A q-analogue of the Kostant partition function and twisted representations

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FPSAC 2005, Taormina

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Outline

- The origins of the problem (geometry)
- Twisted representations
- Making everything symmetric functions
- A branching rule
- A q-analogue of the Kostant partition function

The origins of the problem

Geometric setup

- (M, ω) symplectic manifold
- G compact Lie group
- $\begin{aligned} \tau \colon G &\to \mathrm{Diff}(M) \\ g &\mapsto \tau_g \end{aligned}$ group action with $\tau_g^* \omega = \omega$

Assume that M is prequantizable:

- $\mathbb{L} \to M$ Hermitean line bundle
- τ extends to the line bundle
- we have a connection ∇ with $\mathrm{curv}(\nabla) = \omega$

The spin-$\mathbb{C}$ Dirac operator

- The spin-$\mathbb{C}$ Dirac operator $\not{\partial}_{\mathbb{C}}$ exists on every symplectic manifold.
- For M prequantizable, using the connection ∇, we can lift $\not{\partial}_{\mathbb{C}}$ to a G-invariant operator $\not{\partial}_{\mathbb{C}}^{\mathbb{L}}$ on the line bundle $\mathbb{L} \to M$.
- The index of $\not{\partial}_{\mathbb{C}}^{\mathbb{L}}$ is the virtual vector space

$$\operatorname{Ind} \not{\partial}_{\mathbb{C}}^{\mathbb{L}} = \ker \not{\partial}_{\mathbb{C}}^{\mathbb{L}} \oplus \left(-\operatorname{coker} \not{\partial}_{\mathbb{C}}^{\mathbb{L}} \right).$$

- It is a (virtual) representation of G known as the quantization of τ (with respect to $\not{\partial}_{\mathbb{C}}$).

Quantization and representations

- Let G be a semisimple Lie group and $\mathfrak{g}$ its Lie algebra.
- Let T be a Cartan subgroup (maximal torus) and $\mathfrak{t}$ its Lie algebra.
- $\mathfrak{t}^*_+$ is the fundamental Weyl chamber
- We have the coadjoint representation of G on $\mathfrak{g}^*$.
- Denote by O_λ the coadjoint orbit through a point $\lambda \in \mathfrak{t}^*_+$.

Bott-Borel-Weil

- Suppose that λ is a dominant weight.
- Then O_λ is a prequantizable symplectic manifold.
- The cokernel of $\not\partial_{\mathbb{C}}^{\mathbb{L}}$ vanishes and the index is

$$\mathrm{Ind}\,\not\partial_{\mathbb{C}}^{\mathbb{L}} = \ker \not\partial_{\mathbb{C}}^{\mathbb{L}} = V_\lambda\,,$$

the irreducible representation of G with highest weight λ. (Bott-Borel-Weil theorem)

Twisted representations

The signature Dirac operator

- The signature Dirac operator $\not{\partial}_{\mathrm{sig}}$ exists on any oriented manifold (with a Riemannian metric).

- The index

$$\operatorname{Ind} \not{\partial}_{\mathrm{sig}} = \ker \not{\partial}_{\mathrm{sig}} \oplus \left(-\operatorname{coker} \not{\partial}_{\mathrm{sig}}\right)$$

  is also a virtual representation of G.

- It is also known as the quantization of τ, but with respect to $\not{\partial}_{\mathrm{sig}}$.

Guillemin-Sternberg-Weitsman

- Let G be a semisimple Lie group and T a maximal torus as above
- Let λ be a strictly dominant weight (in the interior of the fundamental Weyl chamber).

Theorem ***(Guillemin-Sternberg-Weitsman)***

The index of $\not{\partial}_{\mathrm{sig}}$ on the coadjoint orbit O_λ is

$$\mathrm{Ind}\,\not{\partial}_{\mathrm{sig}} = (-1)^{\frac{1}{2}\dim O_\lambda}\, V_{\lambda-\delta} \otimes V_\delta\,,$$

where δ is half the sum of the positive roots.

To remember

The thing to remember from all of this:

Quantization for $\not{\partial}_{\mathbb{C}}^{\mathbb{L}}$	$\leftrightsquigarrow$	V_λ
Quantization for $\not{\partial}_{\mathrm{sig}}$	$\leftrightsquigarrow$	$V_{\lambda-\delta} \otimes V_\delta$

So in a certain world and in a certain sense, the representations $V_{\lambda-\delta} \otimes V_\delta$ play the role of the irreducible representations in the classical theory.

Twisted representations

- δ is half the sum of the positive roots.
- V_μ is the irreducible representation with highest weight μ.
- For λ strictly dominant, we define the twisted representation

$$\widetilde{V}_\lambda = V_{\lambda-\delta} \otimes V_\delta \, .$$

Tensor products

- Tensor products of twisted representations can be written in terms of twisted representations again:

$$\widetilde{V}_\lambda \otimes \widetilde{V}_\mu = (V_{\lambda-\delta} \otimes V_\delta) \otimes (V_{\mu-\delta} \otimes V_\delta)$$

$$= (V_{\lambda-\delta} \otimes V_\delta \otimes V_{\mu-\delta}) \otimes V_\delta$$

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$$\widetilde{V}_\lambda \otimes \widetilde{V}_\mu = (V_{\lambda-\delta} \otimes V_\delta) \otimes (V_{\mu-\delta} \otimes V_\delta)$$

$$= \underbrace{(V_{\lambda-\delta} \otimes V_\delta \otimes V_{\mu-\delta})}_{\text{break into sum of irreducibles}} \otimes V_\delta$$

$$= \bigoplus_\nu \widetilde{V}_\nu^{\oplus \widetilde{N}_{\lambda\mu}^{\nu}}$$

Symmetric functions (type A)

Type A_{k-1}

For $\mathrm{GL}_k\mathbb{C}$ things are especially nice because

- A dominant weight is just a partition with k parts (0-parts allowed).
- δ is the staircase partition:

$$\delta = (k-1, k-2, \ldots, 1, 0) \, .$$

- λ strictly dominant means the partition λ has distinct parts ($\lambda - \delta$ is still a partition).

Taking characters

- The character χ_μ of the irreducible (polynomial) representation V_μ is a Schur polynomial:

$$\chi_\mu = s_\mu(x_1, x_2, \ldots, x_k)\,.$$

- This means that the character $\widetilde{\chi}_\lambda$ of the twisted representation $\widetilde{V}_\lambda$ is

$$\begin{aligned} \widetilde{\chi}_\lambda &= \chi_{\lambda-\delta} \cdot \chi_\delta \\ &= s_{\lambda-\delta}(x_1, \ldots, x_k)\, s_\delta(x_1, \ldots, x_k)\,. \end{aligned}$$

Hall-Littlewood polynomials

- The character $\widetilde{\chi}_\lambda$ is the $t = -1$ specialization of the Hall-Littlewood polynomial $P_\lambda(x_1, \ldots, x_k;\, t)$.
- $P_\lambda(x_1, \ldots, x_k;\, 0)$ is the Schur function $s_\lambda(x_1, \ldots, x_k)$, so the character χ_λ .
- We (almost) have, together with J. Weitsman, a geometrical object whose character is $P_\lambda(x_1, \ldots, x_k;\, -(q-1))$.
- This is where the q-analogue of the Kostant partition function comes in.

A branching rule

Branching rules

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- $\mathfrak{S}_{n-1}$ is a subgroup of $\mathfrak{S}_n$ (by leaving the last element fixed).
- $\mathrm{GL}_{k-1}\mathbb{C}$ is the subgroup $\left(\begin{array}{ccc|c} & & & 0 \\ & \mathrm{GL}_{k-1}\mathbb{C} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)$ of $\mathrm{GL}_k\mathbb{C}$.

Branching for $\mathfrak{S}_n$

- Let $\{W_\lambda \ : \ \lambda \vdash n\}$ be the set of irreducible representations of $\mathfrak{S}_n$.

- Let $\{W_\mu \ : \ \mu \vdash n-1\}$ be the set of irreducible representations of $\mathfrak{S}_{n-1}$.

- Then

$$\mathrm{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} W_\lambda \quad = \quad \bigoplus_\mu W_\mu$$

where the sum is over all partitions that can be obtained from λ by removing one corner.

Branching for $\mathrm{GL}_k\mathbb{C}$

- Let $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k)$.
- Let $\nu = (\nu_1, \ldots, \nu_{k-1})$.
- Say that ν interlaces λ ($\nu \lhd \lambda$) if

$$\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \lambda_{k-1} \geq \nu_{k-1} \geq \lambda_k \,.$$

(Obtain ν from λ by removing horizontal strip)

- Then (Weyl's branching rule)

$$\mathrm{Res}^{\mathrm{GL}_k\mathbb{C}}_{\mathrm{GL}_{k-1}\mathbb{C}} V_\lambda \quad = \quad \bigoplus_{\nu \lhd \lambda} V_\nu \,.$$

Branching rule for the $\widetilde{V}_\lambda$

- Let $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k)$ distinct parts.
- Let $\nu = (\nu_1, \ldots, \nu_{k-1})$ distinct parts.
- For $\nu \lhd \lambda$, let

$$\nabla(\lambda, \nu) = \{i \in [k-1] \,:\, \lambda_i > \nu_i > \lambda_{i+1}\} \,.$$

Theorem *(Guillemin-R)*

$$\mathrm{Res}^{\mathrm{GL}_k\mathbb{C}}_{\mathrm{GL}_{k-1}\mathbb{C}} \widetilde{V}_\lambda = \bigoplus_{\substack{\nu \lhd \lambda \\ \nu \text{ distinct parts}}} 2^{\nabla(\lambda,\nu)} \widetilde{V}_\nu \,.$$

Sketch of proof

- In terms of the characters, restricting just means setting the last variable, x_k, equal to 1.

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- Now $s_{\lambda/\mu}(1) = \begin{cases} 1 & \text{if } \lambda/\mu \text{ is horizontal strip,} \\ 0 & \text{otherwise.} \end{cases}$

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- So $s_\lambda(x_1,\ldots,x_{k-1},1) = \sum_{\mu \lhd \lambda} s_\mu(x_1,\ldots,x_{k-1})$.

- On the other hand

$$
\begin{aligned}
s_\delta(x_1,\ldots,x_k) &= \frac{\text{Vandermonde}(x_1^2,\ldots,x_k^2)}{\text{Vandermonde}(x_1,\ldots,x_k)} \\
&= \prod_{1\le i<j\le k} (x_i + x_j)\,.
\end{aligned}
$$

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- Then

$$s_{\delta|_{x_k=1}} = \prod_{1\le i<j\le k-1} (x_i+x_j) \prod_{i=1}^{k-1}(1+x_i) = s_{\delta'}(x_1,\dots,x_{k-1})(e_0+e_1+\cdots+e_{k-1})\,.$$

where $\delta' = (k-2, k-3, \dots, 1, 0)$.

- Define $f^{(m)} = f(x_1, \ldots, x_m)$. We have

$$s^{(k)}_{\lambda-\delta} s^{(k)}_{\delta}\Big|_{x_k=1} = \sum_{\mu \lhd \lambda-\delta} s^{(k-1)}_{\mu} s^{(k-1)}_{\delta'} (e^{(k-1)}_0 + \cdots + e^{(k-1)}_{k-1})$$

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- By a dual version of the Pieri rule,

$$s_\mu e_m = \sum_\nu s_\nu \ ,$$

where the sum is over all ν obtained from μ by adding a vertical strip of size m.

- Finally,

$$s^{(k)}_{\lambda-\delta} s^{(k)}_{\delta}\Big|_{x_k=1} = \sum_{\mu \lhd \lambda-\delta} \sum_{\nu} s^{(k-1)}_{\nu} s^{(k-1)}_{\delta'}$$

$$\widetilde{\chi}_\lambda(x_1, \ldots, x_{k-1}, 1) = \sum_{\mu \lhd \lambda-\delta} \sum_{\nu} \widetilde{\chi}_{\nu+\delta}(x_1, \ldots, x_{k-1}),$$

where the sum is over all ν that can be obtained from μ by adding a vertical strip of size at most $k-1$ (and height at most $k-1$).

- We can relabel things and group terms.

- For partitions with distinct parts λ and ν, let $n(\lambda, \nu)$ be the number of ways that $\nu - \delta'$ can be obtained by a adding a vertical strip of size and height at most $k-1$ to some partition μ interlacing $\lambda - \delta$. Then

$$\widetilde{V}_\lambda = \bigoplus_{\nu \lhd \lambda,\ \text{distinct parts}} n(\lambda, \nu)\, \widetilde{V}_\nu\,.$$

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- For partitions with distinct parts λ and ν, let $n(\lambda,\nu)$ be the number of ways that $\nu-\delta'$ can be obtained by a adding a vertical strip of size and height at most $k-1$ to some partition μ interlacing $\lambda-\delta$. Then

$$\widetilde{V}_\lambda \quad = \bigoplus_{\nu\lhd\lambda,\ \text{distinct parts}} n(\lambda,\nu)\,\widetilde{V}_\nu\,.$$

Lemma

$$n(\lambda,\nu) = 2^{\nabla(\lambda,\nu)}\,.$$

Twisted Gelfand-Tsetlin diagrams

- Since $\mathrm{Res}^{\mathrm{GL}_{k-1}\mathbb{C}}_{\mathrm{GL}_{k-2}\mathbb{C}} \mathrm{Res}^{\mathrm{GL}_k\mathbb{C}}_{\mathrm{GL}_{k-1}\mathbb{C}} = \mathrm{Res}^{\mathrm{GL}_k\mathbb{C}}_{\mathrm{GL}_{k-2}\mathbb{C}}$, we can iterate the branching rule to restrict to $\mathrm{GL}_1\mathbb{C}$.
- This way, we can index one-dimensional subspaces of a twisted representation by a sequence of interlacing partitions with distinct parts. (Indexing not multiplicity-free)
- This sequence forms a triangular array of integers akin to Gelfand-Tsetlin diagrams for the irreducible representations of $\mathrm{GL}_k\mathbb{C}$.

Twisted Gelfand-Tsetlin diagrams

Proposition

$$\mathrm{Res}^{\mathrm{GL}_k\mathbb{C}}_{\mathrm{GL}_1\mathbb{C}}\,\widetilde{V}_\lambda = \bigoplus_{\lambda^{(1)}\lhd\cdots\lhd\lambda^{(k)}=\lambda} 2^{\nabla(\lambda^{(k)},\lambda^{(k-1)},\ldots,\lambda^{(1)})}\,\widetilde{V}_{\lambda^{(1)}}$$

where $\lambda^{(m)}$ is a partition with m distinct parts and

$$\nabla(\lambda^{(k)},\lambda^{(k-1)},\ldots,\lambda^{(1)}) = \sum_{i=1}^{k-1}\nabla(\lambda^{(i+1)},\lambda^{(i)})\,.$$

A q-analogue of the KPF

Characters in general

- For a semisimple Lie group G with Cartan subgroup T, the character of the irreducible representation of G with highest weight $\lambda \in \mathfrak{t}^*_+$ is given by Weyl's character formula:

$$\chi_\lambda = \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda+\delta)-\delta} \prod_{\alpha \in \Delta_+} \frac{1}{1-e^{-\alpha}},$$

  where $\mathcal{W}$ is the Weyl group and Δ_+ is the set of positive roots.

- For $\mathrm{GL}_k\mathbb{C}$, thinking of $x_i = \exp(e_i)$, this is the Schur function $s_\lambda(x_1, \ldots, x_k)$.

The Kostant partition function

- We can define the Kostant partition function K through its generating function:

$$\sum_{\mu} K(\mu)\, e^{\mu} = \prod_{\alpha \in \Delta_+} \frac{1}{1 - e^{\alpha}}\,.$$

- So $K(\mu)$ is the number of ways that μ can be written as a sum of positive roots.

- Kostant's multiplicity formula expresses weight multiplicities in terms of the KPF. For $\mathrm{GL}_k\mathbb{C}$, these are Kostka numbers.

q-analogues

- Our general geometric object has “character”

$$\chi_\lambda^{(q)} = \sum_{\omega\in\mathcal{W}} e^{\omega(\lambda)} \prod_{\alpha\in\Delta_+} \frac{1+(q-1)e^{-\omega(\alpha)}}{1-e^{-\omega(\alpha)}}.$$

- This prompts the definition of a q-analogue K_q of the KPF by

$$\sum_{\mu} K_q(\mu)\, e^{\mu} = \prod_{\alpha\in\Delta_+} \frac{1+(q-1)e^{\alpha}}{1-e^{\alpha}}.$$

Remarks

- We can think of K_q as counting integer points in a polytope with certain weights (in q).
- The $q = 2$ case is the one yielding the twisted representations.
- This is not the classical q-analogue of the KPF defined by Lusztig through

$$\sum_{\mu} \widehat{K}_{q}(\mu)\, e^{\mu} = \prod_{\alpha \in \Delta_+} \frac{1}{1 - q\, e^{\alpha}}$$

to get a q-analogue of weight multiplicities.

- Guillemin, Sternberg and Weitsman have a formula analogous to the Kostant multiplicity formula for twisted representations. Their proof is geometric (Atiyah-Bott formula). We have a purely algebraic proof of this.
- We also have an analogue of the Steinberg formula for decomposing tensor products of twisted representations into twisted representations.
- Both formulas involve the $q = 2$ specialization of the q-analogue of the KPF.

A structural result

Theorem *(Guillemin-R)*

The q-analogue $K_q(\mu)$ for the root system A_k is given by polynomials of degree $\binom{k}{2}$ with coefficients in $\mathbb{Q}[q]$ of degree $\binom{k+1}{2}$ over the relative interior of the cells of the chamber complex for the Kostant partition function.

Example: A_2

- 2-dimensional cones:

$$\begin{aligned} \tau_1 &= \{a_1\alpha_1 + a_2\alpha_2 \ : \ a_1, a_2 > 0 \text{ and } a_1 > a_2\}\,, \\ \tau_2 &= \{a_1\alpha_1 + a_2\alpha_2 \ : \ a_1, a_2 > 0 \text{ and } a_1 < a_2\}\,, \end{aligned}$$

- 1-dimensional cones:

$$\begin{aligned} \tau_3 &= \{a(\alpha_1 + \alpha_2) \ : \ a > 0\}\,, \\ \tau_4 &= \{a_1\alpha_1 \ : \ a_1 > 0\}\,, \\ \tau_5 &= \{a_2\alpha_2 \ : \ a_2 > 0\}\,, \end{aligned}$$

- 0-dimensional cone:

$$\tau_6 = \{0\}\,.$$

Let $\mu = (\mu_1, \mu_2, \mu_3)$ be in the root lattice.

(In particular, $\mu_1 + \mu_2 + \mu_3 = 0$.)

Then

$$K_q(\mu) = \begin{cases} (\mu_1 + \mu_2 - 1)q^3 + 2q^2 & \text{if } \mu \in \tau_1 \,, \\ (\mu_1 - 1)q^3 + 2q^2 & \text{if } \mu \in \tau_2 \,, \\ (\mu_1 - 1)q^3 + q^2 + q & \text{if } \mu \in \tau_3 \,, \\ q & \text{if } \mu \in \tau_4 \text{ or } \mu \in \tau_5 \,, \\ 1 & \text{if } \mu \in \tau_6 \,, \\ 0 & \text{otherwise} \,. \end{cases}$$