > 0, \gamma > 0, \delta > 0, \epsilon > 0$ and $\beta = 0$):

$$u = 0 \text{ mode: always exists} \quad (23)$$

$$a_0 \text{ mode: } \delta > \frac{1}{4} - \epsilon \frac{\gamma}{2} \quad (24)$$

$$b_0 \text{ mode: } \delta > \frac{1}{4} + \epsilon \frac{\gamma}{2} \quad (25)$$

$$a_1 \text{ mode: } \delta > \frac{1}{4} - c^2 - \epsilon \frac{\gamma}{2} \quad (26)$$

$$b_1 \text{ mode: } \delta > \frac{1}{4} - c^2 + \epsilon \frac{\gamma}{2} \quad (27)$$

$$a_0 a_1 \text{ mode: } \delta > \frac{1}{4} + \frac{c^2}{2} - \epsilon \frac{\gamma}{2} \quad (28)$$

$$a_0 b_1 \text{ mode: } \delta > \frac{1}{4} + 2c^2 - \frac{5}{2}\epsilon\gamma \text{ and } c^2 > \epsilon\gamma \quad (29)$$

$$b_0 a_1 \text{ mode: } \delta > \frac{1}{4} + 2c^2 + \frac{5}{2}\epsilon\gamma \quad (30)$$

$$b_0 b_1 \text{ mode: } \delta > \frac{1}{4} + \frac{c^2}{2} + \epsilon \frac{\gamma}{2} \quad (31)$$

In order to determine the stability of these slow-flow equilibria, we linearize about each one and require the real part of at least one of the associated eigenvalues to be positive for an unstable motion. (Since β has been assumed to be zero, there is no dissipation and no asymptotic stability is possible.) In regions of the δ - c plane for which a given mode exists,

this procedure produces the following transition curves separating regions of stability from regions of instability, see Fig.1:

$$u = 0 \text{ mode : } \delta = \frac{1}{4} + \epsilon \frac{\gamma}{2}, \quad \delta = \frac{1}{4} - \epsilon \frac{\gamma}{2}, \quad \delta = \frac{1}{4} - c^2 + \epsilon \frac{\gamma}{2}, \quad \delta = \frac{1}{4} - c^2 - \epsilon \frac{\gamma}{2}, \quad (32)$$

$$a_0 \text{ mode : } \delta = \frac{1}{4} + \frac{c^2}{2} + \epsilon \frac{\gamma}{2} \quad \text{and} \quad c^2 = \epsilon \gamma \quad (33)$$

$$b_0 \text{ mode : } \text{always unstable} \quad (34)$$

$$a_1 \text{ mode : } \delta = \frac{1}{4} + 2c^2 + \frac{5}{2}\epsilon \gamma \quad (35)$$

$$b_1 \text{ mode : } \text{always unstable} \quad (36)$$

$$a_0 a_1 \text{ mode : } \delta = \frac{1}{4} + \frac{4}{7}(-c^2 + \epsilon \gamma), \quad \text{and} \quad (37)$$

$$\begin{aligned} & 33124 \epsilon^4 \gamma^4 + 112112 \delta \epsilon^3 \gamma^3 + 72384 c^2 \epsilon^3 \gamma^3 - 28028 \epsilon^3 \gamma^3 + 54096 \delta^2 \epsilon^2 \gamma^2 \\ & + 61344 c^2 \delta \epsilon^2 \gamma^2 - 27048 \delta \epsilon^2 \gamma^2 - 140096 c^4 \epsilon^2 \gamma^2 - 15336 c^2 \epsilon^2 \gamma^2 + 3381 \epsilon^2 \gamma^2 \\ & - 68992 \delta^3 \epsilon \gamma - 148032 c^2 \delta^2 \epsilon \gamma + 51744 \delta^2 \epsilon \gamma + 2176 c^4 \delta \epsilon \gamma + 74016 c^2 \delta \epsilon \gamma \\ & - 12936 \delta \epsilon \gamma + 44544 c^6 \epsilon \gamma - 544 c^4 \epsilon \gamma - 9252 c^2 \epsilon \gamma + 1078 \epsilon \gamma + 12544 \delta^4 \\ & + 37632 c^2 \delta^3 - 12544 \delta^3 + 3136 c^4 \delta^2 - 28224 c^2 \delta^2 + 4704 \delta^2 - 37632 c^6 \delta \\ & - 1568 c^4 \delta + 7056 c^2 \delta - 784 \delta + 12544 c^8 + 9408 c^6 + 196 c^4 - 588 c^2 + 49 = 0 \end{aligned}$$

$$a_0 b_1 \text{ mode : } \delta = \frac{1}{4} + 4c^2 - 4\epsilon \gamma \quad (38)$$

$$b_0 a_1 \text{ mode : } \delta = \frac{1}{4} + 4c^2 + 4\epsilon \gamma \quad (39)$$

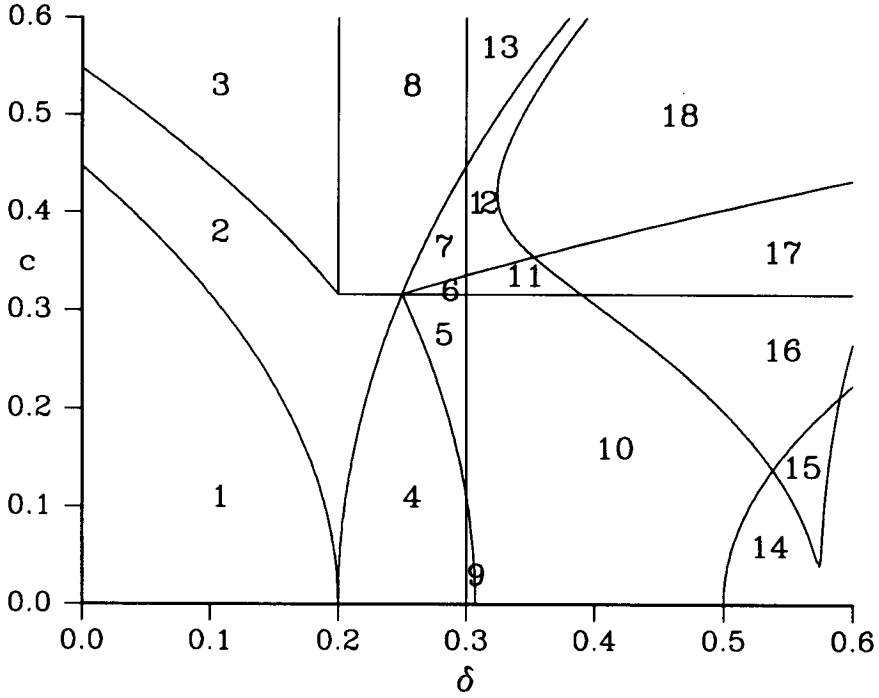
$$b_0 b_1 \text{ mode : } \text{always unstable} \quad (40)$$

Each of these transition curves corresponds to a pair of zero eigenvalues with the exception of the second $a_0 a_1$ mode equation, which corresponds to a pair of double pure imaginary eigenvalues.

Conclusions

Inspection of Fig.1 shows that the dynamics of the averaged two-term truncation eqs.(10)-(13) is complicated. In most of the regions shown there is more than one stable steady state, implying that the long-time behavior is dependent on the initial conditions. Moreover, the $\delta - c$ parameter space is divided up into many regions (of which 18 appear in Fig.1), each of which has different stable steady states than it's neighbors. In addition to the single modes a_0 and a_1 , our analysis predicts the existence of the stable multimodes $a_0 a_1$, $a_0 b_1$, and $b_0 a_1$.

Stable Steady States
 $\alpha = 1, \beta = 0, \gamma = 1, \varepsilon = 0.1$



region	$u = 0$	a_0	a_1	$a_0 a_1$	$a_0 b_1$	$b_0 a_1$
	mode	mode	mode	mode	mode	mode
1	x					
2			x			
3	x		x			
4		x	x			
5		x	x	x		
6			x	x	x	
7			x	x		
8		x	x			
9	x	x	x			
10	x	x	x	x		
11	x		x	x	x	
12	x		x	x		
13	x	x	x			
14	x	x		x		x
15	x	x				x
16	x	x	x			
17	x		x		x	
18	x		x			

Fig.1. The stable steady states of the averaged eqs.(10)-(13).

Numerical integration of eqs.(10)-(13) has confirmed the steady states of Fig.1.

In the presence of damping, $\beta > 0$, eqs.(10)-(13) may also exhibit stable limit cycle solutions. These correspond to quasiperiodic motions in eqs.(6),(7). Although the present paper has not investigated such motions, numerical integrations of eqs.(10)-(13) have shown them to exist.

We note that the approach used in this paper can be applied to three-term and higher order truncations. Such investigations will, however, be encumbered by more complicated algebraic expressions.

The assumption that $\delta = \frac{1}{4} + \delta_1\epsilon$ and $c = c_1\sqrt{\epsilon}$ forces each modal oscillator f_i to be in 2 : 1 resonance with the forcer $\cos t$. If the analysis is extended to $O(\epsilon^2)$, however, then additional resonances are possible, so that c need not be taken $O(\sqrt{\epsilon})$. E.g., the resonances $\omega_0 = \frac{1}{4}$ and $\omega_1 = 1$ correspond to the $\epsilon = 0$ parameter values $\delta = \frac{1}{4}$ and $c^2 = \frac{3}{4}$. Or in the case of a three-term truncation, the resonances $\omega_0 = \frac{1}{4}$ and $\omega_2 = 1$ correspond to the $\epsilon = 0$ parameter values $\delta = \frac{1}{4}$ and $c^2 = \frac{3}{16}$, in which case the ω_1 mode is not resonant, and so we may expect a steady state consisting of modes 0 and 2.

Finally we note that the pde (1) is expected to be much more complicated than Fig.1. We conjecture that there will be many more, perhaps infinitely many, comparable regions, each with many, perhaps infinitely many, steady states.

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