A Bifurcation Analysis of the Quadratically Damped Mathieu Equation and Its Applications to the Dynamics of Submarine Towed-Array Lifting Devices

Deepak V. Ramani
Naval Undersea Warfare Center
Newport, RI 02841

Richard H. Rand
Dept. of Theoretical & Applied Mechanics
Cornell University, Ithaca, NY 14853

William L. Keith
Naval Undersea Warfare Center
Newport, RI 02841

Abstract

This work concerns the dynamics of towed array lifting devices (LFDs), which help to maintain the position of a tow line being dragged behind a vessel. The LFD satisfies the non-dimensional quadratically damped Mathieu equation

$$\ddot{x} + (\delta + \epsilon \cos t)\, x + \dot{x}\, |\dot{x}| = 0.$$

Numerical study of this equation shows a wide array of dynamical features. The dynamical features of the system are exploited to obtain a control strategy for maintaining the LFD in the desired state.

Introduction

Submarine use of passive towed arrays affords increased sonar capability [1]. The objective here is to deploy a multiline array which can be remotely configured for optimum acoustic sensing capabilitiy. That is, a number of individual lines deploy through a single port and fan out to form a three-dimensional, volumetric array of individual sensors. By maintaining a fixed ship bearing and line configuration, composite sensor signals can be analyzed to determine the location and bearing of any acoustic emission source. Deploying and maintaining the position of individual lines comprising a volumetric array requires knowledge of the instantaneous position of each line relative to a fixed point on the ship or relative to the other lines. This must be done in a complex, unsteady ocean environment which is complicated by the turbulent flows associated with the towing vessel and the line themselves. Aperture generation is currently accomplished through the use of small lifting devices, called "lateral force devices" or LFDs.

The dynamics of an LFD are complicated by changes in the tow line tension due to flow-induced vibration caused by coherent turbulent structures. These structures can result from the turbulent boundary layer on the tow line upstream of the LFD and from vortex shedding off of the tow line due to crossflow. Full scale experiments in a towing tank have shown that an LFD can exhibit unstable motions under particular conditions.

Previous work on this problem presented in [2] and [3] derived the equation of motion and carried out both a linear stability analysis of the quadratic Mathieu equation and a nonlinear analysis for small values of ϵ. The goal of the current work is to extend the numerical treatment of the problem to better understand both the bifurcations in the system and their impact on the physical system dynamics.

Simplified Model

We investigate the motion of a simplified model of an LFD. The system along with the acting forces is shown in Figure 1. We assume that the towline connecting the LFD to the submarine is rigid, and can therefore withstand compression. The tension, T, is assumed to have a sinusoidal forcing function

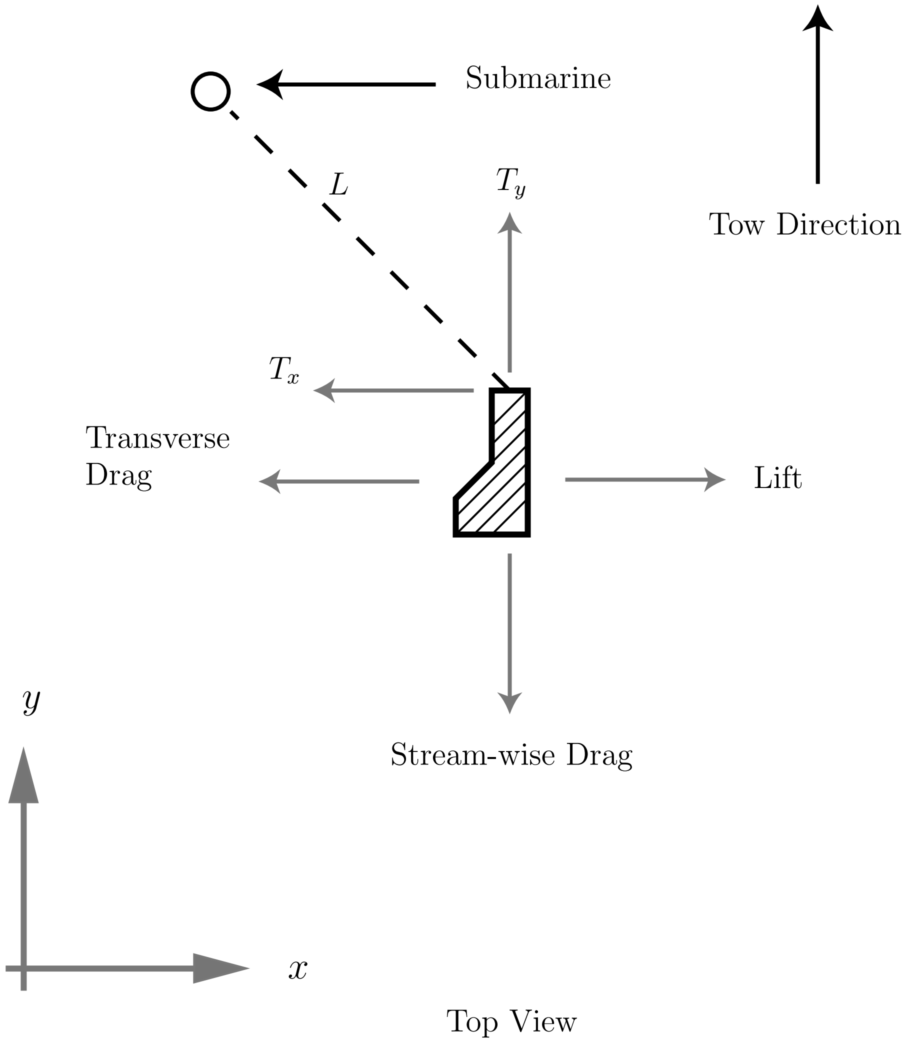

Figure 1: Free body diagram of submarine towing a plate. Figure is in a horizontal plane.

$T = T_0 + T_1 \cos \Omega t$. Using this form of the tension, the equation of motion in the x-direction becomes

$$m\ddot{x} + T\frac{x}{L} + c\dot{x}\,|\dot{x}| = c_0 U(t)^2. \tag{1}$$

The force exerted by the tow cable on the LFD is reflected in the term Tx/L. We neglect changes in the length of the tow cable and treat L as a constant. The LFD is modeled as a plate oriented so that a normal to the plate face will point in the x-direction. The term $c\dot{x}\,|\dot{x}|$ is a fluid drag force, while the term $c_0U(t)^2$ is a fluid lift force. In what follows we assume that the lift force is negligible, which is equivalent to assuming that the angle of attack of the plate with respect to the towing direction is zero.

Equation (1) may be rescaled to take on the following non-dimensional form:

$$\ddot{x} + (\delta + \epsilon \cos t)\, x + \dot{x}\,|\dot{x}| = 0. \tag{2}$$

In Equation (2), δ represents the non-dimensional mean value of the towline tension and ϵ represents the amplitude of the oscillating portion of the towline tension. Equation (2) is the quadratically damped Mathieu equation. Despite the simplifications made in the model of this system, Equation (2) is a nonlinear equation, and as such is expected to exhibit a wide range of dynamical behavior.

Linear Stability And Small ϵ Results

Equation (2) admits the exact solution $x \equiv 0$. The stability of this solution is governed by the linear Mathieu equation, Equation

$$\ddot{x} + (\delta + \epsilon \cos t)\, x = 0. \tag{3}$$

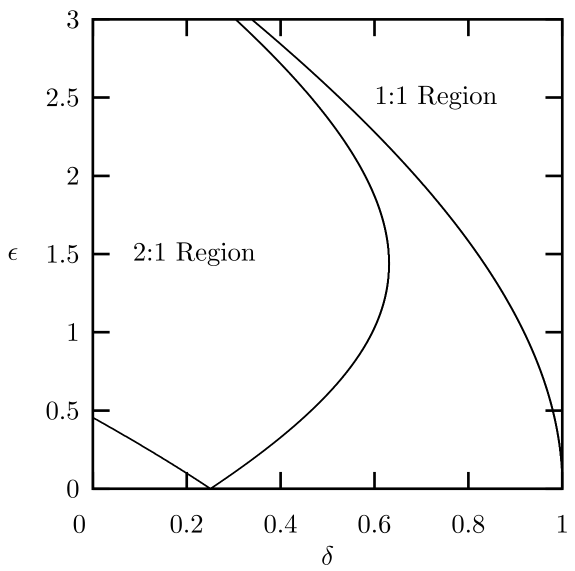

Figure 2: Transition curves of the linear Mathieu equation

The origin is considered stable if all solutions of Equation (3) are bounded, and unstable if an unbounded solution exists. The stability treatment of Equation (3) demonstrates the existence of regions in the δ-ϵ plane, called tongues, which emanate from the δ-axis at points $\delta = n^2/4$, where $n = 0, 1, 2, 3, \cdots$ [4]. Inside the tongues, the origin is unstable, while outside the tongues, the origin is stable. The tongues of instability are said to be bounded by transition curves. Because the linear Mathieu equation governs the stability of the origin in the quadratically-damped Mathieu equation, the transition curves of the linear Mathieu equation represent bifurcation curves for the quadratically-damped Mathieu equation.

Although the linear stability analysis predicts unbounded growth inside the tongues, this is not the case in the nonlinear Equation (2). Inside the tongues, the nonlinear damping in Equation (2) balances the parametric resonance, leading to the existence of a periodic motion. The method of averaging (see [5]) can be used both to show that periodic motions exist inside the instability tongues, and to obtain an approximation to these periodic motions, valid for small ϵ. The details of this calculation are given in [3]. These results predict that at points lying inside the tongue emanating from $\delta = 1/4, \epsilon = 0$, Equation (2) exhibits an attractive 2:1 subharmonic motion having period 4π. For this reason the points lying inside this tongue will be referred to as the 2:1 region. Similarly, at points lying inside the tongue emanating from $\delta = 1, \epsilon = 0$, Equation (2) is predicted to exhibit a pair of attractive 1:1 periodic motions, each having period 2π. This region will be referred to as the 1:1 region.

Determination Of The Secondary Bifurcation

Numerical explorations of the nonlinear quadratically-damped Mathieu Equation (2) may be accomplished by generating a Poincaré map corresponding to a surface of section $t = 0 \bmod 2\pi$. Using this technique, a variety of periodic motions are observed, depending upon where we are in the $\delta - \epsilon$ parameter plane. Figure 3 shows schematically the different Poincaré map portraits that are exhibited by Equation (2). In these diagrams, periodic motions appear as fixed points.

We may summarize the features displayed in Figure 3 as follows: Outside the instability regions, the origin is always stable, as indicated by a lone spiral to the origin. Inside the instability regions, the origin is unstable, as indicated by a saddle-like x at the origin. Inside the 2:1 region the two spiral singularities in the Poincaré map represent a single period 4π motion, whereas in the 1:1 region they represent two period 2π motions. As the transition curves are crossed into 1:1 region or into the 2:1 region below point P, a supercritical pitchfork bifurcation occurs, and two new stable singular points are created in the Poincaré map, while the origin itself becomes unstable. As the 2:1 region is exited above point P into the region marked B (see Figure 3), a subcritical pitchfork bifurcation occurs. In this case, the origin becomes stable and an unstable 2:1 subharmonic periodic motion is created. As region B is exited into region C, the 1:1 transition curve is crossed, and the expected supercritical pitchfork bifurcation curve takes place at the origin. The origin once more becomes unstable, while two stable period 2π motions are born out of the origin.

Perhaps the most interesting feature displayed in Figure 3 corresponds to what happens when we move from either of regions B or C downward across the nearly-straight line bifurcation curve emanating from point P. In this case the two outermost periodic orbits – the stable and unstable period 4π orbits – are destroyed in a saddle-node bifurcation. It is seen that this saddle-node bifurcation does not take place at the origin.

It is desirable that the LFD not oscillate after deployment. This corresponds to the solution $x \equiv 0$. It would seem that a good operating policy would be to operate in regions of the parameter plane where the origin is a stable motion. However, the preceding analysis of the system demonstrates that this may not be an implementable plan. In particular, above the secondary bifurcation curve in Figure 3 there exist two stable states, the origin and the stable period 4π motion. The question of which state the system will settle into depends on the initial condition of the system. However, in most cases, the initial condition for the system is not something that can be prescribed.

In the bistable region, a method of returning the system to the origin is needed if the motion is currently in the basin of attracion of the stable periodic motion. A method of doing this is suggested by the existence of a hysteretic loop in the system. In the bistable region, region B in Figure 3, both the origin and the period 4π motion are stable. These two stable motions are separated by an unstable periodic motion of period 4π.

If the system is undergoing the stable periodic motion, it can be returned to the origin with the following method, illustrated in Figure 4. In the figure dark lines are stable motions and grey lines are unstable motions. The arrows indicate the path that the system takes in following its stable motions. The letters A and B indicate the minimum (starting) and maximum values of δ respectively. If δ is increased, the amplitude of the stable period 4π motion will decrease until eventually it disappears. On the figure this is indicated as the system following the upper black curve until that curve and the shaded grey curve meet in a vertical tangency and both disappear. This corresponds to the saddle-node bifurcation of limit cycles. When the 4π periodic motion disappears, the system will settle onto another stable motion that is present. In this case, it is a period 2π motion that was born when δ crossed into the 1:1 instability region. The value of δ is now reduced to its original value. The system will follow the stable period 2π motion until it disappears at the transition curve bounding the 1:1 instability region. The system will now be at the origin and should remain there.

This method can be tested by direct integration of the system. In Figure 5 the system is begun at $\delta = 0.5$ with the system in the period 4π motion. After establishing the periodic motion, the value of δ is increased until $\delta = 0.8$. At this point the system is in the 1:1 instability region and is locked into a period 2π motion. δ is then decreased back to $\delta = 0.5$. As this happens, the amplitude of the period 2π motion decreases to zero. Finally, Figure 5 shows a hundred-unit integration of the system at $\delta = 0.5$, establishing that the system is in a stable motion at the origin. The value of δ was changed according to a linear rule.

In terms of the physical system, increasing δ corresponds to retracting the tow cable. Decreasing δ corresponds to deploying the tow cable. This control strategy has been submitted for a patent application (see [6]).

CONCLUSIONS

The bifurcations in the quadratically-damped Mathieu equation were studied. Special focus was given to the region of the δ- ϵ parameter plane around point P, the point of infinite slope along the right transition curve of the 2:1 instability region. In this region a bifurcation sequence was numerically identified. It was observed that above P an unstable periodic motion is born by crossing out of the instability region. On the other hand, below P, a stable periodic motion is born by crossing into the instability region. Moreover, a secondary bifurcation curve in which the previously mentioned stable and unstable periodic motions merge, was seen to emanate from point P.

In terms of the LFD, it is desirable to operate in a parameter region for which the origin is asymptotically stable. For values of the parameters δ and ϵ which lie below the secondary bifurcation curve, there is a large region of parameter space for which the origin is an asymptotically stable solution. Above the secondary bifurcation curve, the region of the parameter plane where the origin is stable is quite small. Although it is possible to operate the LFD in this region, it is possible that the LFD might get trapped in a large amplitude oscillation. The system can be brought back to the origin by retracting and then deploying the system.

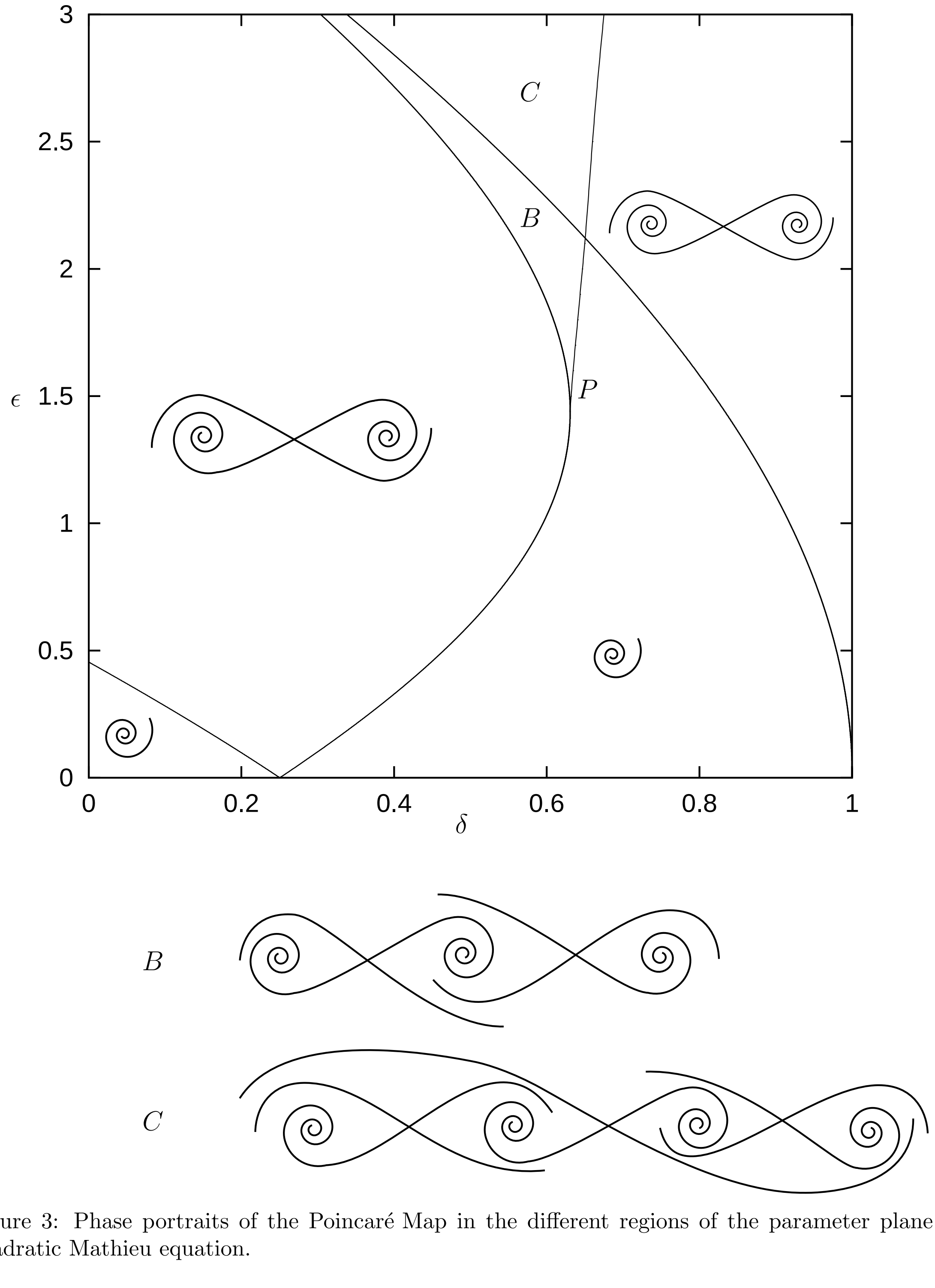

Figure 3: Phase portraits of the Poincaré Map in the different regions of the parameter plane in the quadratic Mathieu equation.

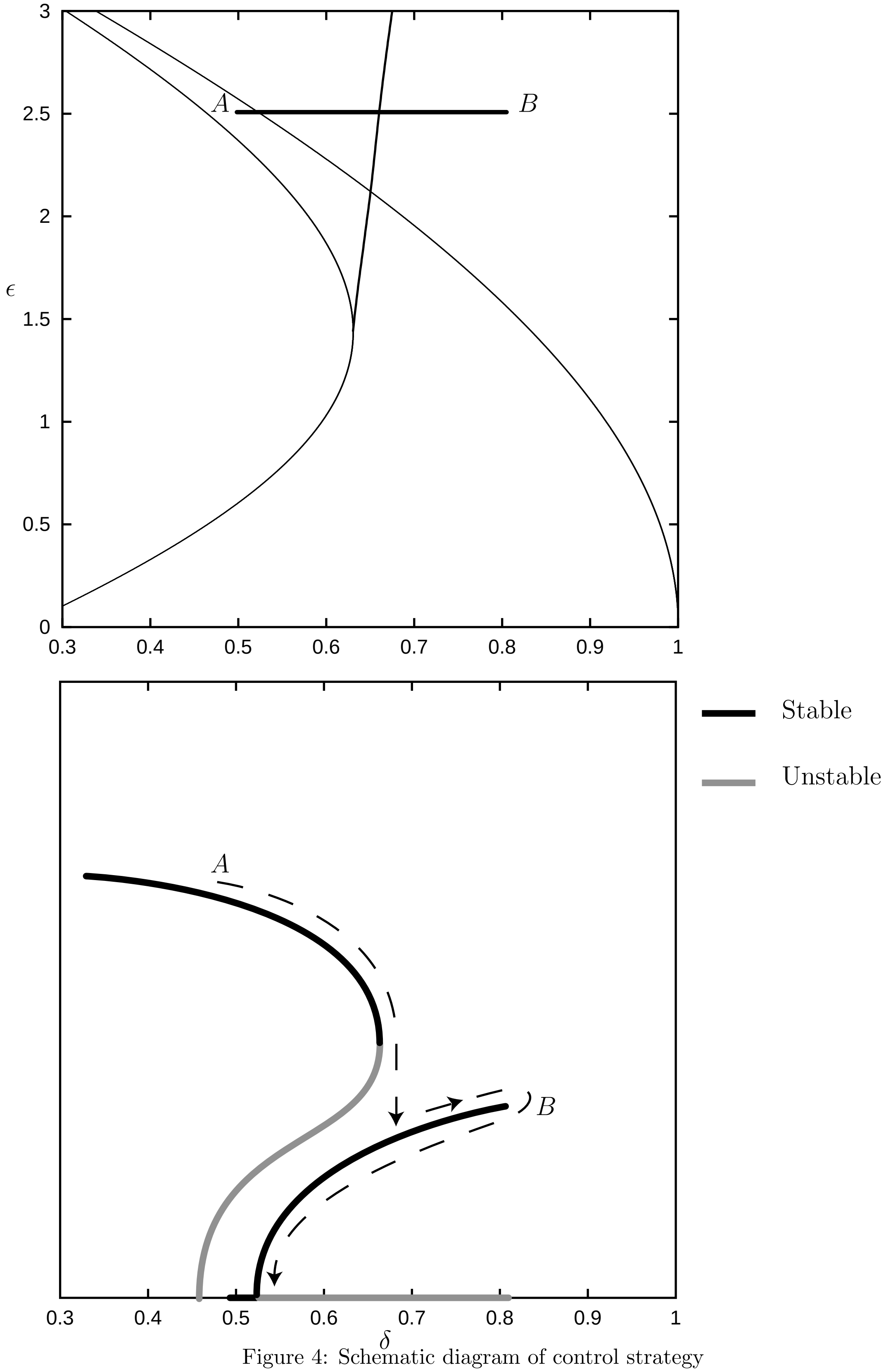

Figure 4: Schematic diagram of control strategy

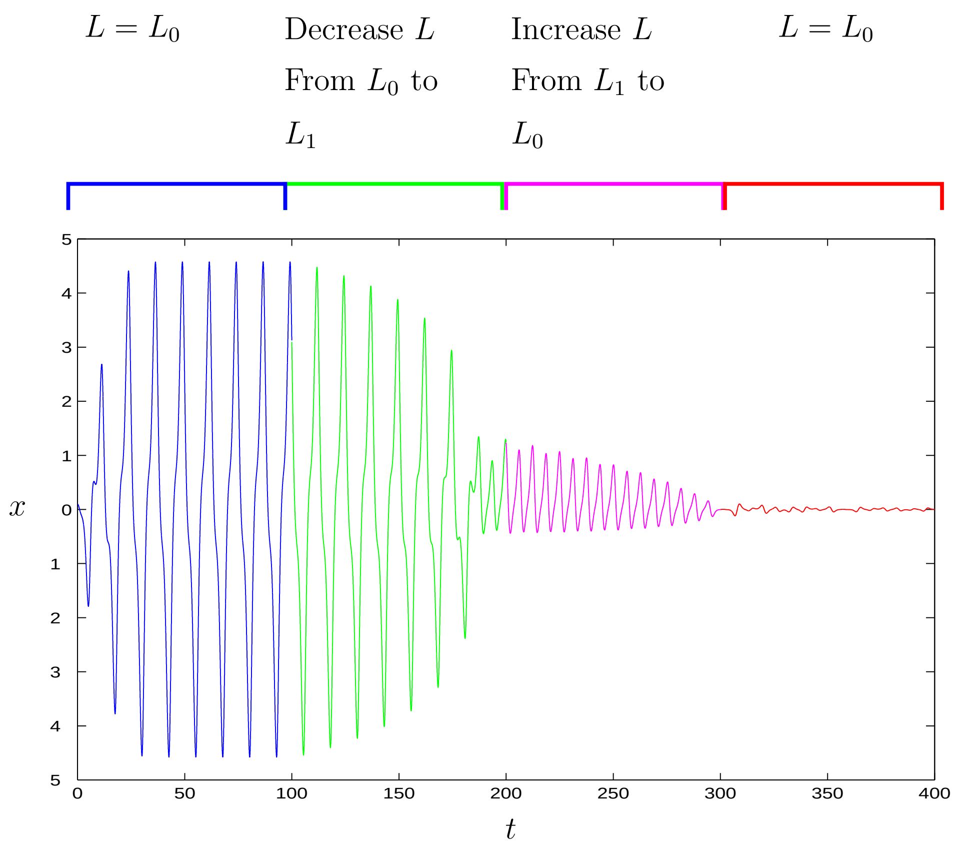

Figure 5: Numerical demonstration of control strategy

Further work that has been done on the system includes a determination of the secondary bifurcation curve by the application of a perturbation method. This calculation is given in [7] and in [8].

Acknowledgment

Author Ramani wishes to acknowledge support for this work in the form of a National Science Foundation Graduate Research Fellowship. This work was partially supported by the Office of Naval Research, Program Officer Dr. Roy Elswick, Code 321.

References

[1] K. M. Cipolla, W. L. Keith, and R. P. Norris, "Passive control of a towed sonar system with hydrodynamic lifting surfaces," *Proceedings of the Third International Symposium on Performance Enhancements for Marine Applications*, 1997.

[2] R. H. Rand, D. V. Ramani, W. L. Keith, and K. M. Cipolla, "Theoretical study of a submarine towed-array lifting device," in *Proceedings of the UDT (Undersea Defence Technology) Europe 2000 Conference (CD-ROM)*, (June 27–29, 2000, Wembley Conference Center, London, UK), June 2000.

[3] R. H. Rand, D. V. Ramani, W. L. Keith, and K. M. Cipolla, "The quadratically damped mathieu equation and its application to submarine dynamics," in *Proceedings of the 2000 ASME International Mechanical Engineering Congress and Exposition* (H. Tzou, M. Golnaraghi, and C. Radcliffe, eds.), vol. AD-Vol. 61 of *Control of Noise and Vibration: New Millenium*, (Nov. 5–10, 2000, Orlando, FL), pp. 39–50, ASME, November 2000.

[4] J. Stoker, *Nonlinear Vibrations*. New York: Wiley-Interscience, 1950.

[5] R. H. Rand, *Topics In Nonlinear Dynamics With Computer Algbera*. USA: Gordon and Breach Science Publishers, 1994.

[6] D. V. Ramani, R. H. Rand, and W. L. Keith, "A technique for nonlinear control of multiline towed arrays." Patent Application, June 2001.

[7] D. V. Ramani, *The Nonlinear Dynamics Of Towed Array Lifting Devices*. PhD thesis, Cornell University, Ithaca, NY, 2001.

[8] D. V. Ramani, R. H. Rand, and W. L. Keith, "Perturbation solution for bifurcation in the quadratically damped mathieu equaion," in *Proceedings of the 2001 ASME Design Engineering Technical Conferences*, (Sept. 9–13, 2001, Pittsburgh, PA), ASME, September 2001.