A $q$-analogue of the Kostant partition function and twisted representations

**Etienne Rassart**
IAS and Cornell University

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**Joint work Victor Guillemin**
Outline

- The origins of the problem (geometry)
- Twisted representations
- Making everything symmetric functions
- A branching rule
- A $q$-analogue of the Kostant partition function
The origins of the problem
Geometric setup

- $(M, \omega)$ symplectic manifold
- $G$ compact Lie group
- $\tau : G \to \text{Diff}(M)$ group action with $\tau_g^* \omega = \omega$

Assume that $M$ is prequantizable:

- $\mathbb{L} \to M$ Hermitean line bundle
- $\tau$ extends to the line bundle
- we have a connection $\nabla$ with $\text{curv}(\nabla) = \omega$
The spin-$\mathbb{C}$ Dirac operator

- The spin-$\mathbb{C}$ Dirac operator $\mathcal{D}_\mathbb{C}$ exists on every symplectic manifold.

- For $M$ prequantizable, using the connection $\nabla$, we can lift $\mathcal{D}_\mathbb{C}$ to a $G$-invariant operator $\mathcal{D}_\mathbb{C}^L$ on the line bundle $L \to M$.

- The index of $\mathcal{D}_\mathbb{C}^L$ is the virtual vector space

$$\text{Ind } \mathcal{D}_\mathbb{C}^L = \ker \mathcal{D}_\mathbb{C}^L \oplus \left(-\text{coker } \mathcal{D}_\mathbb{C}^L\right).$$

- It is a (virtual) representation of $G$ known as the quantization of $\tau$ (with respect to $\mathcal{D}_\mathbb{C}$).
Quantization and representations

- Let $G$ be a semisimple Lie group and $\mathfrak{g}$ its Lie algebra.

- Let $T$ be a Cartan subgroup (maximal torus) and $\mathfrak{t}$ its Lie algebra.

- $\mathfrak{t}^*$ is the fundamental Weyl chamber

- We have the coadjoint representation of $G$ on $\mathfrak{g}^*$.

- Denote by $O_\lambda$ the coadjoint orbit through a point $\lambda \in \mathfrak{t}^*$. 

Suppose that $\lambda$ is a dominant weight.

Then $O_\lambda$ is a prequantizable symplectic manifold.

The cokernel of $\hat{\phi}_C^L$ vanishes and the index is

$$\text{Ind } \hat{\phi}_C^L = \ker \hat{\phi}_C^L = V_\lambda,$$

the irreducible representation of $G$ with highest weight $\lambda$. (Bott-Borel-Weil theorem)
Twisted representations
The signature Dirac operator

- The signature Dirac operator $\hat{\mathcal{D}}_{\text{sig}}$ exists on any oriented manifold (with a Riemannian metric).

- The index

$$\text{Ind } \hat{\mathcal{D}}_{\text{sig}} = \ker \hat{\mathcal{D}}_{\text{sig}} \oplus (-\text{coker } \hat{\mathcal{D}}_{\text{sig}})$$

is also a virtual representation of $G$.

- It is also known as the quantization of $\tau$, but with respect to $\hat{\mathcal{D}}_{\text{sig}}$. 
Let $G$ be a semisimple Lie group and $T$ a maximal torus as above.

Let $\lambda$ be a strictly dominant weight (in the interior of the fundamental Weyl chamber).

**Theorem (Guillemin-Sternberg-Weitsman)**

The index of $\mathcal{O}_{\text{sig}}$ on the coadjoint orbit $O_\lambda$ is

$$\text{Ind } \mathcal{O}_{\text{sig}} = (-1)^{\frac{1}{2} \dim O_\lambda} V_{\lambda-\delta} \otimes V_\delta,$$

where $\delta$ is half the sum of the positive roots.
To remember

The thing to remember from all of this:

Quantization for $\mathcal{O}_C^L \iff V_\lambda$

Quantization for $\mathcal{O}_{\text{sig}} \iff V_{\lambda-\delta} \otimes V_\delta$

So in a certain world and in a certain sense, the representations $V_{\lambda-\delta} \otimes V_\delta$ play the role of the irreducible representations in the classical theory.
Twisted representations

- $\delta$ is half the sum of the positive roots.

- $V_\mu$ is the irreducible representation with highest weight $\mu$.

- For $\lambda$ strictly dominant, we define the twisted representation

$$\tilde{V}_\lambda = V_{\lambda-\delta} \otimes V_\delta.$$
Tensor products

Tensor products of twisted representations can be written in terms of twisted representations again:

\[
\tilde{V}_\lambda \otimes \tilde{V}_\mu = (V_{\lambda-\delta} \otimes V_{\delta}) \otimes (V_{\mu-\delta} \otimes V_{\delta})
\]

\[
= (V_{\lambda-\delta} \otimes V_{\delta} \otimes V_{\mu-\delta}) \otimes V_{\delta}
\]
Tensor products

Tensor products of twisted representations can be written in terms of twisted representations again:

\[ \tilde{V}_\lambda \otimes \tilde{V}_\mu = (V_{\lambda-\delta} \otimes V_\delta) \otimes (V_{\mu-\delta} \otimes V_\delta) \]

\[ = \left( V_{\lambda-\delta} \otimes V_\delta \otimes V_{\mu-\delta} \right) \otimes V_\delta \]

break into sum of irreducibles

\[ = \bigoplus \tilde{V}_\nu \oplus \tilde{N}^\nu_{\lambda\mu} \]
Symmetric functions (type $A$)
For $\text{GL}_k \mathbb{C}$ things are especially nice because

- A dominant weight is just a partition with $k$ parts (0-parts allowed).

- $\delta$ is the staircase partition:

$$\delta = (k - 1, k - 2, \ldots, 1, 0).$$

- $\lambda$ strictly dominant means the partition $\lambda$ has distinct parts ($\lambda - \delta$ is still a partition).
Taking characters

The character $\chi_\mu$ of the irreducible (polynomial) representation $V_\mu$ is a Schur polynomial:

$$\chi_\mu = s_\mu(x_1, x_2, \ldots, x_k).$$

This means that the character $\tilde{\chi}_\lambda$ of the twisted representation $\tilde{V}_\lambda$ is

$$\tilde{\chi}_\lambda = \chi_{\lambda-\delta} \cdot \chi_\delta$$

$$= s_{\lambda-\delta}(x_1, \ldots, x_k) s_\delta(x_1, \ldots, x_k).$$
Hall-Littlewood polynomials

- The character $\tilde{\chi}_\lambda$ is the $t = -1$ specialization of the Hall-Littlewood polynomial $P_\lambda(x_1, \ldots, x_k; t)$.

- $P_\lambda(x_1, \ldots, x_k; 0)$ is the Schur function $s_\lambda(x_1, \ldots, x_k)$, so the character $\chi_\lambda$.

- We (almost) have, together with J. Weitsman, a geometrical object whose character is $P_\lambda(x_1, \ldots, x_k; -(q - 1))$.

- This is where the $q$-analogue of the Kostant partition function comes in.
A branching rule
Branching rules

A branching rule describes how the restriction of a representation of a group $G$ to a subgroup $H$ decomposes into $H$-representations.
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- $\mathcal{S}_{n-1}$ is a subgroup of $\mathcal{S}_n$ (by leaving the last element fixed).

- $\text{GL}_{k-1}\mathbb{C}$ is the subgroup of $\text{GL}_k\mathbb{C}$. 

\[
\begin{pmatrix}
\text{GL}_{k-1}\mathbb{C} & 0 \\
0 & \text{GL}_{k-1}\mathbb{C} \\
\end{pmatrix}
\]
Branching for $\mathfrak{S}_n$

- Let $\{W_\lambda : \lambda \vdash n\}$ be the set of irreducible representations of $\mathfrak{S}_n$.

- Let $\{W_\mu : \mu \vdash n - 1\}$ be the set of irreducible representations of $\mathfrak{S}_{n-1}$.

Then

$$\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} W_\lambda = \bigoplus_{\mu} W_\mu$$

where the sum is over all partitions that can be obtained from $\lambda$ by removing one corner.
Branching for $GL_k \mathbb{C}$

- Let $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k)$.
- Let $\nu = (\nu_1, \ldots, \nu_{k-1})$.
- Say that $\nu$ interlaces $\lambda$ ($\nu \triangleleft \lambda$) if
  $$\lambda_1 \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \cdots \geq \lambda_{k-1} \geq \nu_{k-1} \geq \lambda_k.$$ (Obtain $\nu$ from $\lambda$ by removing horizontal strip)
- Then (Weyl’s branching rule)
  $$\text{Res}_{GL_{k-1} \mathbb{C}}^{GL_k \mathbb{C}} V_{\lambda} = \bigoplus_{\nu \triangleleft \lambda} V_{\nu}.$$
Branching rule for the $\widetilde{V}_\lambda$

- Let $\lambda = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k)$ distinct parts.
- Let $\nu = (\nu_1, \ldots, \nu_{k-1})$ distinct parts.
- For $\nu \triangleleft \lambda$, let
  $$\nabla(\lambda, \nu) = \{ i \in [k-1] : \lambda_i > \nu_i > \lambda_{i+1} \} .$$

**Theorem** *(Guillemin-R)*

$$\text{Res}_{GL_{k-1}\mathbb{C}}^{GL_k\mathbb{C}} \widetilde{V}_\lambda = \bigoplus_{\nu \triangleleft \lambda} 2^{\nabla(\lambda, \nu)} \widetilde{V}_\nu .$$

$\nu$ distinct parts
Sketch of proof

- In terms of the characters, restricting just means setting the last variable, $x_k$, equal to 1.
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- We have $s_{\lambda}(x, y) = \sum_{\mu \subseteq \lambda} s_{\mu}(x) s_{\lambda/\mu}(y)$. 
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  \[ s_\lambda(x, y) = \sum_{\mu \subseteq \lambda} s_\mu(x) s_{\lambda/\mu}(y). \]

- Now
  \[ s_{\lambda/\mu}(1) = \begin{cases} 
  1 & \text{if } \lambda/\mu \text{ is horizontal strip}, \\
  0 & \text{otherwise}. 
\end{cases} \]
Sketch of proof

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- So $s_\lambda(x_1, \ldots, x_{k-1}, 1) = \sum_{\mu \triangleleft \lambda} s_\mu(x_1, \ldots, x_{k-1})$. 
On the other hand

\[ s_\delta(x_1, \ldots, x_k) = \frac{\text{Vandermonde}(x_1^2, \ldots, x_k^2)}{\text{Vandermonde}(x_1, \ldots, x_k)} = \prod_{1 \leq i < j \leq k} (x_i + x_j). \]
On the other hand

\[ s_\delta(x_1, \ldots, x_k) = \frac{\text{Vandermonde}(x_1^2, \ldots, x_k^2)}{\text{Vandermonde}(x_1, \ldots, x_k)} \]

\[ = \prod_{1 \leq i < j \leq k} (x_i + x_j). \]

Then

\[ s_\delta|_{x_k=1} = \prod_{1 \leq i < j \leq k-1} (x_i + x_j) \prod_{i=1}^{k-1} (1 + x_i) \]

\[ = s_{\delta'}(x_1, \ldots, x_{k-1})(e_0 + e_1 + \cdots + e_{k-1}). \]

where \( \delta' = (k - 2, k - 3, \ldots, 1, 0). \)
Define $f^{(m)} = f(x_1, \ldots, x_m)$. We have

$$s^{(k)}_{\lambda - \delta} s^{(k)}_{\delta} |_{x_k = 1} = \sum_{\mu < \lambda - \delta} s^{(k-1)}_{\mu} s^{(k-1)}_{\delta'} (e_0^{(k-1)} + \cdots + e_{k-1}^{(k-1)})$$
Define \( f^{(m)} = f(x_1, \ldots, x_m) \). We have

\[
\left. s^{(k)}_{\lambda-\delta} s^{(k)}_{\delta} \right|_{x_k=1} = \sum_{\mu < \lambda-\delta} s^{(k-1)}_{\mu} s^{(k-1)}_{\delta'} (e^{(k-1)}_{0} + \cdots + e^{(k-1)}_{k-1})
\]

By a dual version of the Pieri rule,

\[
s_{\mu} e_m = \sum_{\nu} s_{\nu},
\]

where the sum is over all \( \nu \) obtained from \( \mu \) by adding a vertical strip of size \( m \).
Finally,

\[ S_{\lambda - \delta}^{(k)} S_{\delta}^{(k)} |_{x_k = 1} = \sum_{\mu < \lambda - \delta} \sum_{\nu} S_{\nu}^{(k-1)} S_{\delta'}^{(k-1)} \]

\[ \tilde{\chi}_\lambda(x_1, \ldots, x_{k-1}, 1) = \sum_{\mu < \lambda - \delta} \sum_{\nu} \tilde{\chi}_{\nu + \delta}(x_1, \ldots, x_{k-1}), \]

where the sum is over all \( \nu \) that can be obtained from \( \mu \) by adding a vertical strip of size at most \( k - 1 \) (and height at most \( k - 1 \)).
We can relabel things and group terms.

For partitions with distinct parts $\lambda$ and $\nu$, let $n(\lambda, \nu)$ be the number of ways that $\nu - \delta'$ can be obtained by adding a vertical strip of size and height at most $k - 1$ to some partition $\mu$ interlacing $\lambda - \delta$. Then

$$\tilde{V}_{\lambda} = \bigoplus_{\nu \triangleleft \lambda, \text{distinct parts}} n(\lambda, \nu) \tilde{V}_\nu.$$
We can relabel things and group terms.

For partitions with distinct parts \( \lambda \) and \( \nu \), let \( n(\lambda, \nu) \) be the number of ways that \( \nu - \delta' \) can be obtained by adding a vertical strip of size and height at most \( k - 1 \) to some partition \( \mu \) interlacing \( \lambda - \delta \). Then

\[
\tilde{V}_\lambda = \bigoplus_{\nu \triangleleft \lambda, \text{distinct parts}} n(\lambda, \nu) \tilde{V}_\nu.
\]

**Lemma**

\[
n(\lambda, \nu) = 2^{\nabla(\lambda, \nu)}.
\]
Twisted Gelfand-Tsetlin diagrams

Since \( \operatorname{Res}_{GL_{k-1}\mathbb{C}}^{GL_k\mathbb{C}} \operatorname{Res}_{GL_{k-2}\mathbb{C}}^{GL_{k-1}\mathbb{C}} = \operatorname{Res}_{GL_{k-2}\mathbb{C}}^{GL_{k}\mathbb{C}} \), we can iterate the branching rule to restrict to \( GL_1\mathbb{C} \).

This way, we can index one-dimensional subspaces of a twisted representation by a sequence of interlacing partitions with distinct parts. (Indexing not multiplicity-free)

This sequence forms a triangular array of integers akin to Gelfand-Tsetlin diagrams for the irreducible representations of \( GL_k\mathbb{C} \).
Proposition

\[ \text{Res}_{\text{GL}_1 \mathbb{C}}^{\text{GL}_k \mathbb{C}} \tilde{V}_\lambda = \bigoplus_{\lambda^{(1)} \prec \ldots \prec \lambda^{(k)} = \lambda} 2^{\nabla(\lambda^{(k)}, \lambda^{(k-1)}, \ldots, \lambda^{(1)})} \tilde{V}_{\lambda^{(1)}} \]

where \( \lambda^{(m)} \) is a partition with \( m \) distinct parts and

\[ \nabla(\lambda^{(k)}, \lambda^{(k-1)}, \ldots, \lambda^{(1)}) = \sum_{i=1}^{k-1} \nabla(\lambda^{(i+1)}, \lambda^{(i)}). \]
A $q$-analogue of the KPF
Characters in general

For a semisimple Lie group $G$ with Cartan subgroup $T$, the character of the irreducible representation of $G$ with highest weight $\lambda \in t^*_+$ is given by Weyl’s character formula:

$$
\chi_\lambda = \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} e^{\omega(\lambda + \delta) - \delta} \prod_{\alpha \in \Delta_+} \frac{1}{1 - e^{-\alpha}},
$$

where $\mathcal{W}$ is the Weyl group and $\Delta_+$ is the set of positive roots.

For $GL_k \mathbb{C}$, thinking of $x_i = \exp(e_i)$, this is the Schur function $s_\lambda(x_1, \ldots, x_k)$. 
The Kostant partition function

- We can define the **Kostant partition function** $\mathbf{K}$ through its generating function:

$$\sum_{\mu} \mathbf{K}(\mu) e^\mu = \prod_{\alpha \in \Delta_+} \frac{1}{1 - e^\alpha}.$$ 

- So $\mathbf{K}(\mu)$ is the number of ways that $\mu$ can be written as a sum of positive roots.

- **Kostant’s multiplicity formula** expresses weight multiplicities in terms of the KPF. For $\text{GL}_k \mathbb{C}$, these are **Kostka numbers**.
"q\text{-analogues}

Our general geometric object has “character”

\[
\chi^{(q)}_\lambda = \sum_{\omega \in \mathcal{W}} e^{\omega(\lambda)} \prod_{\alpha \in \Delta_+} \frac{1 + (q - 1)e^{-\omega(\alpha)}}{1 - e^{-\omega(\alpha)}}.
\]

This prompts the definition of a \(q\text{-analogue} K_q\) of the KPF by

\[
\sum_{\mu} K_q(\mu) e^\mu = \prod_{\alpha \in \Delta_+} \frac{1 + (q - 1)e^\alpha}{1 - e^\alpha}.
\]
Remarks

- We can think of \( K_q \) as counting integer points in a polytope with certain weights (in \( q \)).

- The \( q = 2 \) case is the one yielding the twisted representations.

- This is not the classical \( q \)-analogue of the KPF defined by Lusztig through

  \[
  \sum_{\mu} \hat{K}_q(\mu) \ e^\mu = \prod_{\alpha \in \Delta_+} \frac{1}{1 - q \ e^\alpha}
  \]

  to get a \( q \)-analogue of weight multiplicities.
Guillemin, Sternberg and Weitsman have a formula analogous to the Kostant multiplicity formula for twisted representations. Their proof is geometric (Atiyah-Bott formula). We have a purely algebraic proof of this.

We also have an analogue of the Steinberg formula for decomposing tensor products of twisted representations into twisted representations.

Both formulas involve the $q = 2$ specialization of the $q$-analogue of the KPF.
A structural result

**Theorem** (Guillemin-R)

The $q$-analogue $K_q(\mu)$ for the root system $A_k$ is given by polynomials of degree $\binom{k}{2}$ with coefficients in $\mathbb{Q}[q]$ of degree $\binom{k+1}{2}$ over the relative interior of the cells of the chamber complex for the Kostant partition function.
Example: $A_2$

- 2-dimensional cones:
  \[
  \tau_1 = \left\{ a_1 \alpha_1 + a_2 \alpha_2 : a_1, a_2 > 0 \text{ and } a_1 > a_2 \right\},
  \tau_2 = \left\{ a_1 \alpha_1 + a_2 \alpha_2 : a_1, a_2 > 0 \text{ and } a_1 < a_2 \right\},
  \]

- 1-dimensional cones:
  \[
  \tau_3 = \left\{ a(\alpha_1 + \alpha_2) : a > 0 \right\},
  \tau_4 = \left\{ a_1 \alpha_1 : a_1 > 0 \right\},
  \tau_5 = \left\{ a_2 \alpha_2 : a_2 > 0 \right\},
  \]

- 0-dimensional cone:
  \[
  \tau_6 = \{0\}. 
  \]
Let $\mu = (\mu_1, \mu_2, \mu_3)$ be in the root lattice.
(In particular, $\mu_1 + \mu_2 + \mu_3 = 0$.)

Then

$$K_q(\mu) = \begin{cases} 
(\mu_1 + \mu_2 - 1)q^3 + 2q^2 & \text{if } \mu \in \tau_1, \\
(\mu_1 - 1)q^3 + 2q^2 & \text{if } \mu \in \tau_2, \\
(\mu_1 - 1)q^3 + q^2 + q & \text{if } \mu \in \tau_3, \\
q & \text{if } \mu \in \tau_4 \text{ or } \mu \in \tau_5, \\
1 & \text{if } \mu \in \tau_6, \\
0 & \text{otherwise.}
\end{cases}$$