Polynomiality properties of the Kostka numbers and Littlewood-Richardson coefficients

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Joint work with Sara Billey and Victor Guillemin

About $\mathfrak{sl}_k\mathbb{C}$

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- We can take as the Cartan subalgebra $\mathfrak{h}$ the diagonal subalgebra.
- $\mathfrak{sl}_k\mathbb{C}$ and $\mathfrak{gl}_k\mathbb{C}$ differ very little:

$$\mathfrak{gl}_k\mathbb{C} = \mathfrak{sl}_k\mathbb{C} \oplus \mathbb{C}I \, .$$

Roots

- The dual $\mathfrak{h}^*$ of the Cartan subalgebra can be identified with

$$\{(x_1, \ldots, x_k) \in \mathbb{R}^k \ : \ x_1 + \cdots + x_k = 0\},$$

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the hyperplane of $\mathbb{R}^k$ where the coordinates sum up to zero.

- With this identification, the root system is

$$\Delta = \{e_i - e_j \ : \ 1 \leq i \neq j \leq k\}.$$

- The positive roots are

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- There are $k-1$ simple roots and they form a basis of $\mathfrak{h}^*$.

Weights

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- The root lattice is $\Lambda_R = \mathbb{Z}\Delta$.

- The weight lattice is $\Lambda_W = \mathbb{Z}\{\omega_1, \ldots, \omega_{k-1}\}$.

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$$\delta = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha = \sum_{i=1}^{k-1}\omega_i$$

$$= \frac{1}{2}(k-1, k-3, \ldots, -(k-3), -(k-1)).$$

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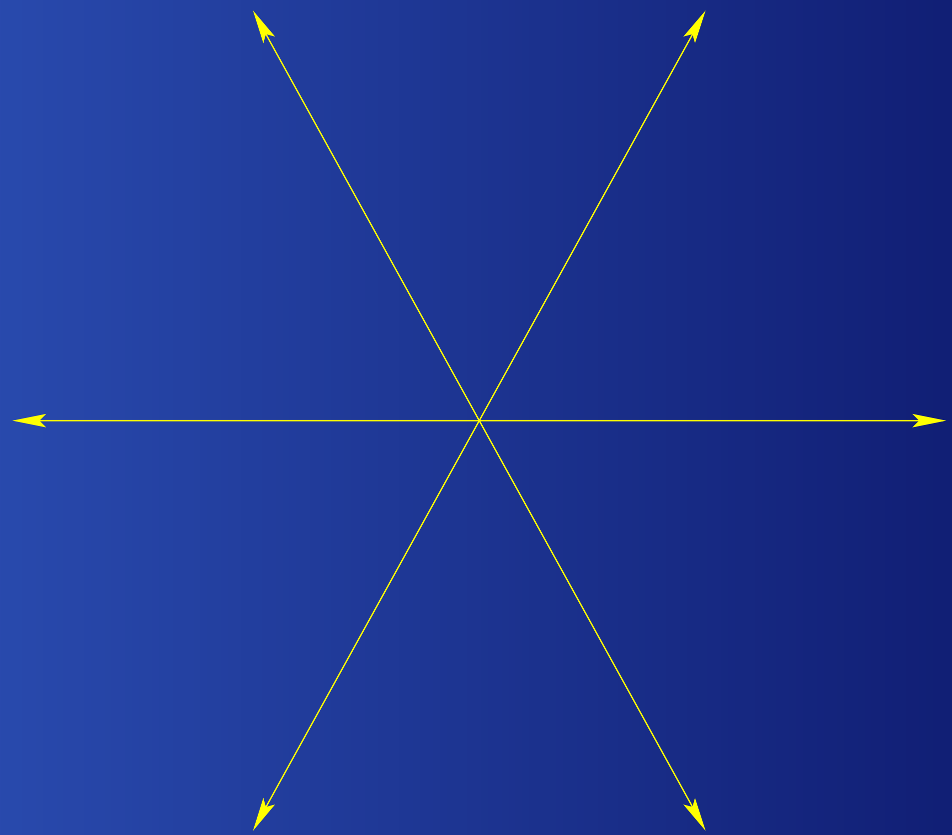
- In the case of $\mathfrak{sl}_k\mathbb{C}$, the reflection through the hyperplane with normal $e_i - e_j$ simply interchanges the ith and jth coordinates.
- Thus the Weyl group for $\mathfrak{sl}_k\mathbb{C}$ is the symmetric group $\mathfrak{S}_k$ acting on $\{e_1, \dots, e_k\}$.
- The lattices Λ_R and Λ_W are invariant under the action of the Weyl group.

Example: A_2



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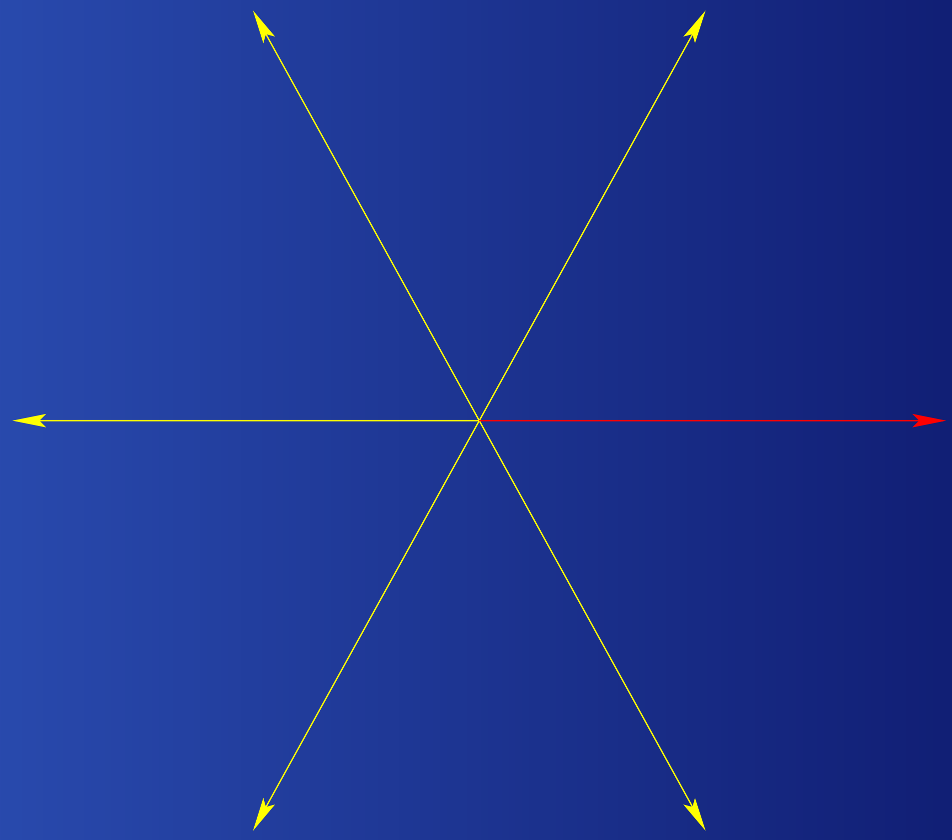

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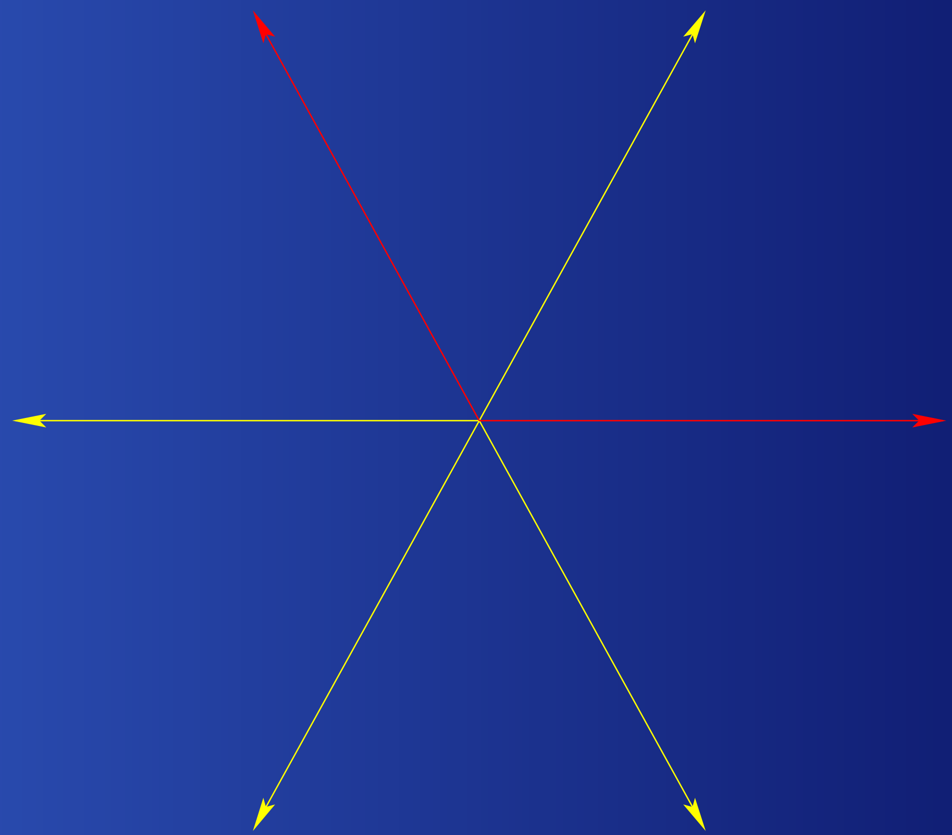

$$\alpha_2 = e_2 - e_3$$

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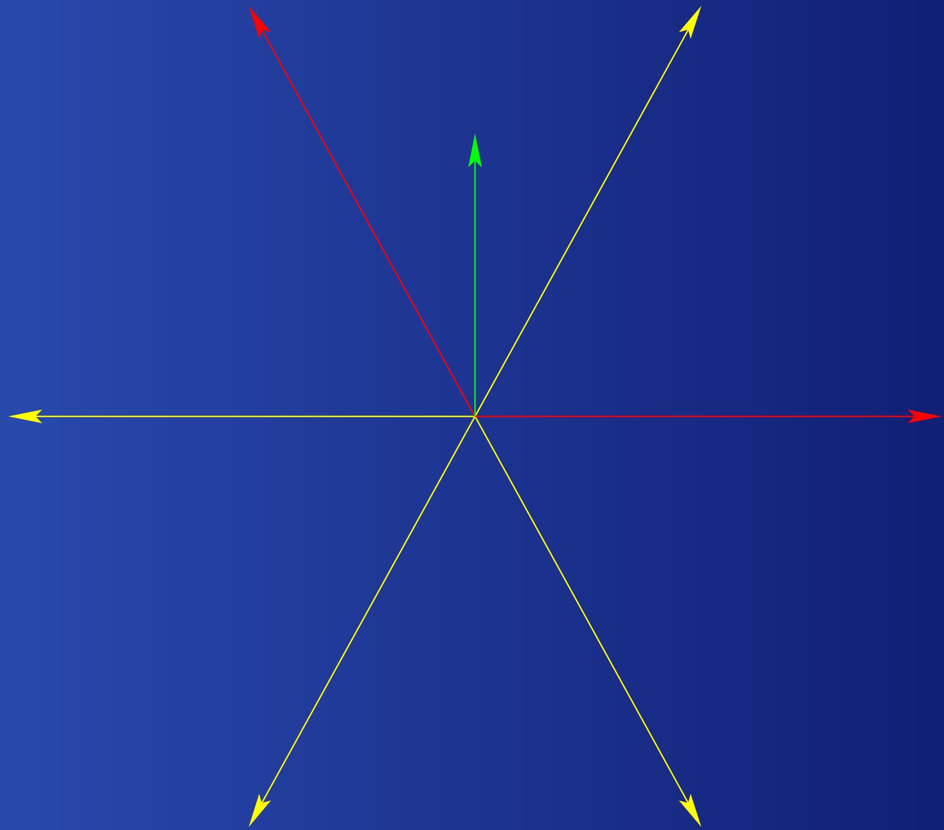

$$\omega_1 = e_1 - \frac{1}{3}(1 \;\; 1 \;\; 1)$$

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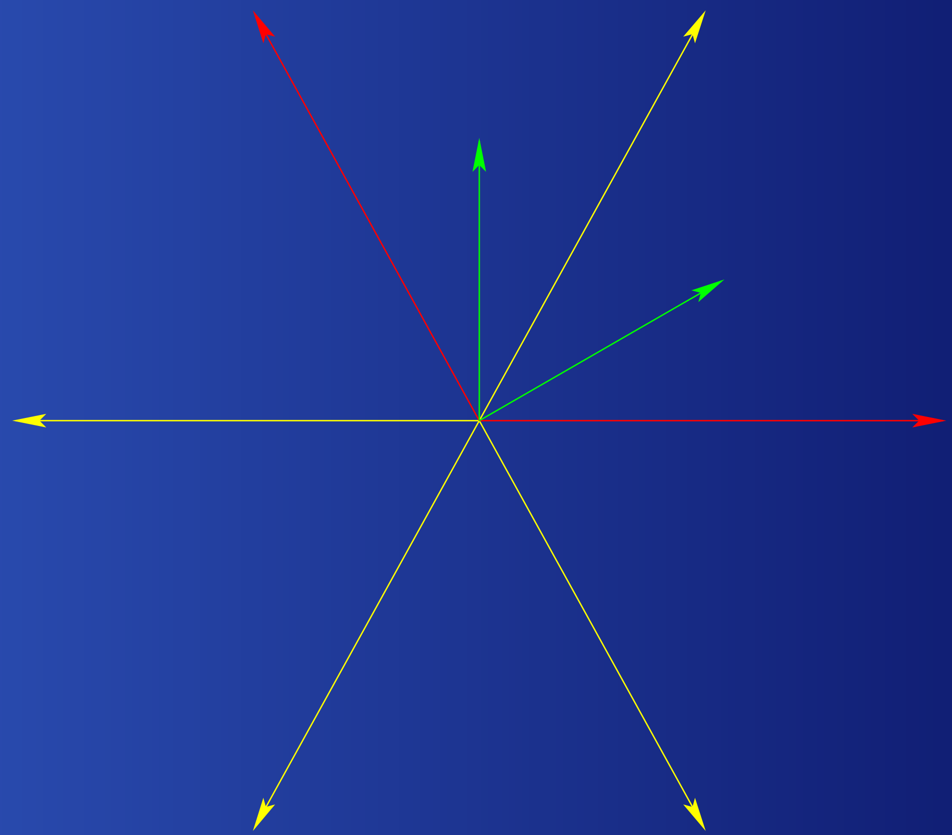

$$\omega_2 = e_1 + e_2 - \frac{2}{3}(1 \;\; 1 \;\; 1)$$

Representations

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- Any representation V of a complex semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ can be broken up into the weight space decomposition

$$V = \bigoplus_{\beta \in \mathfrak{h}^*} V_\beta$$

where

$$V_\beta = \{v \in V \ : \ \rho(h) \cdot v = \beta(h) v \ \ \forall h \in \mathfrak{h}\}.$$

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  - $\beta \in \Lambda_W$,
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  - $\beta \in \operatorname{conv}(\mathfrak{S}_k \cdot \lambda)$.

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- Those are indexed by partitions with at most k parts.
- The character of the representation indexed by λ is the Schur symmetric function $s_\lambda(x_1, \ldots, x_k)$.

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- This means that

$$K_{\lambda\mu} = m_{\bar{\lambda}}(\bar{\mu})$$

where $\bar{\gamma} = \gamma - \frac{|\gamma|}{k}(1 \;\; 1 \;\; \cdots \;\; 1)$.

Kostant's multiplicity formula

The Kostant partition function is the function

$$K(v) = \left| \left\{ (k_\alpha)_{\alpha \in \Delta_+} \in \mathbb{N}^{|\Delta_+|} \ : \ \sum_{\alpha \in \Delta_+} k_\alpha \alpha = v \right\} \right| ,$$

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Kostant's multiplicity formula

$$m_\lambda(\beta) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma)} K(\sigma(\lambda + \delta) - (\beta + \delta)).$$