Chapter 14
Three Ways of Treating a Linear Delay Differential Equation

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Abstract This work concerns the occurrence of Hopf bifurcations in delay differential equations (DDE). Such bifurcations are associated with the occurrence of pure imaginary characteristic roots in a linearized DDE. In this work we seek the exact analytical conditions for pure imaginary roots, and we compare them with the approximate conditions obtained by using the two variable expansion perturbation method. This method characteristically gives rise to a "slow flow" which contains delayed variables. In analyzing such approximate slow flows, we compare the exact treatment of the slow flow with a further approximation based on replacing the delayed variables in the slow flow with non-delayed variables, thereby reducing the DDE slow flow to an ODE. By comparing these three approaches we are able to assess the accuracy of making the various approximations. We apply this comparison to a linear harmonic oscillator with delayed self-feedback.

Keywords Slow flow · Delay · Hopf bifurcation

14.1 Introduction

It is known that ordinary differential equations (ODEs) are used as models to better understand phenomenon occurring in biology, physics and engineering. Although these models present a good approximation of the observed phenomenon, in many cases they fail to capture the rich dynamics observed in natural or technological systems. Another approach which has gained interest in modeling systems is the inclusion of time delay terms in the differential equations resulting in delay-differential

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equations (DDEs). DDE's have found application in many systems, including rotating machine tool vibrations [7], gene copying dynamics [14], laser dynamics [16] and many other examples.

Despite their simple appearance, DDEs have several features that make their analysis a challenging task. For example, when investigating a DDE by use of a perturbation method, one is often confronted with a slow flow which contains delay terms. It is often argued that since the parameter of perturbation, call it ε, is small, $\varepsilon << 1$, the delay terms which appear in the slow flow may be replaced by the same terms without delay, see e.g. [1, 5, 6, 8, 9, 13, 15, 16]. The purpose of the present paper is to compare the exact Hopf bifurcation curves to the approximate curves obtained by analyzing the slow flow. In particular, we compare the exact treatment of the approximate slow flow with a treatment involving a further approximation based on replacing the slow flow delay terms with terms without delay. We consider the case of a linear delay differential equation and look for the smallest delay T such that the following system has pure imaginary eigenvalues, a setup for a Hopf, depending on the nonlinear terms (omitted here):

$$x'' + x = \varepsilon\, k\, x_d, \qquad \text{where} \qquad x_d = x(t-T) \tag{14.1}$$

To this aim we adopt three methods. The first method consists of solving for the exact solution of the characteristic equation. In the second and third methods, we use a perturbation method, the two variables expansion method, to produce a slow flow. In the second method we keep the delayed variables in the slow flow, while in the third method we replace the delayed variables by non-delayed variables.

14.2 First Method

For the first method we adopt an exact treatment of (14.1) without assuming ε is small. To begin we assume a solution to (14.1) in the form

$$x = \exp(rt) \tag{14.2}$$

Substituting (14.2) into (14.1) yields

$$r^2 + 1 - k\,\varepsilon \exp(-rT) = 0 \tag{14.3}$$

It is known that for a Hopf bifurcation to occur the real parts of a pair of eigenvalues of the characteristic equation, i.e. r's of (14.3), must cross the imaginary axis leading to a changing of sign of this pair of eigenvalues (Rand [10], Strogatz [12]). In other words the origin, which is an equilibrium point for (14.1), changes stability from source to sink or vice-versa. Therefore to find the critical delay T_{cr} causing a Hopf bifurcation, we set $r = i\omega$ giving

$$1 - \omega^2 - k\,\varepsilon \cos \omega t = 0, \qquad \sin \omega t = 0 \tag{14.4}$$

Solving the second equation in (14.4), we obtain $\omega T_{cr} = n\pi, \quad n = 1, 2, 3, \ldots$. However since we are only interested in the smallest delay T causing Hopf bifurcation, we only consider the two solutions $\omega T_{cr} = \pi$ and 2π. Replacing these two solutions in the first equation in (14.4) and solving for ω and T we obtain:

$$\omega = \sqrt{1 + k\,\varepsilon} \tag{14.5}$$

$$T_{cr} = \frac{\pi}{\omega} = \frac{\pi}{\sqrt{1 + k\varepsilon}} \tag{14.6}$$

and

$$\omega = \sqrt{1 - k\,\varepsilon} \tag{14.7}$$

$$T_{cr} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1 - k\varepsilon}} \tag{14.8}$$

Note that (14.6) and (14.8) are exact values for the delay T for which a Hopf bifurcation may occur. In the two next sections we consider the parameter ε in (14.1) to be small, i.e. $\varepsilon << 1$, and we use a perturbation method to derive a slow flow. We note that using the two variable expansion method gives the same slow flow as would be obtained by using the averaging method [11].

14.3 Second Method

The two variable method posits that the solution depends on two time variables, $x(\xi, \eta)$, where $\xi = t$ and $\eta = \varepsilon t$. Then we have

$$x_d = x(t - T) = x(\xi - T, \eta - \varepsilon T) \tag{14.9}$$

Dropping terms of $O(\varepsilon^2)$, (14.1) becomes

$$x_{\xi\xi} + 2\varepsilon x_{\xi\eta} + x = \varepsilon\, k\, x(\xi - T, \eta - \varepsilon T) \tag{14.10}$$

Expanding x in a power series in ε, $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$, and collecting terms, we obtain

$$Lx_0 \equiv x_{0\xi\xi} + x_0 = 0 \tag{14.11}$$

$$Lx_1 \equiv k\, x_0(\xi - T, \eta - \varepsilon T) - 2x_{0\xi\eta} \tag{14.12}$$

From (14.11) we have that

$$x_0(\xi, \eta) = A(\eta)\cos\xi + B(\eta)\sin\xi \tag{14.13}$$

In (14.12) we will need $x_0(\xi - T, \eta - \varepsilon T)$:

$$x_0(\xi - T, \eta - \varepsilon T) = A_d \cos(\xi - T) + B_d \sin(\xi - T) \tag{14.14}$$

where $A_d = A(\eta - \varepsilon T)$ and $B_d = B(\eta - \varepsilon T)$.

Substituting (14.13) and (14.14) into (14.12) and eliminating resonant terms gives the slow flow:

$$\frac{dA}{d\eta} = -\frac{k}{2} A_d \sin T - \frac{k}{2} B_d \cos T \tag{14.15}$$

$$\frac{dB}{d\eta} = -\frac{k}{2} B_d \sin T + \frac{k}{2} A_d \cos T \tag{14.16}$$

where $A_d = A(\eta - \varepsilon T)$ and $B_d = B(\eta - \varepsilon T)$. We set

$$A = a\exp(\lambda\eta), \quad B = b\exp(\lambda\eta), \quad A_d = a\exp(\lambda\eta - \varepsilon\lambda T), \quad B_d = b\exp(\lambda\eta - \varepsilon\lambda T) \tag{14.17}$$

where a and b are constants. This gives

$$\begin{bmatrix} -\lambda - \frac{k}{2}\exp(-\lambda\varepsilon T)\sin T & -\frac{k}{2}\exp(-\lambda\varepsilon T)\cos T \\ \frac{k}{2}\exp(-\lambda\varepsilon T)\cos T & -\lambda - \frac{k}{2}\exp(-\lambda\varepsilon T)\sin T \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{14.18}$$

For a nontrivial solution (a, b) we require the determinant to vanish:

$$\left(-\lambda - \frac{k}{2}\exp(-\lambda\varepsilon T)\sin T\right)^2 + \frac{k^2}{4}\exp(-2\lambda\varepsilon T)\cos^2 T = 0 \tag{14.19}$$

We set $\lambda = i\omega$ for a Hopf bifurcation and use Euler's formula $\exp(-i\omega\varepsilon T) = \cos\omega\varepsilon T - i\sin\omega\varepsilon T$. Separating real and imaginary parts we obtain

$$4k^2\cos 2\varepsilon\omega T + 16k\omega\sin T\sin\varepsilon\omega T - 16\omega^2 = 0 \tag{14.20}$$

$$-4k^2\sin 2\varepsilon\omega T + 16k\omega\sin T\cos\varepsilon\omega T = 0 \tag{14.21}$$

At this stage we adopt the technique used in [11] to analytically solve for the pair (ω, T). Following [11] we obtain the exact solutions of (14.20)–(14.21), giving the delay for which Hopf bifurcation occurs

$$T_{cr_1} = \frac{\pi}{1 + \varepsilon\, k/2} \tag{14.22}$$

$$|\varepsilon\, k/2| < 1$$

$$T_{cr_2} = \frac{2\pi}{1 - \varepsilon\, k/2} \tag{14.23}$$

We note that the denominator of (14.22)–(14.23) are just the Taylor expansions to the first order of the exact frequency given by (14.5)–(14.7). In the next section we replace the delayed variables in (14.15)–(14.16), i.e. A_d and B_d, by the non-delayed variables A and B.

14.4 Third Method

The slow flow given by (14.15)–(14.16) is replaced by a slow flow with no delayed variables such that:

$$\frac{dA}{d\eta} = -\frac{k}{2} A \sin T - \frac{k}{2} B \cos T \tag{14.24}$$

$$\frac{dB}{d\eta} = -\frac{k}{2} B \sin T + \frac{k}{2} A \cos T \tag{14.25}$$

In order to find the critical delay we proceed as in the previous section. The corresponding characteristic equation has the form:

$$\lambda^2 + k\,\lambda\, \sin T + \frac{k^2}{4} = 0 \tag{14.26}$$

For a Hopf bifurcation, we require imaginary roots of the characteristic equation. This gives

$$k \sin T = 0 \tag{14.27}$$

Solving the above equation for the critical delay T we obtain:

$$T_{cr} = \pi \tag{14.28}$$

$$T_{cr} = 2\pi \tag{14.29}$$

Figure 14.1 shows a plot of the Hopfs in the $k - T$ parameter plane. From the figure we remark that the exact delay that is obtained from solving (14.4) agrees with the numerical Hopf bifurcation curves (blue) that is obtained by using the DDE-BIFTOOL continuation software [2–4]. The Hopf curves given by the second method offer a good approximation to the numerical Hopf curves, in particular for small values of the feedback magnitude k. However, the curves obtained from the third method do not agree with the numerical Hopf curves.

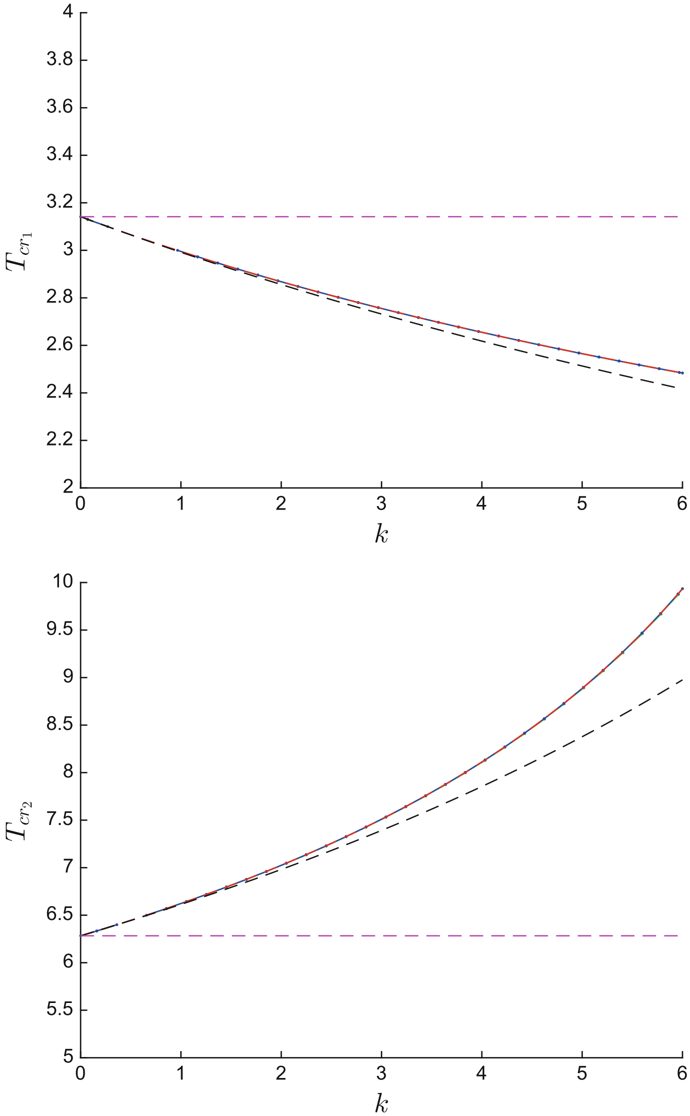

Fig. 14.1 Critical delay versus the feedback magnitude k. *Top*: *blue* (BIFTOOL), *red* (14.6), *black* (14.22), *magenta* (14.28). *Bottom*: *blue* (BIFTOOL), *red* (14.8), *black* (14.23), *magenta* (14.29). Parameter $\varepsilon = 0.1$

14.5 Conclusion

When a DDE with delayed self-feedack is treated using a perturbation method (such as the two variable expansion method, multiple scales, or averaging), the resulting slow flow typically involves delayed variables. In this work we compared the behavior of the resulting DDE slow flow with a related ODE slow flow obtained by replacing the delayed variables in the slow flow with non-delayed variables and comparing the resulting approximate critical delays causing Hopf bifurcation with the exact analytical Hopf curves. We studied a sample system based on the harmonic oscillator with delayed self-feedback, (14.1). We found that replacing the delayed variables in the slow flow by non-delayed variables fails to give a good approximation. However, keeping the delayed variables in the slow flow gives better results.

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