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Integrability of plane quadratic vector fields

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Abstract. In this article we give a clear, fairly self-contained account of the problem of the center and the question of integrability for plane quadratic vector fields. The literature on this subject contains many inaccuracies and errors and there exists no survey covering all aspects of the problem and warning the reader about these pitfalls. This article is designed to fill this gap. However, we also give a new proof for the sufficiency of the conditions for the center, a proof which simultaneously shows the "global" integrability of the systems satisfying these conditions. Their first integrals are elementary functions, defined on $\mathbb{R}^2$ or the complement of some algebraic curve.

A. Introduction. This article is written with the purpose of providing a clear, rapid and self-contained account of the problem of the center and the question of integrability for real plane quadratic vector fields.

The problem of the center is an important one in the theory of plane polynomial vector fields. For example, its solution for the case of quadratic vector fields is used in works devoted to Hilbert's 16th problem, second part. This is one of the few problems on Hilbert's list which is still open.

While working on the problem of the creation of limit cycles from a homoclinic loop in perturbations of quadratic integrable systems, the authors needed a full list of first integrals of such systems. Any such vector field must have a center in the region bounded by the homoclinic loop and we also needed a compact form for the conditions for a center in terms of the coefficients of the system. We found such conditions stated without proof in [6], where references are listed in order of their publication. The first two references on this list were published in 1911 in (old) dutch. The next reference given is [10] (in German) where Frommer states, as necessary and sufficient, conditions for the center which are neither necessary nor sufficient! Four other references ([24],[4],[25],[26]) appeared in Russian (the last two references in a journal with limited

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circulation) and were never translated. While we were researching this literature, a new book [30] appeared in print in the series Translations of the A.M.S. Half of a chapter in this book is devoted to Frommer's work [10] but Frommer's error is not mentioned. Moreover, the "theorem" supposed to summarize Frommer's work is different from Frommer's, and states, as necessary and sufficient, conditions which are only sufficient! It is unlikely that the authors of the book consulted certain important contributions to the subject [12],[13],[25],[26], since these are not listed in the bibliography of the book and may have been unavailable to its authors. A useful reference on the subject [7, ch. 7], also not mentioned in the book, came to our attention later. Frequent inaccuracies and errors, and main sources scattered through publications in several languages, each covering only some aspect of the problem - these difficulties confront a person who needs to use, with understanding, the results on this subject. These observations indicate that there is a need for a clear survey of the subject carefully scrutinizing its literature.

This article thus serves a double purpose. First, it gives the reader a clear and rapid overview of the literature, warning him about its pitfalls. Second, we direct attention to the question of the integrability of the systems, discussing some interesting aspects of this question, and we give a new, simultaneous proof of both the sufficiency of the conditions for the center and the integrability of the quadratic systems satisfying them. In this proof we show how the failed "proofs" of Frommer [10] and Ye [30] could be made to work. Also, we are interested here in "global" integrability, in the sense that we need the first integrals to be defined not just locally around the center but on an area as large as possible and containing the homoclinic orbits whenever these exist. The first integrals of the systems, are elementary functions expressed here in terms of the variables and the coefficients of the systems, and they are defined on the whole plane or on the complement of some algebraic curve. This is an invariant curve of the system in all but one of the cases, passing through all the remaining singular points of the system if they happen to be sinks, sources or saddle-nodes.

The remainder of the article is organized as follows:

In section B we consider the general problem of the center, we survey the literature following the historical development of the subject, and we call attention to the various meanings of the terminology used. In section C we consider the particular case of quadratic systems, following again the historical development, and we draw attention to the errors in the literature. We show how the compact form of the conditions for the center is obtained. In section D we discuss the question of integrability and we give the proof of the main result.

B. The problem of the center. Brief survey of the literature.

The concept of a center was defined by Poincaré (cf. [21], p. 17 and 11). A singular point (x_0,y_0) of a system

$$\text{(B1)} \qquad \dot{x} = P(x,y), \qquad \dot{y} = Q(x,y)$$

is a center if there exists a neighborhood U of (x_0,y_0) such that every point in U other than (x_0,y_0) is nonsingular and the integral curve passing through the point is closed.

In what follows, we shall consider a system (B1) with a singularity, which without loss of generality we may suppose to be the origin. We shall assume this singularity to be nondegenerate i.e. the linear part of the system at the origin has a nonsingular matrix.

In [22], Poincaré considered the problem of giving necessary and sufficient conditions for the existence of a center for a system (B1) with P,Q polynomials. He gives an infinite set of necessary and sufficient conditions for such a system to have a center at the origin. In his memoir on the stability of motion [18], Lyapunov studies systems of differential equations in n variables. When applied to the case n= 2, his results also give an infinite set of necessary and sufficient conditions for the system (B1) with P,Q polynomials to have a center. (Actually, Lyapunov's result is more general since it is stated for the case where P and Q are analytic functions). In searching for sufficient conditions for a center, both Poincaré and Lyapunov's work involve the idea of trying to find a constant of the motion F(x,y) for (B1) in a neighborhood of the origin, with F expressed as a power series

$$F(x,y) = F_2(x,y) + \dots + F_k(x,y) + \dots$$

where F_k is a homogeneous polynomial of order k and F_2 is a positive definite quadratic form, thus assuring us that the curves are closed in a sufficiently small neighborhood of the origin. (A function $F: U \to \mathbb{R}$, $U \subset \mathbb{R}^2$, U open, is a constant of the motion on U of (B1) if $F(x(t),y(t)) = \text{constant}$ for all t, for all solution curves $(x(t),y(t))$ of (B1).)

In general, for a system (B1) to have a constant of motion $F(x,y) = \Sigma_{i=2}^{\infty} F_i(x,y)$, we must have

$$\text{(B2)} \qquad dF/dt = (\partial F/\partial x)P(x,y) + (\partial F/\partial y)Q(x,y) = 0$$

Since

$$\text{(B3)} \qquad \begin{aligned} P(x,y) &= P_1(x,y) + P_2(x,y) + \dots + P_n(x,y) \\ Q(x,y) &= Q_1(x,y) + Q_2(x,y) + \dots + Q_n(x,y) \end{aligned}$$

where P_i, Q_i are homogeneous polynomials of order i, we have:

$$\text{(B4)} \qquad \begin{aligned} dF/dt &= \sum_{k=2}^{\infty} \sum_{i=1}^{n} [(\partial F_k/\partial x) P_i(x,y) + (\partial F_k/\partial y) Q_i(x,y)] \\ &= \sum_{q=3}^{\infty} \sum_{k+i=q} \Phi_{k,i} \end{aligned}$$

where

$$\text{(B5)} \qquad \Phi_{k,i} = (\partial F_k/\partial x)P_i + (\partial F_k/\partial y)Q_i$$

In order to obtain an F with $dF/dt = 0$ it is sufficient to take F such that

(B6) $$\Sigma_{k+i=q} \Phi_{k,i} = 0 \quad \text{for } q = 3,4\ldots$$

i.e.,

(B7) $$\Phi_{2,1} = 0,\ \Phi_{3,1} = -\Phi_{2,2},\ \ldots,\ \Phi_{k,1} = -\Phi_{(k-1),2} - \cdots -\Phi_{2,(k-1)},\ \ldots$$

The first equation in (B7) is a condition on the coefficients of F_2. Assuming we can solve it, then having F_2 we can use the second equation in (B7) to find the coefficients of $F_3, \ldots$, etc.

Assuming that we can solve $\Phi_{2,1} = 0$ and with a positive definite form F_2, the eigenvalues of the linear part of the system at zero must be purely imaginary. Indeed, since F_2 is definite, we may assume it to be $x^2 + y^2$ in some coordinate system. Let $X_1 = ax + by$, $Y_1 = cx + dy$. Then $\Phi_{2,1} = 0$ is the equation $2x(ax + by) + 2y(cx + dy) = 0$. Hence $a = d = 0$ and $b = -c$.

When we try to find the F_i, $i \geq 3$ from (B7), two possibilities present themselves to us: either we can annhilate all the terms (B6) in (B4), and hence solve all equations (B7), finding all F_i, or we can annihilate such terms only up to a certain order.

In the first case, Poincaré and Lyapunov showed that we can obtain a series which is convergent in a neighborhood of the origin, defining a function $F(x,y)$ that is a constant of the motion of the system (B1) in the neighborhood of the origin, and hence the origin is a center.

Thus, the infinite set of conditions (B6) are sufficient conditions for the system (B1), with the origin as singular point, to have a center at the origin.

In the second case, using polar coordinates, Poincaré showed that the first q for which we cannot solve the equation

$$\Phi_{q,1} = -\Phi_{(q-1),2} - \cdots - \Phi_{2,(q-1)}$$

in order to find F_q, must be even [21, p. 98]. For this q, however, he showed that one can find F_q such that

$$\Phi_{q,1} + \Phi_{(q-1),2} + \cdots + \Phi_{2,(q-1)} = C\,(x^2 + y^2)^{q/2}$$

with C a constant. Writing $F = F_2 + F_3 + \ldots + F_q$, one obtains a Lyapunov function for the origin. Since $dF/dt = C\,(x^2 + y^2)^{q/2}$, if C is positive, then the origin is unstable, while if C is negative, then the origin is asymptotically stable.

Lyapunov was the first to give a complete proof of the following result [18] for real analytic systems:

THEOREM A necessary and sufficient condition for the analytic system

$$(*)\qquad \begin{aligned} dx/dt &= ax + by + \sum_{k=2}^{\infty} X_k(x,y) \\ dy/dt &= cx + dy + \sum_{k=2}^{\infty} Y_k(x,y) \ , \end{aligned}$$

where the coefficients a,b,c,d are such that the linearization at the origin has purely imaginary eigenvalues, to have a center at the origin, is that the system have a constant of the motion F(x,y) in a neighborhood of the origin, with F real analytic, F not a constant.

We note that the condition in the hypothesis of this theorem, that the eigenvalues of the linearization at the origin be purely imaginary, is a necessary condition for the system (*) above to have a center. This was shown by Poincaré in [21].

A. Maier wrote a beautifully clear article [19] in which he gives the proof of the above result.

Implicit in Poincaré's argument in [22], which we have described briefly in the preceeding pages, is the fact that for a system (B1) with P and Q polynomials, with purely imaginary eigenvalues for the linear part, one can construct a power series

$$F(x,y) = F_2(x,y) + \dots + F_k(x,y) + \dots$$

such that

$$\text{(B8)}\qquad dF/dt = V_1(x^2+y^2)^2 + \dots + V_k(x^2+y^2)^{k+1} + \dots$$

with $V_1, V_2, \dots, V_k, \dots$ constants. The first non-zero V_i gives the asymptotic stability or instability of the origin according to its negative or positive sign. Indeed, stopping the series at F_k, we obtain a polynomial which is a Lyapunov function for the system. Due to this, the V_i 's are called by some authors the Lyapunov constants [28], while others use the term focal values for them (cf.[16],[5]). Although the term focal value varies in meaning from author to author, work done in [11] shows that there is consistency among the meanings in a sense we explain below.

In [11] the authors claim that Andronov et al. define the focal values in [1] by the formula $\alpha_i = d^i(o)/i!$ (cf. [11] p. 343, (5.11)). In fact, in [1], Andronov et al. define the notion of the i-focal value at the origin for a system of the form

$$\begin{aligned} \dot{x} &= ax - by + \varphi(x,y) \\ \dot{y} &= bx + ay + \psi(x,y) \end{aligned}$$

as the i-th derivative $d^i(o)$ of the function $d(\rho_o) = P(\rho_o) - \rho_o$ where P is the Poincaré return map. (cf. [1], p. 243, Definition 25). The functions φ and ψ are assumed to be of the class C^k, $k \geq 1$ or analytic. Andronov et al. use the notation $\alpha_i = d^i(o)/i!$ for $i = 2, 3, \dots$ and $\alpha_1 = d^1(o) + 1$ i.e. α_i are the Taylor series coefficients of the map P at the origin. The first non-zero focal value of Andronov corresponds to an odd number $i = 2n + 1$. Andronov calls this the n-th Lyapunov value [1, p. 244]. The relationship between the focal values of Andronov and the Lyapunov constants was studied in [11] where it is proved that the first non-zero Lyapunov constant V_n differs only by a positive constant factor from the first non-zero focal value, which is $d^{(2n+1)}(0)$. Hence the identification in terminology is natural.

In terms of the V_i 's the conditions for a center become $V_k = 0$, for $k = 1,2,\dots$. Now $V_1, \dots, V_k, \dots$ are polynomials with rational coefficients in the coefficients of P and Q. By Hilbert's basis theorem [29], the ideal generated by these polynomials has a finite basis $B_1, B_2, \dots, B_m$. Hence we have a finite set of necessary and sufficient conditions for a center, i.e. $B_i = 0$ for $i = 1,2,\dots,m$. To calculate this basis, we reduce each V_k modulo $\langle V_1,\dots,V_{k-1}\rangle$, the ideal generated by $V_1,\dots, V_{k-1}$ as indicated in [5]. The elements of the basis thus obtained are called the Lyapunov quantities [5] or the focal quantities [23].

We remark here that there are systems of planar differential equations which have continuous families of periodic orbits, but do not have a center. A cubic example is the system

$$\dot{x} = \frac{1}{2}(1 - y - x^2), \qquad \dot{y} = x(x^2 - 1 - y)$$

which is invariant under the symmetry $(x,y,t) \rightarrow (x,y,-t)$. Such examples cannot occur for quadratic systems, however. Periodic orbits of quadratic systems are convex and contain only one singular point in their interior [6], so a continuous family of periodic orbits for a quadratic system must surround a center. Thus, a solution of the problem of the center for quadratic systems gives all quadratic systems with continuous families of periodic orbits.

C. The problem of the center for quadratic systems.

The problem of the center for systems (B1) with P,Q quadratic polynomials was first considered by Dulac. In [8], Dulac lists all possible cases of quadratic equations (B1) having a center at the origin, then shows that for each such equation, one can integrate it with elementary functions as integrals. Dulac's work involves quadratic equations over the complex field and he does not stop to draw conclusions for what happens over the reals. The notion of the center as defined in part B does not apply to systems over the complex field. Dulac starts with the equation over the complex field

$$X(x,y)dx + Y(x,y)dy = 0$$

(with X,Y holomorphic functions) having a singular point at the origin. Assuming that the singular point is non-degenerate, the equation above can be brought by a suitable change of axes over the complex field to the form

$$(x + \dots)\, dy + (-\lambda y + \dots)\, dx = 0$$

Dulac looks for a general integral $F(x,y) = \sum_{i=2}^{\infty} F_i(x,y) =$ constant, of such an equation. For Dulac the origin is a center in the case when λ is a negative rational number and the equations $\sum_{k+i=q} \Phi_{k,i} = 0$, $q = 3,4,\ldots$, can all be solved for F_i.
In [8] Dulac treats the case where $\lambda = -1$ for quadratic equations.

The problem of the center for quadratic equations over the reals was considered for the first time by Kapteyn [12]. In [12], Kapteyn gives a set of algebraic conditions on the coefficients of a real quadratic system, for the system to have a center at the origin. These conditions are only sufficient since not all quadratic systems with a center are covered in [12]. In [13], Kapteyn first observes that Dulac's paper [8] deals with complex quadratic systems, listing all such systems which have a center. Then he extends his method used in [12] to complex quadratic systems and obtains in [13] necessary and sufficient algebraic conditions formulated in terms of the coefficients of the systems for these systems to have a center ($\lambda = -1$). In particular, if we restrict ourselves to the real case, from [13], we obtain the following important result:

THEOREM 1 i) Every quadratic system

$$\text{(C1)} \qquad \begin{aligned} dx/dt &= -y - b\,x^2 - (2c + \beta)\,x\,y - d\,y^2, \\ dy/dt &= x + a\,x^2 + (2b + \alpha)\,x\,y + c\,y^2 \end{aligned} \qquad (x,y) \in \mathbb{R}^2$$

can be brought by a rotation to a system of the form (C1) with $a + c = 0$

ii) Assuming that $a + c = 0$, a necessary and sufficient condition for such a system to have a center at the origin is to satisfy one of the following conditions:

I) $b + d = 0$

II) $\beta = 0 = a\,\alpha$

III) $\beta = 0 = \alpha + 5(b + d) = a^2 + 2d^2 + b\,d$

The proof of i) is immediate. The proof of ii) goes as follows: first one shows the sufficiency of the condition "I or II or III". Then using the sufficiency of this condition and work done by Bautin in [3], one proves the necessity. To prove the sufficiency of the conditions I, II, III, we shall show that any system satisfying one of these three conditions can be integrated with elementary functions. We obtain in this way constants of motion for the corresponding systems and hence the origin is a center. Hence the proof works for both the sufficiency of each one of the conditions I, II, III and the integrability of the system. We discuss the various aspects of integrability in the next paragraph and since the sufficiency of the condition is related to the integrability of the system, we shall prove the sufficiency in the next paragraph.
In the remaining part of this paragraph we describe Bautin's work in [3] and give the proof of the necessity of the condition "I or II or III", assuming its sufficiency.

In his paper [3], Bautin uses the sufficiency of each one of the conditions I, II, III in order to obtain the general expression of the (focal) values α_k for $k \neq 1$ for quadratic systems.

Bautin's result is stated for systems with the origin a focus or center for their linearization. We focus our attention on the case we need here i.e. the case of systems with a center at the origin for their linearization and hence systems which can be written in the form (C1) with $a + c = 0$. We state Bautin's result applied to such systems. The calculations are simpler in this case for the first part of the proof and we give them below.

THEOREM. (Bautin). For a system (C1) with $a + c = o$, the (focal) values α_k are homogeneous polynomials of degree k-1 in the coefficients of the system. The ideal generated by α_k, $k > 1$ has the following basis:

$$B_1 = (b + d)\,\beta$$
$$B_2 = (b + d)\, a\, \alpha\, (\alpha + 5\,(b + d))$$
$$B_3 = (b + d)^2\, a\, \alpha\, (a^2 + 2d^2 + bd)$$

The values α_k have the form:

$$\alpha_1 = 1,\quad \alpha_2 = 0,\quad \alpha_3 = \frac{\pi}{4} B_1,\quad \alpha_4 = B_1\,\theta_4^{(1)},\quad \alpha_5 = \frac{\pi}{24} B_2 + B_1\theta_5^{(1)}$$

$$\alpha_6 = B_2\,\theta_6^{(2)} + B_1\,\theta_6^{(1)},\qquad \alpha_7 = \frac{25\pi}{32} B_3 + B_2\,\theta_7^{(2)} + B_1\,\theta_7^{(1)}$$

$$\alpha_k = B_3\,\theta_k^{(3)} + B_2\,\theta_k^{(2)} + B_1\,\theta_k^{(1)} \qquad \text{for } k > 7$$

where $\theta_k^{(j)}$ are homogeneous polynomials in the coefficients of the system.

For the proof, we use the polar coordinate equation associated to the system which turns out to be:

$$\frac{d\rho}{d\varphi} = \frac{D\rho^2}{1-\rho B}$$

with

$$B = -\ a\cos^3\varphi\ - (3b + \alpha)\cos^2\varphi\,\sin\varphi + (3a - \beta)\cos\varphi\,\sin^2\varphi\ - d\sin^3\varphi$$
$$D = -\ a\sin^3\varphi\ - b\cos^3\varphi + (3a - \beta)\cos^2\varphi\,\sin\varphi + (2b + \alpha - d)\sin^2\varphi\,\cos\varphi$$

Writing $1/(1 - \rho B) = \sum_{n=o}^{\infty} (\rho B)^n$ for ρ small, the equation becomes

$$\frac{d\rho}{d\varphi} = R_2\rho^2 + R_3\rho^3 + \ldots + R_n\rho^n + \ldots$$

where $R_n = B^{n-2} D$. Since B and D are linear functions in the coefficients of the system, clearly R_n is a homogeneous polynomial of degree n-1 in these coefficients. From Lyapunov's work (cf [18], [19]), it follows that the solution of the above equation, considered as a function of the initial condition ρ_o , has in the neighborhood of the origin an expansion

$$\rho(\varphi) = \alpha_1(\varphi)\rho_o + \alpha_2(\varphi)\rho_o^2 + \dots$$

where the α_i's are functions of φ and of the coefficients of the system (C1). Substituting this expansion in the equation and identifying coefficients we obtain a recursive set of equations

$$\frac{d\alpha_1}{d\varphi} = 0 \ , \quad \frac{d\alpha_2}{d\varphi} = \alpha_1^2 R_2 \ , \quad \frac{d\alpha_3}{d\varphi} = 2\alpha_1\alpha_2R_2 + \alpha_1^3R_3 \ , \ \dots$$

Integrating these equations and using the initial condition condition $\rho\,(o) = \rho_o$, we obtain:

$$\alpha_1(\varphi) = 1, \quad \alpha_2(\varphi) = \int_o^{\varphi} R_2(\xi)\,d\xi \ , \quad \alpha_3(\varphi) = \int_o^{\varphi} (2R_2(\xi)\,\alpha_2(\xi) + R_3(\xi))\,d\xi \ , ,\dots$$

The Poincaré return map is:

$$P(\rho_o) = \rho(2\pi) = \alpha_1(2\pi)\rho_o + \alpha_2(2\pi)\,\rho_o^2 + \dots$$

Hence $\alpha_1 = \alpha_1(2\pi) = 1$, $\alpha_2 = \alpha_2(2\pi)$ is a linear function in the coefficients of the system , $\alpha_3 = \alpha_3\,(2\pi)$ is a quadratic homogeneous polynomial in the coefficients of the system, etc. α_k is then a homogeneous polynomial of degree k-1 . We now show that the ideal generated by α_k , $k > 1$ has as a basis $\{B_1 , B_2 , B_3\}$. To do this Bautin uses the sufficiency for the center of each one of the conditions, I, II, III . Since III is a sufficient condkition for the center, III implies the annihilation of all the focal values $d^{(i)}(o)$. Hence $\alpha_1 = 1$ and $\alpha_k = 0$ $\forall k > 1$. Hence for $k > 1$, α_k must be in the polynomial ideal generated by the polynomials

$$\beta \ , \ \alpha + 5\,(b + d)\ , \ a^2 + 2d^2 + bd$$

of the ring $\mathbb{R}\,[a, b, d, \alpha, \beta]$ of real polynomials in the coefficients of the system. So

$$\alpha_k = (a^2 + 2d^2 + bd)\,\overline{\theta}_k^{(3)} + (\alpha + 5(b+d))\,\overline{\theta}_k^{(2)} + \beta\,\overline{\theta}_k^{(1)}$$

with $\overline{\theta}_k^{(j)}$ homogeneous polynomials in the same coefficients. By regrouping the terms containing β within the term $\beta\overline{\theta}_k^{(1)}$ we may assume that $\overline{\theta}_k^{(3)}$ and $\overline{\theta}_k^{(2)}$ do not contain β. Then, using the sufficiency of the condition II and then of I we further factor $\overline{\theta}_k^{(j)} = a\alpha\,(b + d)\theta_k^{(j)}$, $j = 2, 3$ and $\overline{\theta}_k^{(1)} = (b + d)\ \theta_k^{(1)}$. We obtain

$$\alpha_k = a\,\alpha\,(b + d)\,(a^2 + 2d^2 + bd)\theta_k^{(3)} + a\,\alpha\,(b + d)\,(\alpha + 5\,(b + d))\,\theta_k^{(2)} + \beta\,(b + d)\,\theta_k^{(1)} \qquad k > 1$$

We now put $k = 2$ in the above expression for α_k . Taking into account that α_2 must be a linear function in the coefficients of the system, we obtain that $\theta_k^{(j)} = 0$, for $j = 1, 2, 3$ and hence $\alpha_2 = 0$. Putting $k = 3$ in the above expression for α_k , we find that $\theta_k^{(j)} = 0$ for $j = 2, 3$ and that $\theta_k^{(1)}$ must be a constant. By calculation this constant turns out to be $\pi/4$. So $\alpha_3 = (\pi/4)\,\beta\,(b + d)$. For $k = 4$, we see that $\theta_4^{(2)} = 0 = \theta_4^{(3)}$ and hence

$\alpha_4 = \beta(b+d)\,\theta_k^{(1)} = B_1\theta_4^{(1)}$. For $k=5$ we see that $\theta_5^{(3)} = 0$ and $\theta_5^{(2)}$ must be a constant (since α_4 must be a polynomial of degree 3). By calculation this constant turns out to be $\pi/24$. We have

$$\alpha_5 = \frac{\pi}{24}\, a\alpha\,(b+d)\,(\alpha + 5\,(b+d)) + \beta\,(b+d)\,\theta_5^{(1)} = \frac{\pi}{24}\,B_2 + B_1\,\theta_5^{(1)}$$

Analogously one obtains the expression for α_6 and α_7 i.e.

$$\alpha_6 = B_2\theta_6^{(2)} + B_1\theta_6^{(1)} \quad , \quad \alpha_7 = \frac{25\pi}{32}B_3 + B_2\theta_7^{(2)} + B_1\theta_7^{(1)}$$

It remains to prove the formula for α_k, $k > 7$ from the one we already obtained above. For this, one may assume that $\theta_k^{(3)}$ in the above formula, contains neither α nor β. We already know that $\theta_k^{(3)}$ may be assumed not to contain β. By writing

$$\theta_k^{(3)} = (a + 5(b+d))P_k^{(3)}(a, b, d, \alpha) + Q_k^{(3)}(a, b, d)$$

with $P_k^{(3)}$ a homogeneous polynomial in a, b, d, α and $Q_k^{(3)}$ a homogeneous polynomial in a, b, d only and by regrouping together the terms containing the factor $\alpha + 5(b+d)$, one may assume that $\theta_k^{(3)}$ does not contain α. To finish the proof of the theorem, it remains to show that $\theta_k^{(3)}$ contains $b+d$ as a factor. It suffices to check this for the case $\beta = \alpha + 5(b+d) = 0$. The details of this last calculation are contained in Bautin's paper [3, p. 403-408].

As a corollary of the above proposition we prove the necessity of the condition "I or II or III" in the theorem assuming its sufficiency:

If the system has a center at the origin then all the focal values $d^i(o)$ are zero i.e $\alpha_1 = 1$ and $\alpha_k = 0$ for all $k > 1$. Hence $B_1 = B_2 = B_3 = 0$. But this is equivalent to having one of the conditions I, II, III satisfied. Indeed, if we call $a\alpha$ the second factor in B_2, B_3 then the conditions I, II, III in the theorem, correspond respectively to annihalating the first factor in B_1, B_2, B_3, then the second factor, and finally the second factor in B_1 and the third factor in B_2, B_3.

REMARK We point out that in [3], the sign for the third focal quantity ($\overline{v_7}$ in [3]) is wrong as was shown by other independent calculations (cf. [15], [23]. [5]). (Shi Songling was apparently the first to notice this sign error.)

To use Kapteyn's conditions I, II, III, one needs to perform a rotation in order to bring the system to the form (C1) with $a + c = 0$. Naturally, one would like to have necessary and sufficient conditions for a center in terms of the coefficients of the system, without having to perform this transformation. Frommer studied this problem. In his paper [10], he claimed to have found such necessary and sufficient conditions. His conditions are of the form:

(*) $\qquad I_F$ or II_F or III_F

where we call I_F, II_F, III_F the three algebraic conditions given by Frommer (cf. [10], p.418). Frommer's condition (*) turned out to be neither necessary nor sufficient. Frommer's condition (*) is not necesssary since for $a = d = 0$, $b = 1$, $c = 2$, $\alpha = -5$, $\beta = -10$ we have a system (C1) which has a center at the origin, as will be shown later, but does not satisfy (*). Frommer's condition (*) is not sufficient since the system (C1) with $c = d = 1$, $a = -7/16$, $b = 1/8$, $\alpha = 1$, $\beta = 1/2$ satisfies II_F and hence also (*) but the origin is not a center in this case as will become clear later.

Frommer's error was noticed by Bautin in [2] but it is ignored in some of the more recently published literature. For instance the recently revised and augmented book on limit cycles of Ye Yan Qian et al. [30], which appeared in english translation in 1986 contains a chapter on the work of Frommer and Bautin where the author states that Frommer gave necessary and sufficient conditions for the center and then proceeds to describe Frommer's approach. The Theorem supposed to summarize this approach (Theorem 9.1, p. 201), is different from the one stated by Frommer, but this theorem states as necessary and sufficient, conditions which are only sufficient. Indeed, $a = d = 0$, $b = 1$, $c = 2$, $\alpha = -5$, $\beta = -10$ gives a counter-example to necessity.

In [24], Sakharnikov gives necessary and sufficient conditions for the center. While 5 of his conditions are stated in terms of the coefficients of (C1), in order to apply his sixth condition one has to perform first a rotation to bring the system (C1) to the form (C1) with $a + c = 0$.

Further work based on [24] was done by Belyustina [4] and Sibirskii [25], [26] who finally obtained necessary and sufficient conditions for the center in terms of the coefficients of the system (C1) directly.

The most compact form we have for the conditions for the center appeared (without proof) for the first time in the western literature in Coppel's paper [6], in the following form:

THEOREM 2 The system (Cl) has a center at the origin if and only if one of the following conditions is satisfied:

(C3)

I) $a + c = b + d = 0$

II) $\alpha(a + c) = \beta(b + d)$ and
$a\alpha^3 - (3b + \alpha)\alpha^2\beta + (3c + \beta)\alpha\beta^2 - d\beta^3 = 0$

III) $\alpha + 5(b + d) = \beta + 5(a + c) = 2(a^2 + d^2) + ac + bd = 0$

The papers [25], [26] have not appeared in English translation. We managed to obtain only one [25] of the above papers, where part of this result is proved. To make the paper self-contained and also to make the proof available in the western literature, we give it below:

Proof: We first note that if $a + c = 0$, these give exactly the conditions I, II, III of Theorem 1. So consider the case $a + c \neq 0$, when only conditions II and III have to be checked. Let us denote by a', b', c', d', α', β' the parameters of the system after we have applied the rotation of axes

$$x = \xi \cos\theta - \eta \sin\theta$$

(C4)

$$y = \xi \sin\theta + \eta \cos\theta$$

so as to make $a' + c' = 0$. The new coefficients are as follows:

(C5)
$$a' = a\cos^3\theta + (3b+\alpha)\cos^2\theta\sin\theta + (3c+\beta)\cos\theta\sin^2\theta + d\sin^3\theta$$
$$b' = b\cos^3\theta + (2c+\beta-a)\cos^2\theta\sin\theta - (2b+\alpha-d)\cos\theta\sin^2\theta - c\sin^3\theta$$
$$c' = c\cos^3\theta - (2b+\alpha-d)\cos^2\theta\sin\theta - (2c+\beta-a)\cos\theta\sin^2\theta + b\sin^3\theta$$
$$d' = d\cos^3\theta - (3c+\beta)\cos^2\theta\sin\theta + (3b+\alpha)\cos\theta\sin^2\theta - a\sin^3\theta$$
$$\alpha' = \alpha\cos\theta - \beta\sin\theta$$
$$\beta' = \alpha\sin\theta + \beta\cos\theta$$

From these we obtain:

(C6) $$a' + c' = (a+c)\cos\theta + (b+d)\sin\theta$$

So that to have $a' + c' = 0$ we need to take θ such that

(C7) $$(a+c)\cos\theta + (b+d)\sin\theta = 0$$

So we can take

(C8) $$\cos\theta = -(b+d)\,[(a+c)^2 + (b+d)^2]^{-1/2}, \quad \sin\theta = (a+c)\,[(a+c)^2 + (b+d)^2]^{-1/2}$$

All that is needed is to express conditions II and III of Kapteyn which are formulated in terms of a', b', c', d', α', β' here, as functions of a, b, c, d, α, β and to verify that they correspond exactly to II and III above, respectively. Consider first the case II of Kapteyn, i.e. $\beta' = a'\alpha' = 0$ here. By (C5), $\beta' = 0$ means

(C9) $$\beta\cos\theta + \alpha\sin\theta = 0$$

Eq. (C7) together with (C9) then yields

(C10) $$\alpha(a+c) = \beta(b+d)$$

i.e., the first part of II in (C3). If $\beta = 0$ then $\alpha = 0$ and so the second part of II holds. Suppose now that $\beta \neq 0$ and assume that $a' = 0$. Using the formula for a' in (C5) and replacing $\cos\theta$ by $-(\alpha/\beta)\sin\beta$, from (C9), we obtain

(C11) $$a' = [-a\alpha^3 + (3b+\alpha)\alpha^2\beta - (3c+\beta)\alpha\beta^2 + d\beta^3](\sin^3\theta)/\beta^3$$

so since by (C4) we have that $\theta \neq k\pi$, as $a + c \neq 0$, $a' = 0$ iff the second part of condition II above holds. Assume now that condition III in Theorem 1 holds, i.e. $\beta' = \alpha' + 5(b' + d') = a'^2 + 2d'^2 + b'd' = 0$. The calculations yield

(C12) $$\alpha' + 5(b' + d') = (\alpha + 5(b+d))\cos\theta - (\beta + 5(a+c))\sin\theta$$

Using the expression for β' in (C5), and using (C12) and $\beta' = \alpha' + 5(b' + d') = 0$, we have

(C13) $$\alpha[\alpha + 5(b+d)] + \beta[\beta + 5(a+c)] = 0$$

Since $\beta' = 0$ also means $\alpha (a + c) = \beta (b + d)$ and replacing α by $\beta(b + d)/(a+c)$ in (C13), (C13) becomes

(C14) $$[\beta + 5(a + c)] [(a + c)^2 + (b + d)^2] = 0$$

In fact, $\alpha' + 5(b' + d') = 0 = \beta'$ is equivalent to $\beta + 5(a+ c) = 0$ and $\alpha(a + c) = \beta(b + d)$. Replacing β by $-5(a + c)$ in the last equation, we obtain $\alpha + 5(b + d) = 0$ and hence $\alpha'+5(b'+d') = 0 = \beta'$ is equivalent to $\beta + 5(a + c) = \alpha + 5(b + d) = 0$. We now turn to the last part of condition III of condition III, i.e.

(C15) $$a'^2 + 2d'^2 + b'd' = 0$$

After replacing β by $- 5 (a + c)$ and α by $- 5 (b + d)$ in the equations for a', b' and d' in (C5) and using (C8), the calculations for the left hand side of (C15) yield (This calculation is done in [14].):

(C16) $$a'^2 + 2d'^2 + b'd' = 2(a^2 + d^2) + ac + bd$$

So that (C15) when expressed in terms of a, b, c, d, α, β becomes

(C17) $$2(a^2 + d^2) + ac + bd = 0$$

It is now easy to check that the system $a = d = 0$, $b = 1$, $c = 2$, $\alpha = - 5$, $\beta = - 10$, previously considered, has a center at the origin. Also we can check that the system (C1) with $c = d = 1$, $a = -7/16$, $b = 1/8$, $\alpha = 1$, $\beta = 1/2$, does not have a center at the origin.

D. Integrability and the conditions for the center

We shall use the notion of integrability in the following sense: A system of real differential equations

(D1) $$\dot{x} = F(x,y), \quad \dot{y} = G(x,y)$$

is integrable on an open subset U of $\mathbb{R}^2$, if there is a non-constant C^1 function $H: U \to \mathbb{R}$ which is constant along any solution curve of the system in U. Such a function is called a first integral of the system (this is the classical terminology), or a constant of motion. We have an analogous definition for systems over the complex numbers if we assume H to be an analytic function on $U \subset \mathbb{C}^2$.

The linear system $\dot{x} = x$, $\dot{y} = y$ is integrable with constant of motion y/x on $U = \mathbb{R}^2 - \{(x,y) \in \mathbb{R}^2 \mid x = 0\}$. As the reader could easily check, the cubic example (A1) is section A is integrable on the set $U = \{(x,y) \in \mathbb{R}^2 \mid x \neq \pm 1\}$ with constant of motion:

$$\frac{1}{2} \ell n (y^2 + (x^2 - 1)^2) + \arctan \frac{y}{x^2 - 1}$$

Caution is required in the geometric interpretation of integrability. If the integral H is defined on the complement of a certain curve in $\mathbb{R}^2$ containing a singular point of the system

(D1), then the singular point may well be a sink or source. In some of the cases described below, this does indeed occur. There are quadratic vector fields whose phase portraits contain both a sink and a center, despite the fact that such systems have integrals which are elementary functions.

In [8], Dulac gave the list of all quadratic systems over the complex domain which have a center at the origin and he showed that each such system can be integrated with elementary functions. The general integrals he obtains for these systems are of the form $J(X,Y,C) = 0$ where J is a function defined on $\mathbb{C}^2 \times \mathbb{C}$. Some of these general integrals are of the form $H(x,y) = C$ with H a polynomial or an analytic function on the whole of $\mathbb{C}^2$. For others we could have an explicit $H(x,y) = C$ only on some open subset U of $\mathbb{C}^2$. In these cases U is the complement of a curve in $\mathbb{C}^2$. Dulac was not concerned with what happens over the reals but he made the observation that a real quadratic system could also be studied by using his methods and the technique of complexification. In [10], Frommer found the general integrals for two specific cases of real quadratic systems. By using the approach of Dulac in [8], in [30], Ye obtains the general integral for another real quadratic case.

Integrals for the general quadratic system with a center were obtained in [17] by Lunkevich and Sibirskii. But their work is based on Sibirskii's book on algebraic invariants [27] which is published in Russian and has not been translated. Here we want to give direct calculations for all the general integrals for real quadratic systems with a center. Doing this, apart from the proof of the integrability of such systems, we obtain at the same time the sufficiency of the conditions I, II, III in Theorem 1, section B, as we show below.

By part i) of Theorem 1, section B, we may assume that the quadratic system is given in the form:

$$\text{(D2)} \qquad \frac{dy}{dx} = -\frac{x + ax^2 + (2b+\alpha)\,xy + c\,y^2}{y + bx^2 + (2c+\beta)\,xy + d\,y^2} \qquad \text{with } c = -a$$

So we only have to give a complete list of constants of the motion for such systems. While doing this we shall also be proving the sufficiency part of part ii) of Theorem 1, section B.

THEOREM. i) A necessary and sufficient condition for an equation (D2) to have a center at the origin is that one of the following conditions be satisfied:

I) $b + d = 0$

II) $\beta = 0 = a\alpha$

III) $\beta = 0 = \alpha + 5\,(b + d) = a^2 + 2d^2 + bd$

ii) For each one of the cases exibited above we have a general constant of the motion $H(x,y)$ on an open set U of $\mathbb{R}^2$, U being either the whole of $\mathbb{R}^2$ or a complement of some algebraic curve in $\mathbb{R}^2$.

Proof: Assuming the sufficiency of the conditions, we proved in Section C, their necessity. It remains to be shown that the three conditions are sufficient. Consider first the case I), i.e. $b + d = 0$. This case was treated by Frommer [10] who observed that a rotation of axes preserves the condition $a + c = b + d = 0$. So by conveniently choosing the angle θ, we can make the new coefficients a', c' be zero. Hence we may assume the equation to be:

(D3) $$\frac{dy}{dx} = - \frac{x + Axy}{y + Bx^2 + Cxy - By^2}$$

If $B = 0$, the equation is separable. In case also $A = 0$, a constant of the motion is

(D4) $$y^2/2 + x/C - (1/C^2)\ \ell n\,|\,1 + Cx\,|$$

(Analogously if $C = 0$ and $A \neq 0$). If $B = 0$ and $A \neq 0 \neq C$, a constant of the motion is

(D5) $$y/A + x/C - (1/A^2)\ \ell n\,|\,1 + Ay\,| - (1/C^2)\ \ell n\,|\,1 + Cx\,|$$

If $B \neq 0$, then besides the origin, the equation will also have the singular point $x = 0$, $y = 1/B$. We can move $(0,1/B)$ to infinity by the projective transformation

(D6) $$x = \xi / (1 + B\eta), \quad y = \eta / (1 + B\eta)$$

and we obtain the separable equation:

(D7) $$\frac{d\eta}{d\xi} = - \frac{\xi\,[\,1 + (A+2B)\eta + B(A+B)\eta^2]}{\eta\,[\,1 + C\xi - B(A+B)\xi^2\,]}$$

Separating the variables and factoring we obtain

(D8) $$\frac{\eta\, d\eta}{(1+B\eta)\,[1+(A+B)\eta]} = - \frac{\xi\, d\xi}{1 + C\xi - B(A+B)\xi^2}$$

This equation may be integrated to yield a constant of the motion. By developing in a power series around the origin we obtain a constant of the motion in the form

(D9.1) $$(\xi^2+\eta^2)/2 - [\,C\xi^3 + (A+2B)\eta^3\,]/3 + \ldots$$

and going back to the original coordinates

(D9.2) $$(x^2+y^2)/2 - [Cx^3 - 3Bx^2y + (A-B)y^3]/3 + \ldots$$

This shows that the origin is a center. The constant of the motion of (D8), is given by:

(D10.1) $$\frac{1}{B}\eta + \frac{1}{C}\xi - \frac{1}{B^2}\ \ell n\,|\,1+B\eta\,| - \frac{1}{C^2}\ \ell n\,|\,1+C\xi\,|, \qquad \text{if } B = -A$$

(D10.2) $$\frac{1}{A}\ \ell n \left[\frac{|\,1+B\eta\,|^{A+B}}{|\,1+(A+B)\eta|^{B}}\right] - \frac{1}{\xi_1 - \xi_2}\ \ell n \left[\frac{|\xi-\xi_1|^{\xi_1}}{|\xi-\xi_2|^{\xi_2}}\right], \qquad \text{if } B \neq -A \text{ and } \Delta > 0,$$

where $\Delta = C^2 + 4B(A+B)$

$$\xi_1 = \frac{C+\sqrt{\Delta}}{2B(A+B)} \qquad \xi_2 = \frac{C-\sqrt{\Delta}}{2B(A+B)}$$

(D10.3) $$\frac{1}{A}\,\ell n\left[\frac{|1+B\eta|^{A+B}}{|1+(A+B)\eta|^{B}}\right] + \frac{\xi_0}{\xi-\xi_0} - \ell n\,|\xi-\xi_0|\,, \qquad \text{if } B \neq -A \text{ and } \Delta = 0$$

where $\xi_0 = \dfrac{C}{2B(A+B)}$

(D10.4) $$\frac{1}{A}\,\ell n\left[\frac{|1+B\eta|^{A+B}}{|1+(A+B)\eta|^{B}}\right] - \frac{C}{\sqrt{-\Delta}}\arctan\frac{2B(A+B)\xi - C}{\sqrt{-\Delta}}$$
$$- \frac{1}{2}\,\ell n\left[(2B(A+B)\xi-C)^2-\Delta\right], \quad \text{if } B \neq -A \text{ and } \Delta < 0$$

Expressing the formula (D10.1) in terms of x and y we obtain:

(D10.I) $$(1-By)^{-1}[BC^2y + CB^2x] + (B^2+C^2)\ln|1-By| - B^2\ln|1-By+Cx|$$
$$(\text{if } B = -A)$$

We observe that this constant of the motion is defined for $(1-By)(1+Cx-By) \neq 0$. Analogously we can express the other constants of the motion in terms of x, y. Clearly in all the cases, the constants of the motion are defined on the complement of some algebraic curve.

REMARK 1 We want to point out some facts about the systems discussed above. The system (D3) with $B = -A$ $(C \neq 0 \neq B)$ has three singular points: the origin which is a center, the point $(0,1/B)$ which is a degenerated singular point and the point $(-C/B^2, 1/B)$ which is a node.

The constant of the motion (D10.I) is undefined on the line $y = 1/B$ which passes through the second and third singular points, and on the line $1-By+Cx = 0$. These are invariant lines of the system and they determine four open sectors, one of which is covered with a family of closed curves surrounding the origin. All the closed curves of the system are in this sector.

Analogous observations could be made about the other cases described above. In all the other cases the constants of the motion are also defined on the complement of an algebraic curve which is either a straight invariant line or it is composed of two or three straight lines.

Case II). We separate this case into two cases, i.e. IIα): $\beta = 0 = \alpha$. The case IIα is that of a Hamiltonian system, the general constant of the motion being:

(D11) $$(x^2+y^2)/2 + ax^3/3 + bx^2y + cxy^2 + dy^3/3$$

Hence it's clear the origin is a center. Consider now the case IIa): $\beta = 0 = a$. The equation in this case is of the form:

$$\frac{dy}{dx} = -\frac{x + Axy}{y + bx^2 + dy^2} \tag{D12}$$

We put $A = -2b'$. The direction field is symmetric with respect to reflection in the y-axis. We use Kapteyn's method ([12], p. 1451) to obtain the general integral in this case. Assume first that $A \neq 0$. Due to the symmetry, we limit ourselves first to the half plane $x > 0$. We make the change of variables

$$v = x^2/2\,, \quad z = 1 - 2b'y \tag{D13}$$

This is a diffeomorphism: $\mathbb{R}^+ x \mathbb{R} \to \mathbb{R}^+ x \mathbb{R}$. Written in the variables v, z the equation becomes:

$$\frac{dv}{dz} - \frac{b}{b'z}\, v = \frac{2b' + d - 2(b' + d)\, z + dz^2}{8b'^3 z} \tag{D14}$$

If $b \neq -A = 2b'$ and $b \neq b'$, this equation has a particular integral

$$v = \mu + \nu z + \gamma z^2 \tag{D15}$$

where

$$\mu = -(2b' + d)/(8bb'^2), \quad \nu = -(b' + d)/(4b'^2(b' - b)), \quad \gamma = d/(8b'^2(2b' - b)) \tag{D16}$$

Thus the constant of the motion is

$$[x^2 - 2(\mu + \nu + \gamma) - 2A(\nu + 2\gamma)\, y - 2A^2\gamma y^2]\, |\, 1 + Ay\, |^{2b/A} \tag{D17}$$

We observe that this constant of the motion is defined also for $x < 0$ and because of the symmetry of the direction field and of (D17) with respect to the y-axis, (D17) turns out to be a constant of the motion for eq.(D12) in its whole domain of definition. Thus, for small values of x and y we obtain the development of this constant of the motion in the form:

$$x^2 + y^2 + 2bx^2y + (4b + 4b' + 2d)y^3/3 + \ldots \tag{D18}$$

Hence the origin is a center in this case.

If $b = b' = -d$, the equation (D14) has as a particular integral $v = \mu + \gamma z^2$ with constant of motion (D17) with $\nu = 0$, $\gamma = \mu = -1/(8b^2)$.

If $b = b' \neq -d$, the variation of constants for the linear equation (D14) gives the constant of motion

$$(1 + Ay)^{-1}[4b^3x^2 + 2b + d + 2(b + d)(1 + Ay)\, \ell n\, |1 + Ay| - d(1 + Ay)^2] \tag{D17.1}$$

If $b = 2b' = -A$, we obtain for (D14) the constant of motion

$$(1 + Ay)^{-2}(b^3x^2 + (b + d) - 2(b + 2d)(1 + Ay) - 2d(1 + Ay)^2\, \ell n\, |\, 1 + Ay\, |) \tag{D17.2}$$

The constants of motion (D17.1) and (D17.2) have developments around the origin similar to (D18). The origin is a center for these cases too. We point out that in all the cases described above the constants of motion are "global" being defined for $1+Ay \neq 0$.

If $A = 0$, using the change $v = x^2/2$ as before we obtain the equation

$$\frac{dv}{dy} + 2bv = -y - dy^2$$

The constant of motion in this case is

(D17.3) $$(2b^3x^2 + 2b(b-d)y + 2b^2dy^2 + d - b)e^{2by}$$

and it is defined on the entire plane.

Case III: The constant of motion in this case can be computed using Dulac's method as it was done in [30]. For the sake of completeness we give here this calculation which illustrates well the technique of complexification. We assume here $a = -c \neq 0$ (the case $a = c = 0 = \beta$ was considered previously in IIa). By a similarity transformation we can make $a = -c = 1$. Thus the equation in this case becomes

(D19) $$\frac{dy}{dx} = -\frac{x + y^2 + (2b+\alpha)xy - y^2}{y + bx^2 - 2xy + dy^2}$$

By hypothesis we have $\beta = 0 = \alpha+5(b+d) = a^2+2d^2+bd$. Since $a \neq 0$, we have $d \neq 0$. Hence

(D20) $$\alpha = -5(b+d) \text{ and } b = -(2d^2+1)/d$$

Using (D20), we eliminate α in (D19) and rewriting the equation we obtain:

(D21) $$[x+x^2-(3b+5d)xy-y^2]\,dx + [y+bx^2-2xy+dy^2]\,dy = 0$$

We now consider this equation over the complex numbers. We choose a basis of eigenvectors and we make the corresponding coordinate change:

(D22) $$x = (x' - y')/2i, \qquad y = (x' + y')/2$$

We obtain the equation:

(D23) $$[y' + (b+3d+2i)x'^2/2 + (b+d)x'y'/2 - (b+d)y'^2]\,dx'$$
$$+ [x' - (b+d)x'^2 + (b+d)x'y'/2 + (b+3d-2i)y'^2/2]\,dy' = 0$$

Dividing by 2 and then applying the transformation

(D24) $$x' = -2x_1/(b+d), \qquad y' = -2y_1/(b+d)$$

We obtain:

(D25) $$[y_1 - (b+3d+2i)x_1^2/(b+d) - x_1y_1 + 2y_1^2]\,dx_1$$
$$+ [x_1 + 2x_1^2 - x_1y_1 - (b+3d-2i)y_1^2/(b+d)]\,dy_1 = 0$$

We put

(D26) $a' = -(b+3d+2i)/(b+d)$, $b' = -(b+3d-2i)/(b+d)$

Using the expression for b in (D26) we prove that

(D27) $a'b' = 1$

Using a', b' in the equation (D25) we have

(D28) $[y_1 + 2y_1^2 - x_1y_1 + a'x_1^2]\, dx_1 + [x_1 + 2x_1^2 - x_1y_1 + b'y_1^2]\, dy_1 = 0$

The equation (D28) subject to the condition (D27) has an integrating factor

(D29) $[f(x_1,y_1)]^{-5/2} = [1+2x_1 + 2y_1 + a'x_1^2 + 2x_1y_1 + b'y_1^2]^{-5/2}$

and the general integral (from which we can then obtain the constant of motion) is:

(D30) $[f(x_1,y_1)]^3 = C[F(x_1,y_1)]^2$

where

(D31) $F(x_1, y_1) = 1 + 3(x_1 + y_1) + (3/2)[(a'+1)x_1^2 + (a'+b'+2)x_1y_1 + (b'+1)y_1^2]$
$+ (1/2)[a'(a'+1)x_1^3 + 3(a'+1)x_1^2y_1 + 3(b'+1)x_1y_1^2 + b'(b'+1)y_1^3]$

Returning to the variables x,y we get the general integral

(D32) $[d^2+2d(d^2+1)y+(d^2+1)(x-dy)^2]^3 =$
$C[d^2 + 3d(d^2+1)y+3d^2(d^2+1)y^2 - 3d(d^2+1)xy+(d^2+1)(dy-x)^3]^2$

On the complement of an algebraic curve this integral is of the form H(x,y) = C. Developing H around the origin we get

(D33) $d^2 + 6(d^2+1)(x^2+y^2) + 12(d^2+1)x^3 - (12/d)(d^2+1)(2d^2+1)x^2y +$
$6(d^2+1)xy^2 - (12/d)(d^2+1)(4d^2+5)y^3 + \ldots = C$

Hence the origin is a center in case III too and thus we have proved the sufficiency of the three conditions.

REMARK 2 We point out that we have two cases when the constants of motion are polynomial functions, i.e. the hamiltonian case II_α and the case II_a with b a non-negative multiple of A.

CONCLUDING COMMENT

After this article was completed, our attention was drawn by T. Blows to the reference [7]. A clear analysis of the problem of the center based on the work of Kapteyn is given there (cf. ch. 7). The account in this article is different and more extensive since it covers systems in the general form (C1) with a and - c not necessarily equal. We first showed how the failed "proofs" of Frommer [10] and Ye could be corrected and completed. Furthermore, the constants of motion we found are explicitly expressed in terms of the variables and the coefficients of the system by means of simple formulas. (These may be of use later on, in the study of perturbations of integrable systems).

Not all formulas in [7] satisfy these requirements.Also we should point out that the formula (7.85) of [7], obtained for the case I is valid only locally around the origin, where the real polynomial functions f_1, f_2, f_3 are all positive and hence $f_1^{\lambda 1}\ f_2^{\lambda 2}\ f_3^{\lambda 3}$ is only a local constant of motion. An analogous observation can be made for the formula (7.72) given in [7] for the case IIa. More importantly, we should point out to the reader that the formula for α_7 in [7] is exactly the one of Bautin in [3] and is therefore incorrect as we indicated in Section C. To correct it, one needs to make a change of sign.

In addition, we were concerned with the problem of "global" integrability in the sense that we wanted our formulas to be valid not just locally but in as large a region as possible, a region containing the homoclinic orbits whenever these exist. We have discussed this question in detail, showing that in all cases the formulas work on either $\mathbb{R}^2$ or on the complement of some curve. In fact using our observations for the cases I and II as well as observations we made about the case III as it is treated in [7] we obtain the following:

THEOREM Any quadratic system with a center is integrable on $\mathbb{R}^2$ or on the complement of an algebraic curve C . The curve C on which the integrability breaks down is an invariant curve of the system in all but one of the cases and it passes through all the remaining singular points which happen to be sinks, sources or saddle-nodes.

Proof: We covered the cases I and II before and the analysis of the singular points can easily be completed by the reader.
For case III, the authors of [7] observe that by writing $\lambda = d^2/(d^2+a^2)$ and by making the coordinate change $X = -ax+dy$, $Y = dy$, the equation (D2) reduces in case III to the equation

(D34) $$\frac{dX}{dY} = \frac{\lambda X - Y + X^2 - XY}{\lambda X - \lambda Y - \lambda X^2 + 3XY - 2Y^2}$$

Using the transformation

(D35) $$\xi = X^2 + 2Y + \lambda \quad , \quad \eta = X^3 + 3XY + 3Y + \lambda$$

one can show that the equation (D34) leads to the equation

(D36) $$\frac{1}{2}\,\eta\, d\xi \ - \frac{1}{3}\,\xi d\eta \ = \ 0$$

with general integral $\eta^2 =$ (const) ξ^3 and hence

(D37) $$(X^3+3XY+3Y+\lambda)^2 = \text{(const)}\ (X^2+2Y+\lambda)^3$$

is the general integral for (D34). Thus we obtain a constant of motion which is defined for $X^2+2Y+\lambda \neq 0$.

The curves

(D38) $$\begin{aligned} dX^3+3XY+3Y+\lambda &= 0 \\ X^2+2Y+\lambda &= 0 \end{aligned}$$

are clearly invariant curves of (D34) and they have only one point of intersection (X_0,Y_0). This is the only singular point of (D34) which is different from (0,0). The transformation (D35) is a diffeomorphism in a neiborhood of this point. It follows that this point is a node.

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