A polynomiality property for Littlewood-Richardson coefficients

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FPSAC 2004, Vancouver

Outline

- Littlewood-Richardson coefficients
- The hive model
- Partition functions
- The Steinberg arrangement
- Polynomiality in the chamber complex

Littlewood-Richardson coefficients

- Symmetric functions. LR coefficients express the multiplication rule for Schur functions:

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu \,.$$

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- Symmetric functions. LR coefficients express the multiplication rule for Schur functions:

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu \, .$$

- They also appear when writing skew Schur functions in terms of the Schur function basis:

$$s_{\nu/\lambda} = \sum_\mu c^\nu_{\lambda\mu} s_\mu \, .$$

- Representation theory of $\mathrm{GL}_k\mathbb{C}$. The character of the irreducible polynomial representation V_λ is the Schur function $s_\lambda(x_1, \ldots, x_k)$.

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- As such, the $c^\nu_{\lambda\mu}$ gives the multiplicity with which the irreducible representation V_ν of $\mathrm{GL}_k\mathbb{C}$ appears in the tensor product of the irreducible representations V_λ and V_μ:

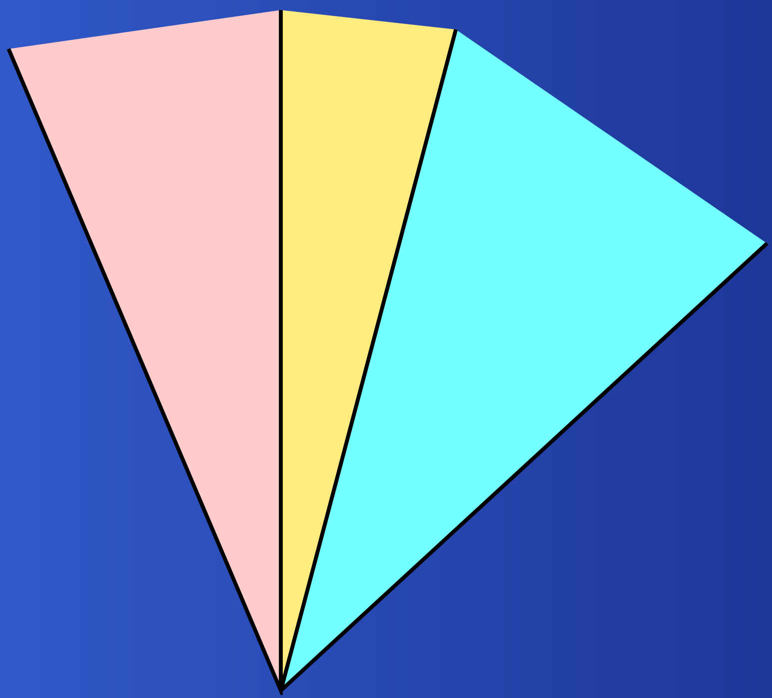
$$V_\lambda \otimes V_\mu \quad = \quad \bigoplus_\nu c^\nu_{\lambda\mu} V_\nu \quad = \quad \bigoplus_\nu V_\nu{}^{\oplus\, c^\nu_{\lambda\mu}}\,.$$

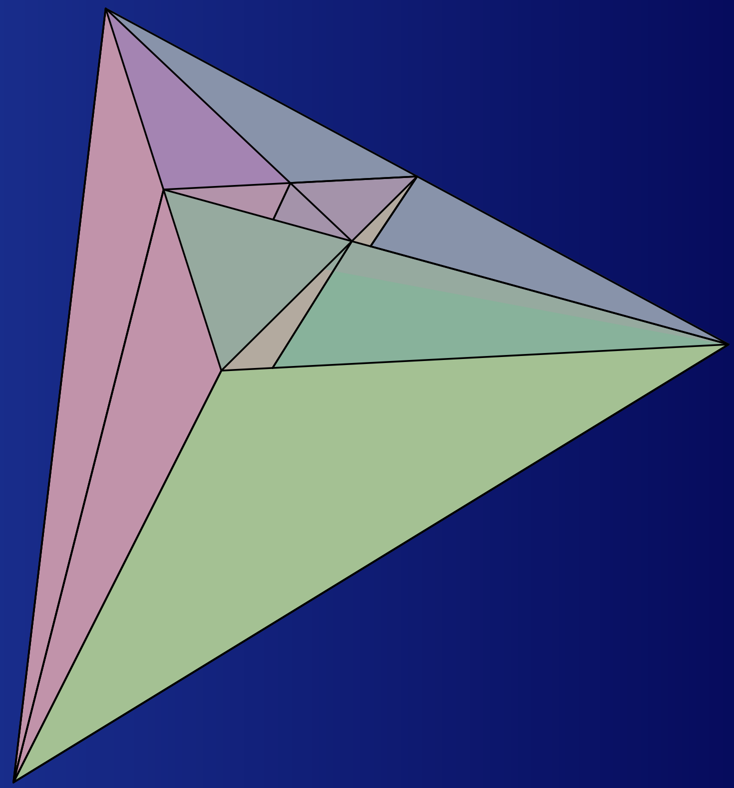
- Schubert calculus. Schubert classes form a linear basis of the cohomology ring of the Grassmannian, and the LR coefficients again express the multiplication rule:

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu \,.$$

The main result

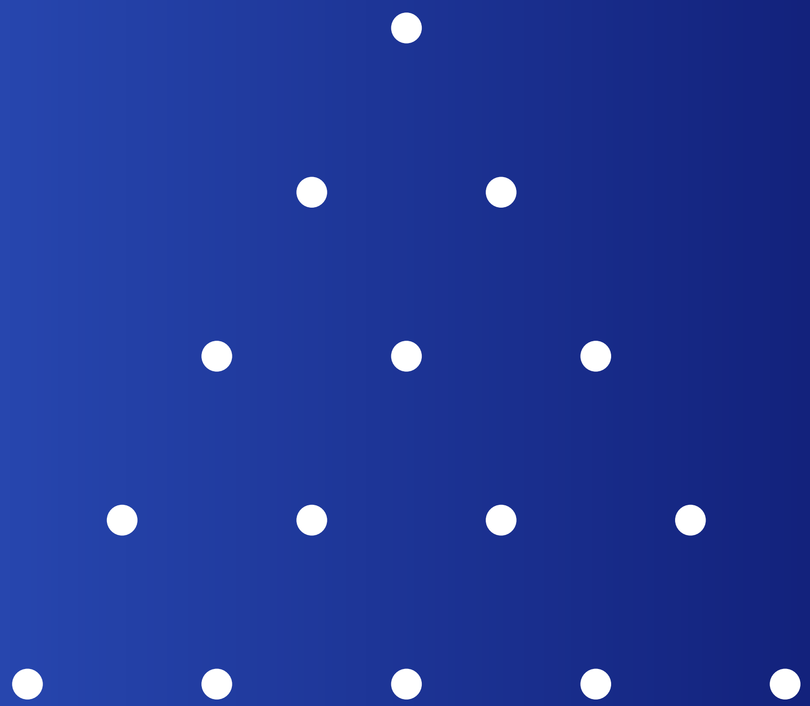
$$\left.\begin{array}{rcl} \lambda & = & (\lambda_1, \ldots, \lambda_k) \\ \mu & = & (\mu_1, \ldots, \mu_k) \\ \nu & = & (\nu_1, \ldots, \nu_k) \end{array}\right\} \quad |\lambda| + |\mu| = |\nu|$$



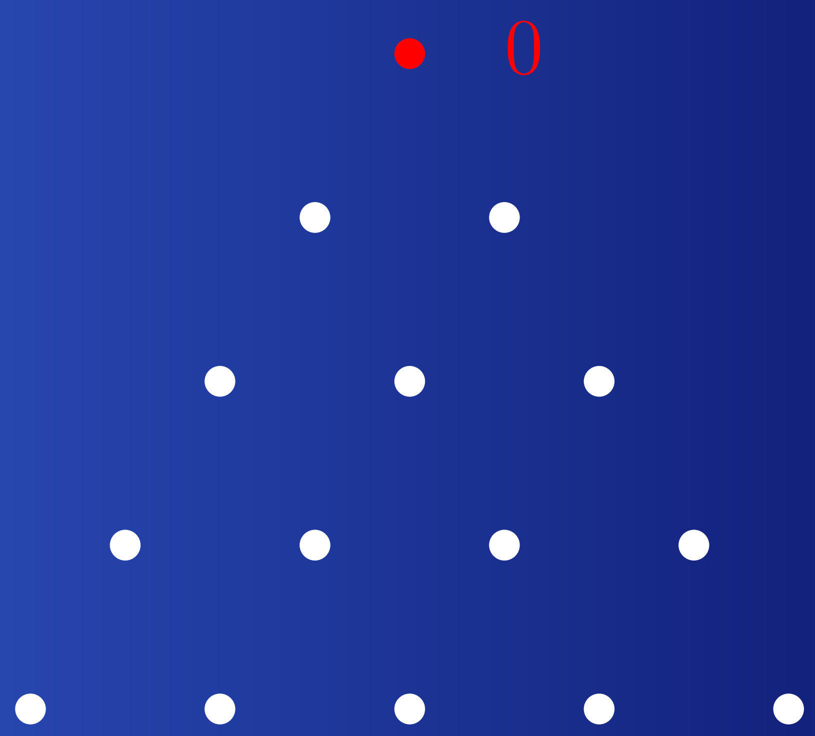


The hive model

An integral k-hive is an array of nonnegative integers of the form (here for $k = 4$):



Boundary conditions



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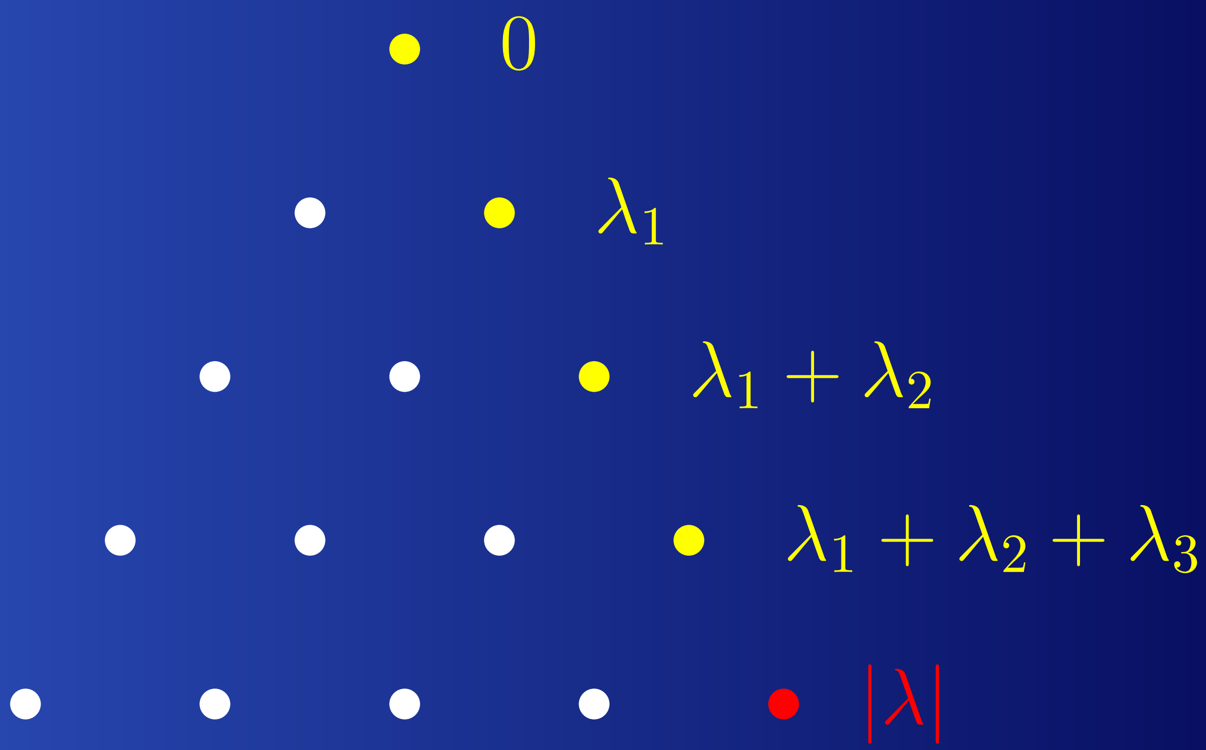

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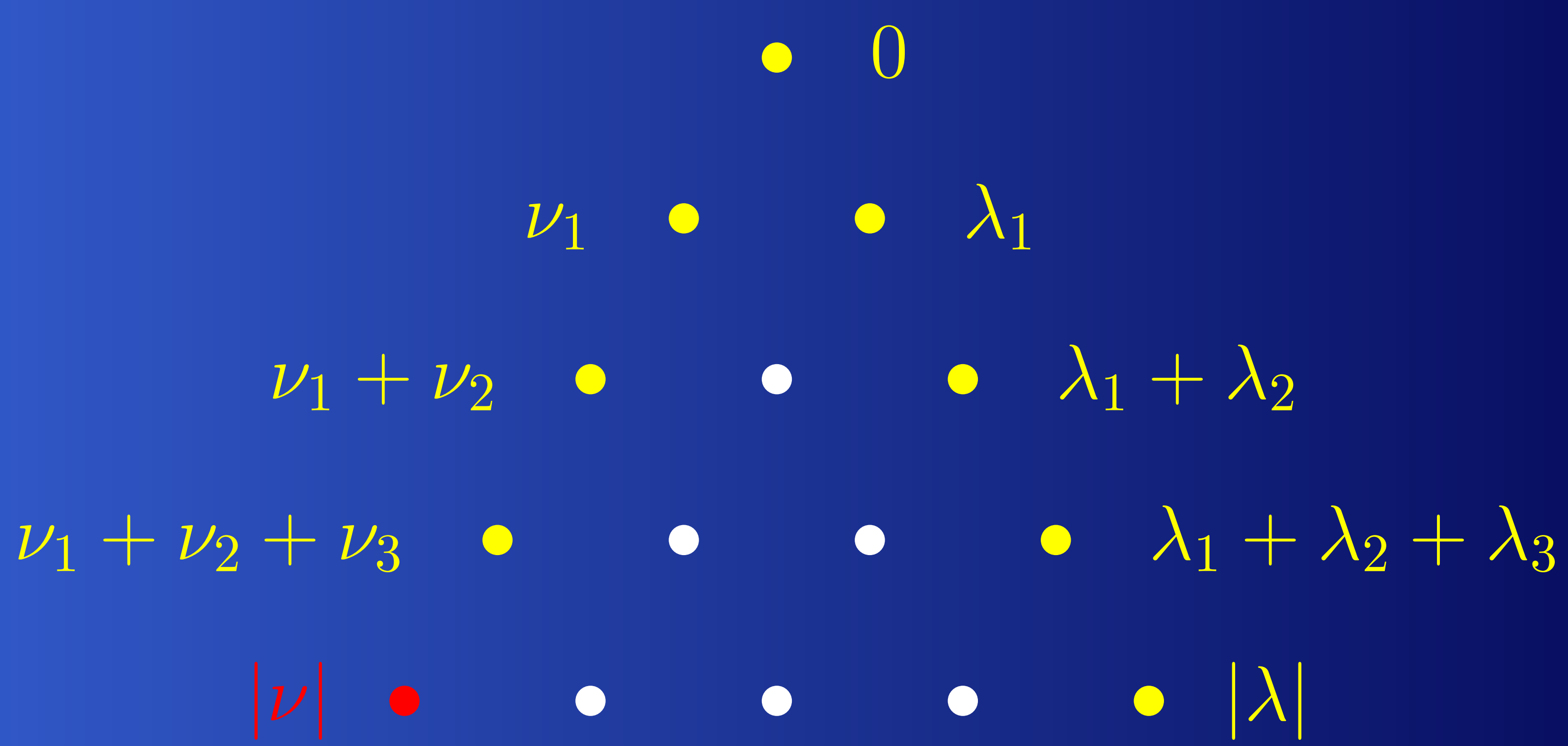

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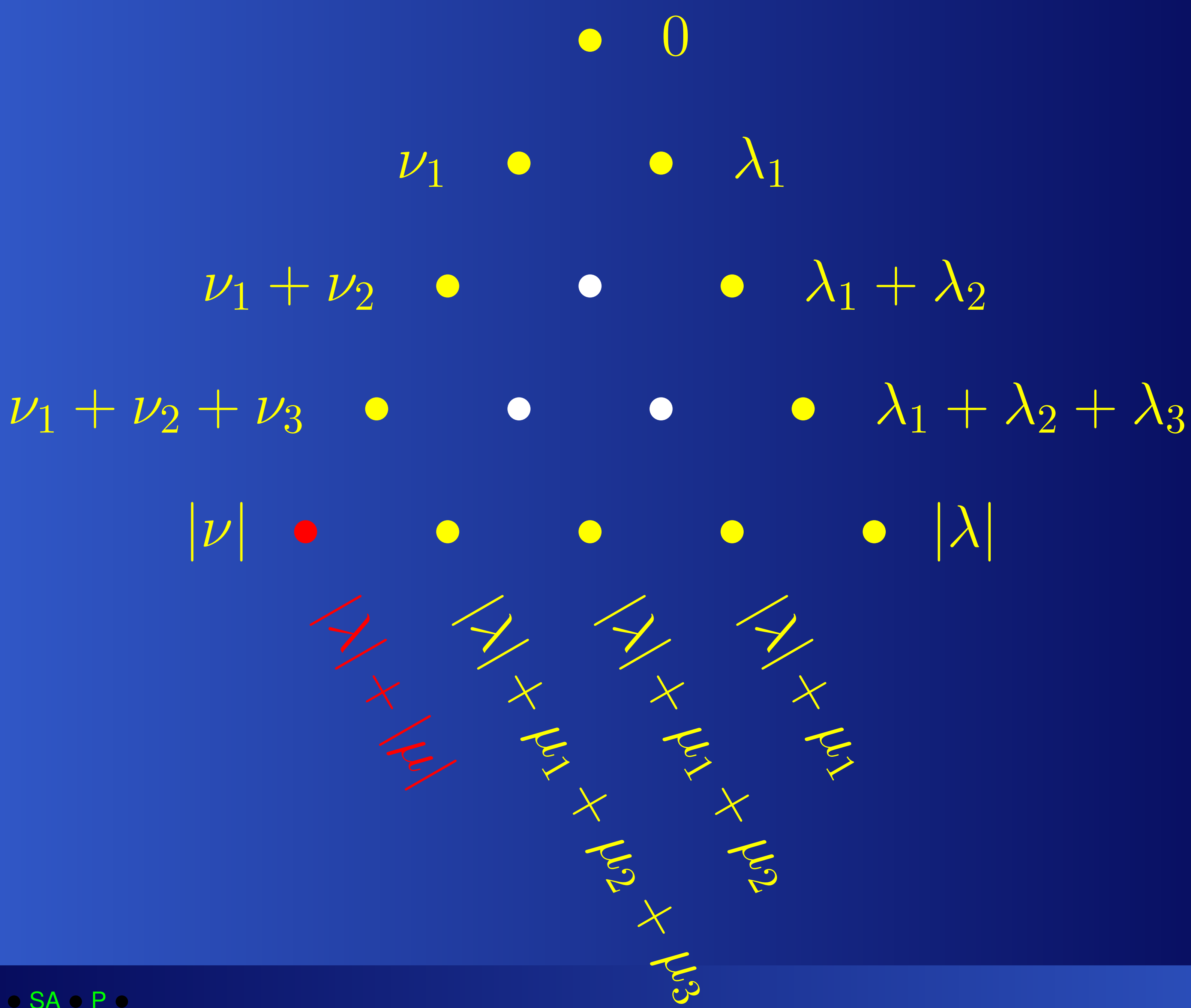

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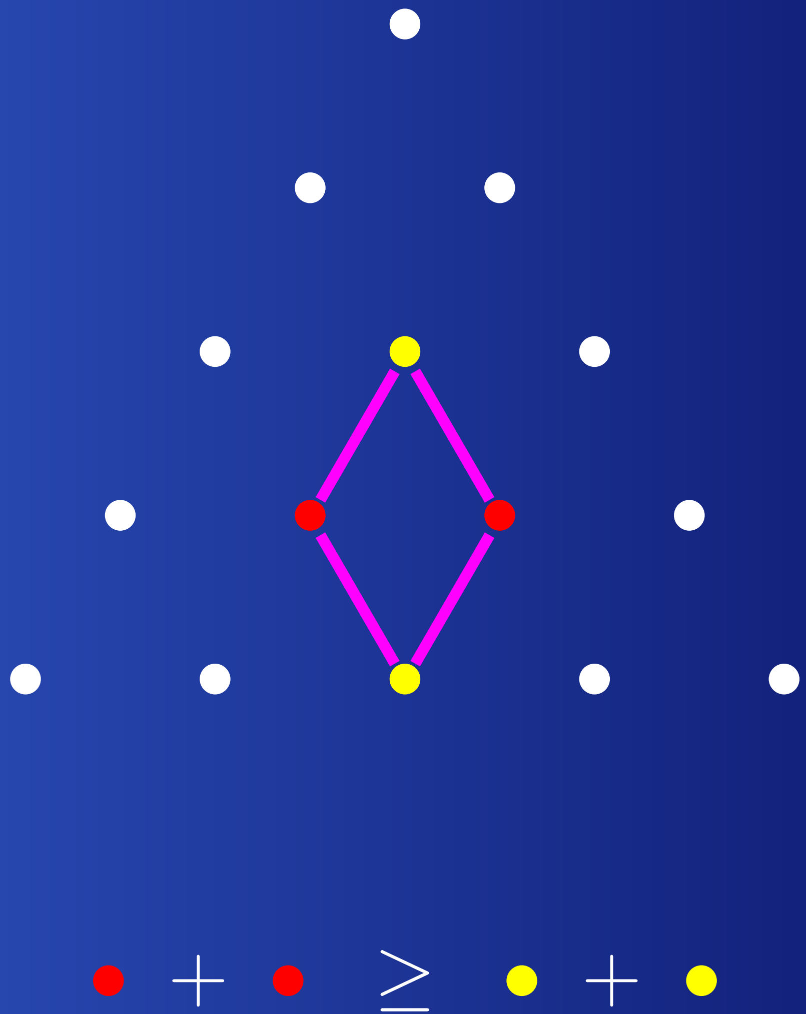

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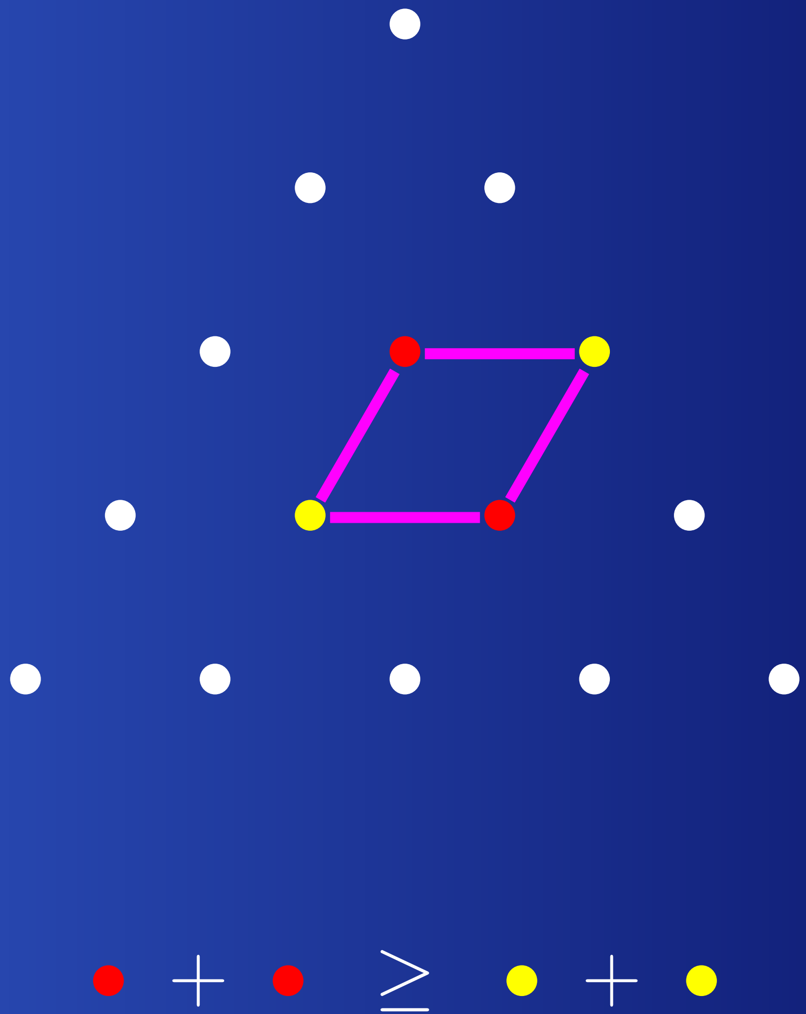

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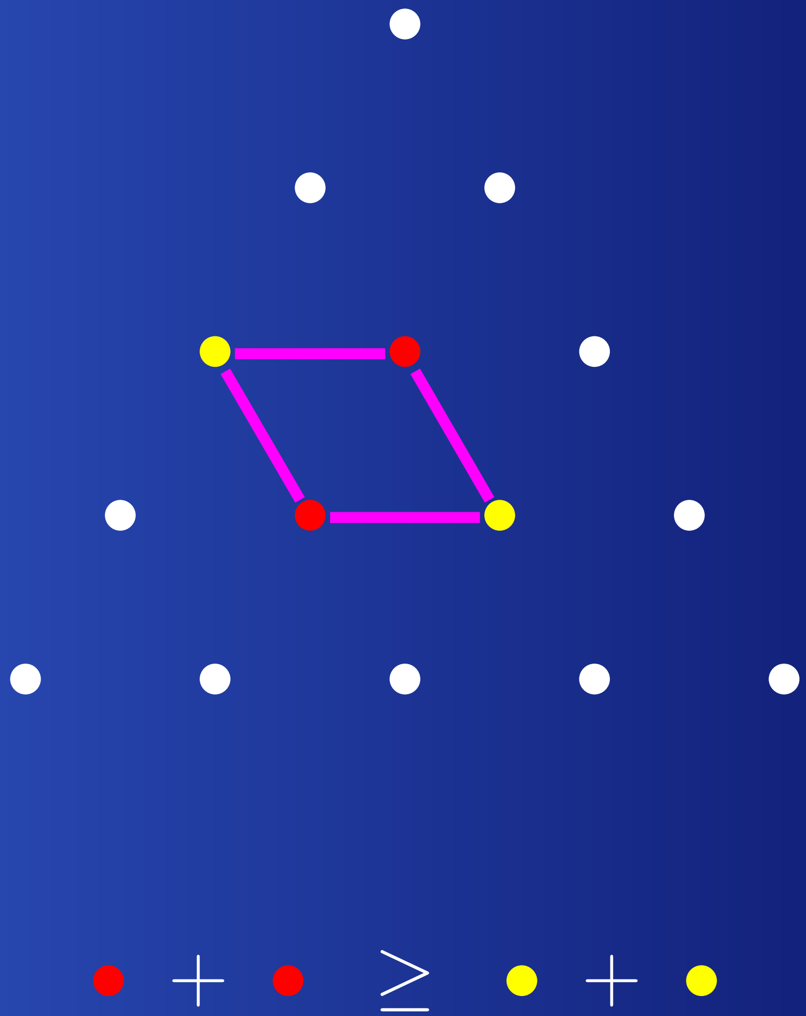

Theorem **(Knutson-Tao, Fulton)**

Let λ, μ and ν be partitions with at most k parts such that $|\lambda| + |\mu| = |\nu|$.

The Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is the number of integral k-hives satisfying the boundary conditions and the hive conditions.

The root system A_{k-1}

- Roots

$$\Delta = \{e_i - e_j \ : \ 1 \leq i \neq j \leq k\}\,.$$

- Positive roots

$$\Delta_+ = \{e_i - e_j \ : \ 1 \leq i < j \leq k\}\,.$$

- δ

$$\delta \quad = \quad \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$$

δ corresponds to the staircase partition

$$(k-1, k-2, \ldots, 1, 0)\,.$$

The Kostant partition function

- The Kostant partition function is the function

$$K(v) = \left| \left\{ (k_\alpha)_{\alpha \in \Delta_+} \in \mathbb{N}^{|\Delta_+|} \ : \ \sum_{\alpha \in \Delta_+} k_\alpha \alpha = v \right\} \right| .$$

- $K(v)$ is the number of ways that v can be written as a sum of positive roots.

Steinberg's formula

There is a formula due to Steinberg that gives the multiplicities of the irreducible factors in the tensor product of two irreducible representations of a complex semisimple Lie algebra.

Steinberg's formula (for $\mathrm{GL}_k\mathbb{C}$)

$$c^{\nu}_{\lambda\mu} = \sum_{\sigma\in\mathfrak{S}_k}\sum_{\tau\in\mathfrak{S}_k}(-1)^{\mathrm{inv}(\sigma\tau)}K(\sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta))$$

Vector partition functions

Let M be a $d \times n$ matrix over the integers. The vector partition function associated to M is the function

$$\begin{aligned} \phi_M : \quad \mathbb{Z}^d & \longrightarrow \mathbb{N} \\ b & \mapsto |\{x \in \mathbb{N}^n \ : \ Mx = b\}| \end{aligned}$$

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Example

If $M = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and $b = \binom{3}{1}$ then $\phi_M(b) = 3$

since $b = \binom{1}{0} + \binom{0}{1} + \binom{2}{0} = \binom{1}{0} + \binom{2}{1} = 3\binom{1}{0} + \binom{0}{1}$.

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- The domains of quasipolynomiality form a complex of convex polyhedral cones, the chamber complex of ϕ_M .

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- ϕ_M is piecewise quasipolynomial of degree $n - \mathrm{rank}(M)$. (Sturmfels)
- The domains of quasipolynomiality form a complex of convex polyhedral cones, the chamber complex of ϕ_M .
- Alekseevskaya, Gelfand and Zelevinsky described how to determine the chamber complex of a partition function from its matrix.

Unimodularity

A $d \times n$ matrix of full rank d is unimodular if all its $d \times d$ submatrices have determinant 0 or ± 1.

Vector partition functions of unimodular matrices are polynomial over the cones of their chamber complexes. (Sturmfels)

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Lemma (well-known) *The root system A_n is unimodular for all n.*

Corollary *The Kostant partition function for A_{k-1} is polynomial of degree $\binom{k-1}{2}$ over the cones of its chamber complex.*

A partition function for the $c_{\lambda\mu}^{\nu}$

Theorem A

For every k, we can find integer matrices E_k and B_k such that the Littlewood-Richardson coefficients for partitions with at most k parts can be written as

$$c_{\lambda\mu}^{\nu} = \phi_{E_k}\left(B_k \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} \right).$$

Example: $k = 3$

We want to find a vector partition function counting the integral 3-hives of the form

$$0$$

$$\nu_1 \qquad\qquad \lambda_1$$

$$\nu_1+\nu_2 \qquad\qquad m \qquad\qquad \lambda_1+\lambda_2$$

$$|\nu| \qquad\qquad |\lambda|+\mu_1+\mu_2 \qquad\qquad |\lambda|+\mu_1 \qquad\qquad |\lambda|$$

The hives conditions are given by

$$\begin{aligned}
m &\le \lambda_1 + \nu_1 \\
-m &\le -\lambda_1 - \lambda_3 - \mu_1 \\
-m &\le -\lambda_1 - \lambda_2 - \lambda_3 - \mu_1 - \mu_2 + \nu_2
\end{aligned}$$

$$\begin{aligned}
-m &\le -\lambda_2 - \nu_1 \\
m &\le \lambda_1 + \lambda_2 + \mu_1 \\
-m &\le \mu_2 - \nu_1 - \nu_2
\end{aligned}$$

$$\begin{aligned}
-m &\le -\lambda_1 - \nu_2 \\
-m &\le -\lambda_1 - \lambda_2 - \mu_2 \\
m &\le \lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 - \nu_3
\end{aligned}$$

This corresponds to the matrix system

$$E_3 \cdot \begin{pmatrix} m \\ s_1 \\ s_2 \\ \vdots \\ s_9 \end{pmatrix} = B_3 \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

with

$$E_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$B_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

A chamber complex for the $c_{\lambda\mu}^{\nu}$

- Theorem A implies that the Littlewood-Richardson coefficients are given by quasipolynomials over the cells of a chamber complex $\mathcal{C}^{(k)}$.

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- Theorem A implies that the Littlewood-Richardson coefficients are given by quasipolynomials over the cells of a chamber complex $\mathcal{C}^{(k)}$.
- The vector partition function ϕ_{E_k} puts λ, μ and ν on an equal footing: $\mathcal{C}^{(k)}$ is a complex in (λ, μ, ν)-space.

Example: $k = 3$ (continued)

$a_1 = (\,1\;1\;1\,|\,0\;0\;0\,|\,1\;1\;1\,)$ $\qquad$ $a_2 = (\,0\;0\;0\,|\,1\;1\;1\,|\,1\;1\;1\,)$

$$b = (\,2\;1\;0\,|\,2\;1\;0\,|\,3\;2\;1\,)$$

$$c = (\,1\;1\;0\,|\,1\;1\;0\,|\,2\;1\;1\,)$$

$d_1 = (\,1\;1\;0\,|\,1\;0\;0\,|\,1\;1\;1\,)$ $\qquad$ $d_2 = (\,1\;0\;0\,|\,1\;1\;0\,|\,1\;1\;1\,)$

$e_1 = (\,1\;1\;0\,|\,0\;0\;0\,|\,1\;1\;0\,)$ $\qquad$ $e_2 = (\,0\;0\;0\,|\,1\;1\;0\,|\,1\;1\;0\,)$

$$f = (\,1\;0\;0\,|\,1\;0\;0\,|\,1\;1\;0\,)$$

$g_1 = (\,1\;0\;0\,|\,0\;0\;0\,|\,1\;0\;0\,)$ $\qquad$ $g_2 = (\,0\;0\;0\,|\,1\;0\;0\,|\,1\;0\;0\,)$

Cone	Positive hull description	Polynomial
κ_1	$\mathrm{pos}(a_1, a_2, b, c, d_1, d_2, e_1, e_2)$	$1 - \lambda_2 - \mu_2 + \nu_1$
κ_2	$\mathrm{pos}(a_1, a_2, b, c, d_1, d_2, g_1, g_2)$	$1 + \nu_2 - \nu_3$
κ_3	$\mathrm{pos}(a_1, a_2, b, c, e_1, e_2, g_1, g_2)$	$1 + \lambda_1 + \mu_1 - \nu_1$
κ_4	$\mathrm{pos}(a_1, a_2, b, d_1, d_2, e_1, e_2, f)$	$1 + \nu_1 - \nu_2$
κ_5	$\mathrm{pos}(a_1, a_2, b, d_1, d_2, f, g_1, g_2)$	$1 + \lambda_2 + \mu_2 - \nu_3$
κ_6	$\mathrm{pos}(a_1, a_2, b, e_1, e_2, f, g_1, g_2)$	$1 - \lambda_3 - \mu_3 + \nu_3$
κ_7 κ_8	$\mathrm{pos}(a_1, a_2, b, c, d_1, d_2, e_1, g_1)$ $\mathrm{pos}(a_1, a_2, b, c, d_1, d_2, e_2, g_2)$	$1 + \lambda_3 + \mu_1 - \nu_3$ $1 + \lambda_1 + \mu_3 - \nu_3$

κ_9	$\mathrm{pos}(a_1, a_2, b, c, d_1, e_1, e_2, g_2)$	$1 + \lambda_1 - \lambda_2$
κ_{10}	$\mathrm{pos}(a_1, a_2, b, c, d_2, e_1, e_2, g_1)$	$1 + \mu_1 - \mu_2$
κ_{11}	$\mathrm{pos}(a_1, a_2, b, c, d_1, e_1, g_1, g_2)$	$1 - \lambda_2 - \mu_3 + \nu_2$
κ_{12}	$\mathrm{pos}(a_1, a_2, b, c, d_2, e_2, g_1, g_2)$	$1 - \lambda_3 - \mu_2 + \nu_2$
κ_{13}	$\mathrm{pos}(a_1, a_2, b, d_1, d_2, e_1, f, g_1)$	$1 - \lambda_1 - \mu_3 + \nu_3$
κ_{14}	$\mathrm{pos}(a_1, a_2, b, d_1, d_2, e_2, f, g_2)$	$1 - \lambda_3 - \mu_1 + \nu_3$
κ_{15}	$\mathrm{pos}(a_1, a_2, b, d_1, e_1, f, g_1, g_2)$	$1 + \mu_2 - \mu_3$
κ_{16}	$\mathrm{pos}(a_1, a_2, b, d_2, e_2, f, g_1, g_2)$	$1 + \lambda_2 - \lambda_3$
κ_{17}	$\mathrm{pos}(a_1, a_2, b, d_1, e_1, e_2, f, g_2)$	$1 + \lambda_1 + \mu_2 - \nu_2$
κ_{18}	$\mathrm{pos}(a_1, a_2, b, d_2, e_1, e_2, f, g_1)$	$1 + \lambda_2 + \mu_1 - \nu_2$

The Steinberg arrangement

Steinberg's formula:

$$c^{\nu}_{\lambda\mu} = \sum_{\sigma\in\mathfrak{S}_k} \sum_{\tau\in\mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma\tau)} K(\sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta)) \,.$$

Kostant partition function K polynomial over cones

$\Downarrow$

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The $c^{\nu}_{\lambda\mu}$ are locally polynomial

We find a hyperplane arrangement, the Steinberg arrangement, over whose regions the $c^{\nu}_{\lambda\mu}$ are given by polynomials.

Theorem B

The quasipolynomials giving the Littlewood-Richardson coefficients in the cones of $\mathcal{C}^{(k)}$ are polynomials of total degree $\binom{k-1}{2}$ in the sets of variables λ, μ and ν.

$$\begin{aligned} \lambda &= (\lambda_1, \ldots, \lambda_k) \\ \mu &= (\mu_1, \ldots, \mu_k) \\ \nu &= (\nu_1, \ldots, \nu_k) \end{aligned}$$

Lemma

For each cone C of the chamber complex $\mathcal{C}^{(k)}$, we can find a region R of the Steinberg arrangement such that $C \cap R$ contains an arbitrarily large ball.

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- Then the polynomial on R and the quasipolynomial on C agree on all the lattice points (λ, μ, ν) in that ball.

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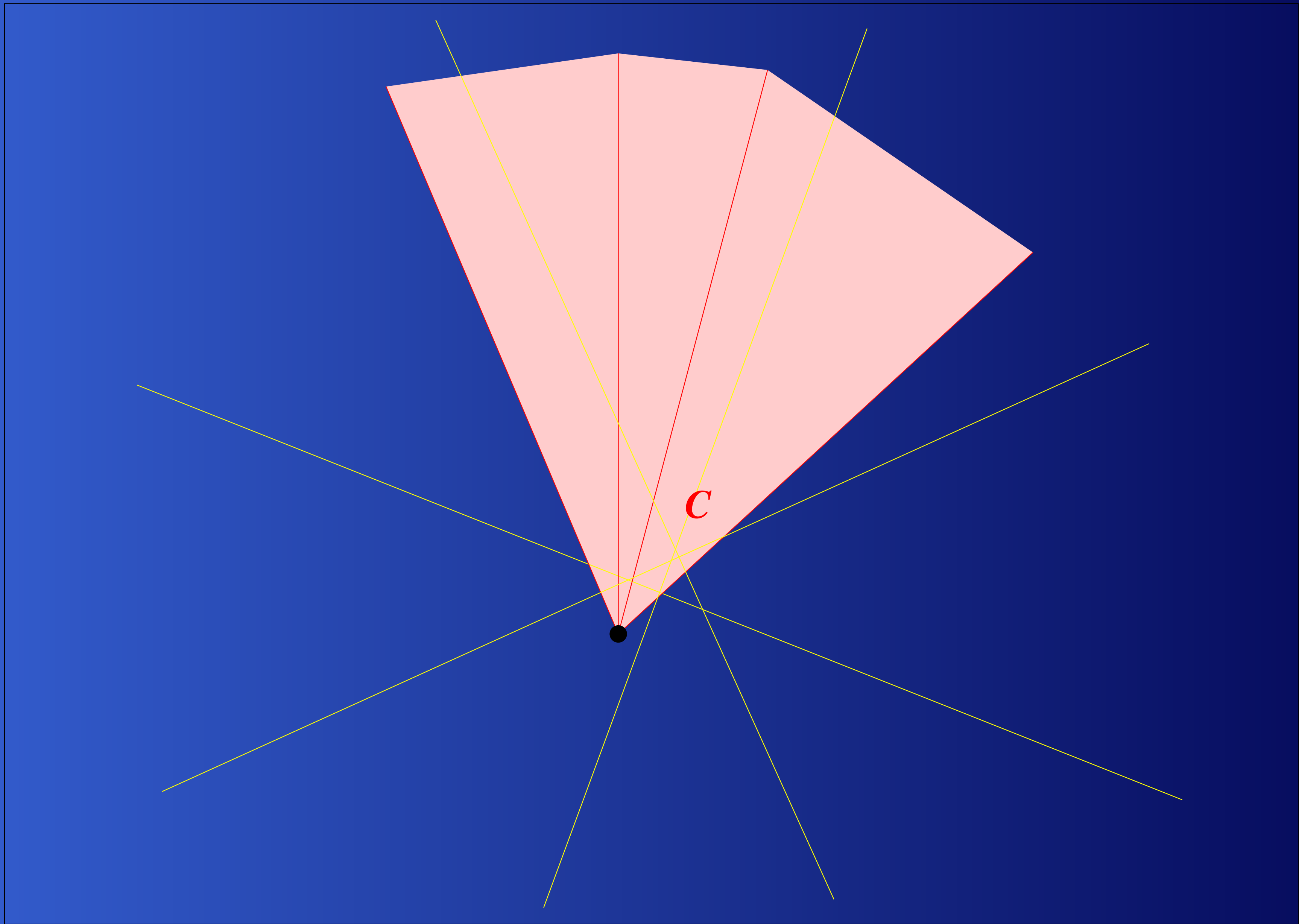
- Then the polynomial on R and the quasipolynomial on C agree on all the lattice points (λ, μ, ν) in that ball.
- The degree bounds follow from the degree bounds on the Kostant partition function.

Idea of proof



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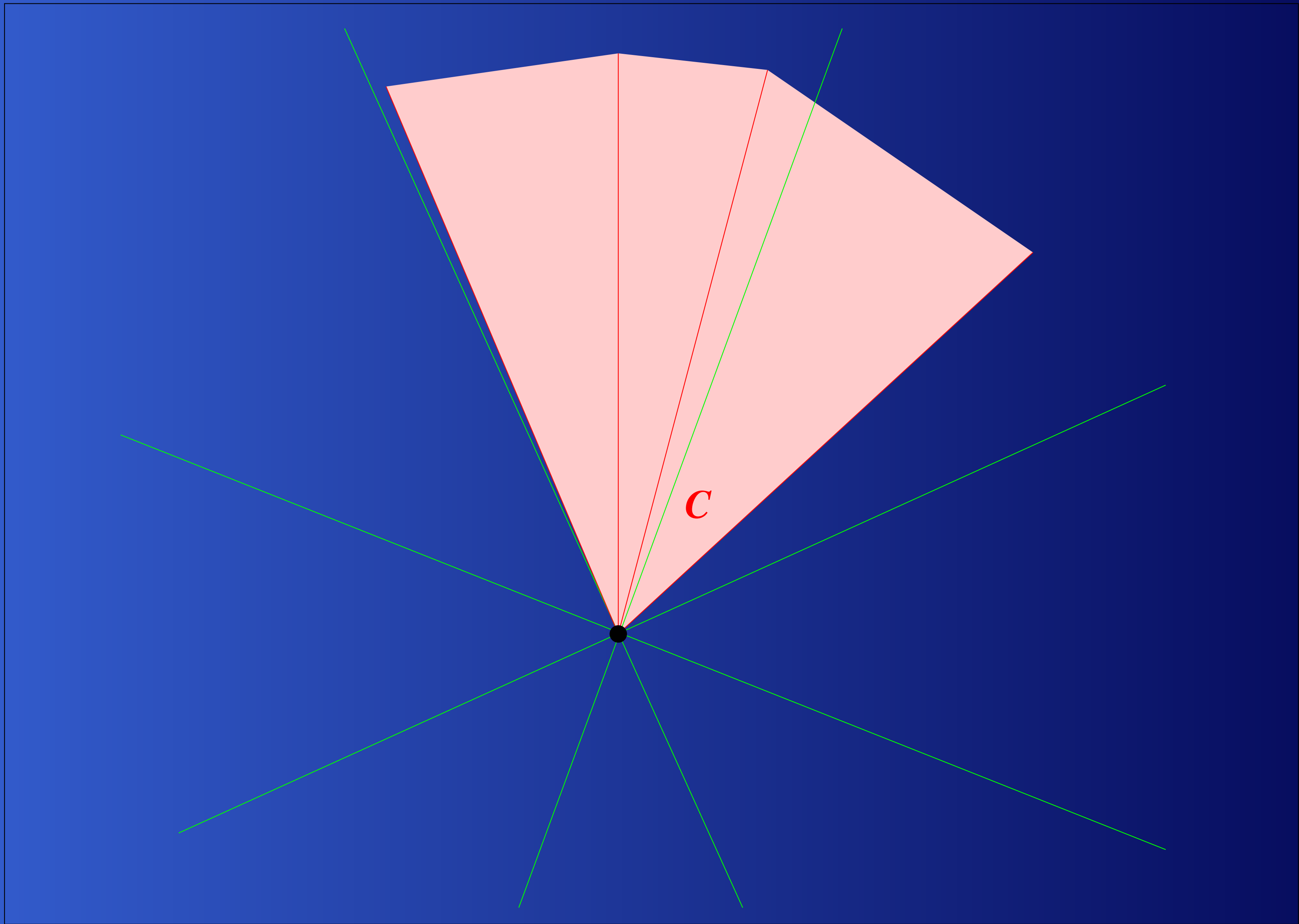

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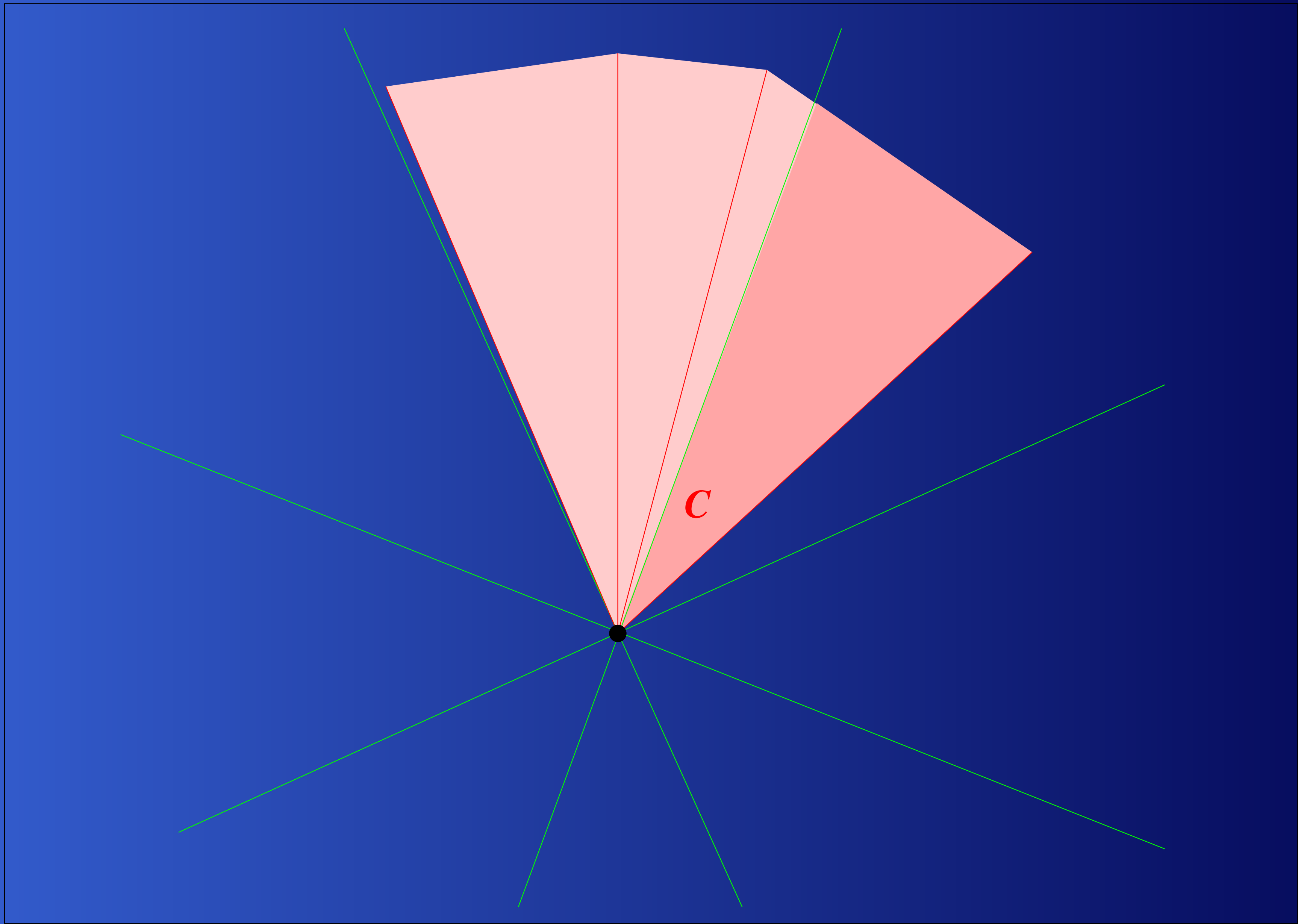

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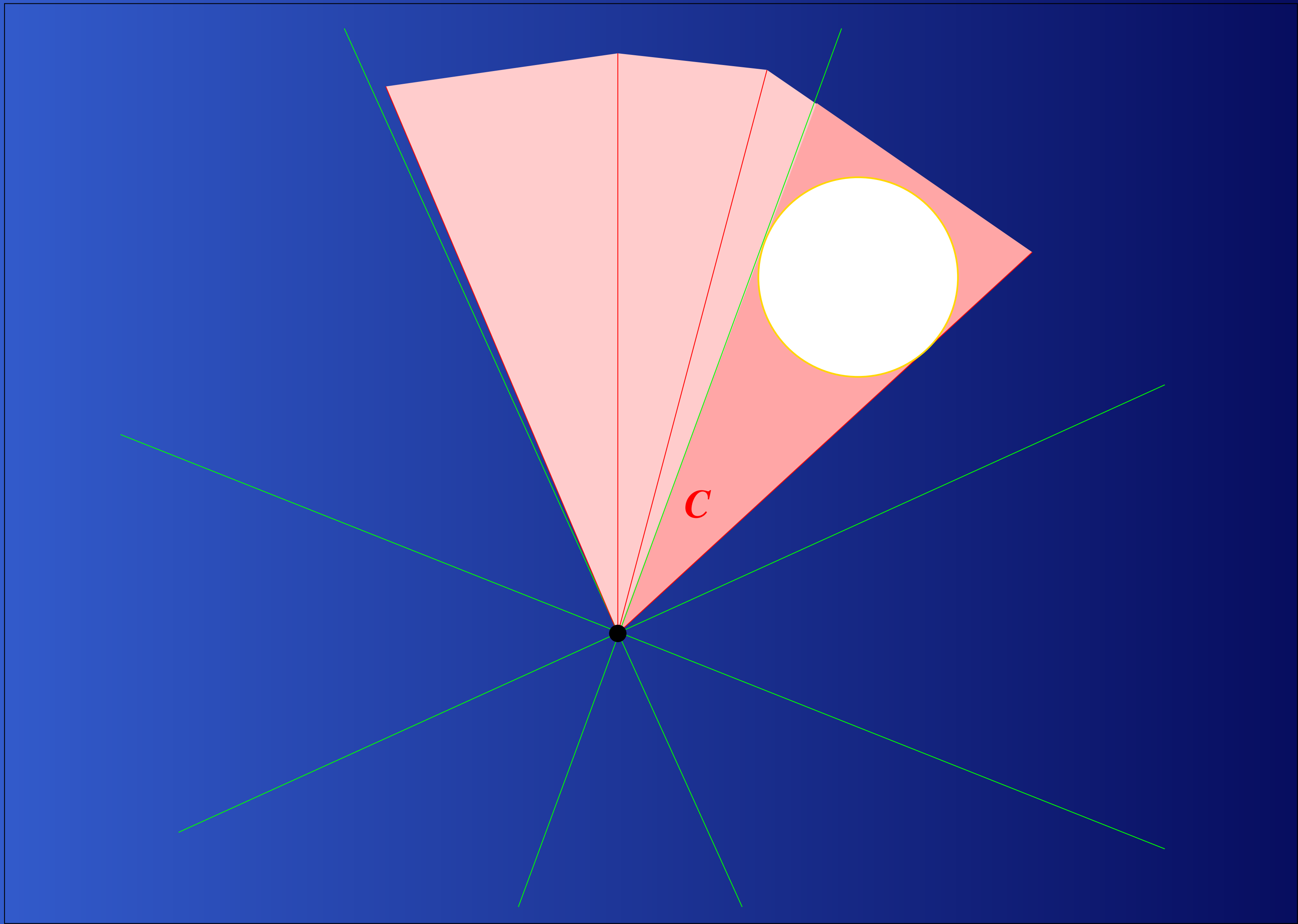

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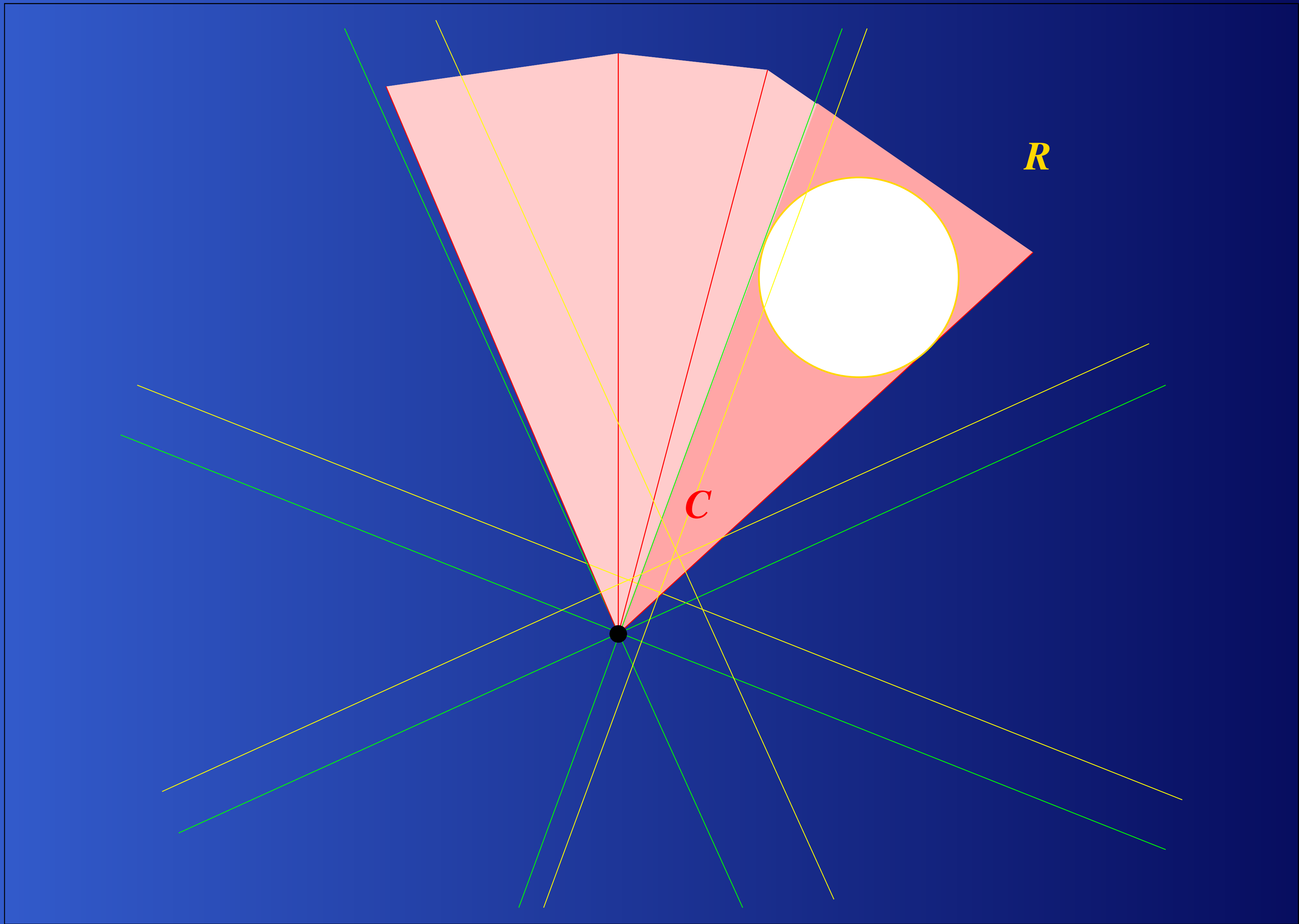

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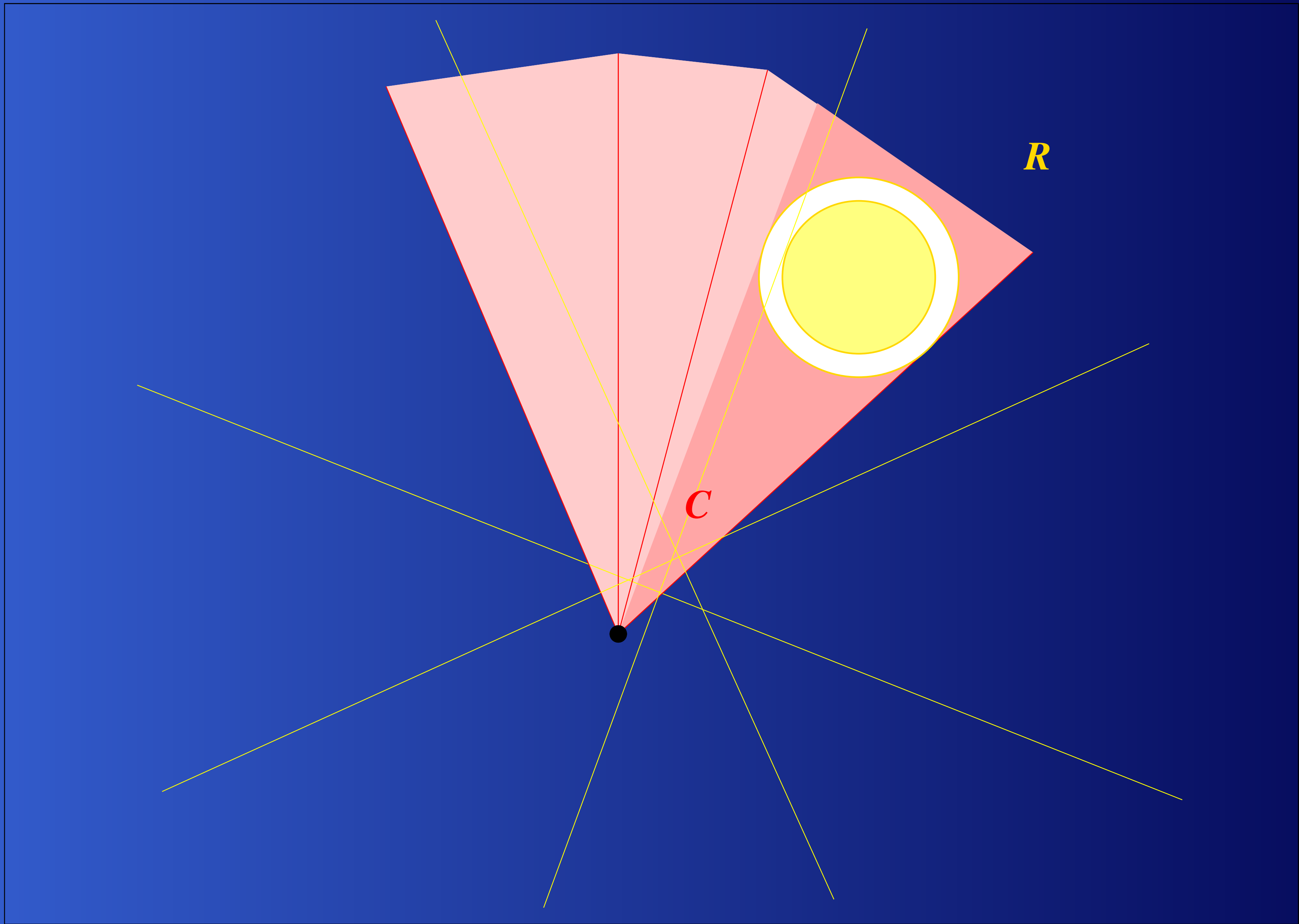

Stretching for LR coefficients

- Theorem B shows in particular that the function

$$N \in \mathbb{N} \quad \longmapsto \quad c_{N\lambda\ N\mu}^{N\nu}$$

is polynomial in N.

- This was known previously (Derksen-Weyman, Knutson).

Hive polytopes

- If we consider hives with real entries with the same boundary and hive conditions, we get a polytope, the hive polytope, whose number of integer points gives the corresponding $c_{\lambda\mu}^{\nu}$.

- The function $N \mapsto c_{N\lambda\ N\mu}^{N\nu}$ is the Ehrhart polynomial of the hive polytope for λ, μ and ν.

- Hive polytopes are not integral in general (King-Tollu-Toumazet).

Conjecture

Conjecture **(King-Tollu-Toumazet)**

For all partitions λ, μ and ν such that $c^{\nu}_{\lambda\mu} > 0$ there exists a polynomial $P^{\nu}_{\lambda\mu}(N)$ in N with nonnegative rational coefficients such that $P^{\nu}_{\lambda\mu}(0) = 1$ and $P^{\nu}_{\lambda\mu}(N) = c^{N\nu}_{N\lambda\ N\mu}$ for all positive integers N.