Polynomiality properties of the Kostka numbers and Littlewood-Richardson coefficients

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October 8, 2003

Joint work with Sara Billey and Victor Guillemin

Outline

- Introduction with pictures
- A partition function for the Kostka numbers
- Some symplectic geometry
- The Kostant arrangements
- Polynomiality in the chamber complex
- Factorization patterns
- Littlewood-Richardson coefficients

Introduction

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- The Kostka number $K_{\lambda\beta}$ is the number of semistandard Young tableaux of shape λ and content β.

- $K_{\lambda\beta}$ is also the multiplicity with which the weight β appears in the irreducible representation of $\mathrm{GL}_k\mathbb{C}$ (or $\mathrm{SL}_k(\mathbb{C})$) with highest weight λ.

Schur functions

$$s_\lambda(x_1,\ldots,x_k) = \sum_{T \in \mathrm{SSYT}(\lambda;k)} \mathbf{x}^T .$$

$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$
$x_1^2x_2$	$x_1^2x_3$	$x_1x_2^2$	$x_1x_2x_3$	$x_1x_2x_3$	$x_1x_3^2$	$x_2^2x_3$	$x_2x_3^2$

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$$\begin{ytableau} 1 & 1 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 1 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \quad \begin{ytableau} 1 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 2 & 3 \\ 3 \end{ytableau}$$

$$x_1^2x_2 \quad x_1^2x_3 \quad x_1x_2^2 \quad x_1x_2x_3 \quad x_1x_2x_3 \quad x_1x_3^2 \quad x_2^2x_3 \quad x_2x_3^2$$

$$\begin{aligned} s_{\ydiagram{2,1}}(x_1, x_2, x_3) &= x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 \\ &\quad + 2\, x_1x_2x_3 . \end{aligned}$$

Kostka numbers

- From the definition of the Schur functions, we have that

$$s_\lambda = \sum_\beta K_{\lambda\beta}\, \mathbf{x}^\beta \,,$$

where $K_{\lambda\beta}$ is the number of ways of filling a SSYT of shape λ with integers distributed according to composition β.

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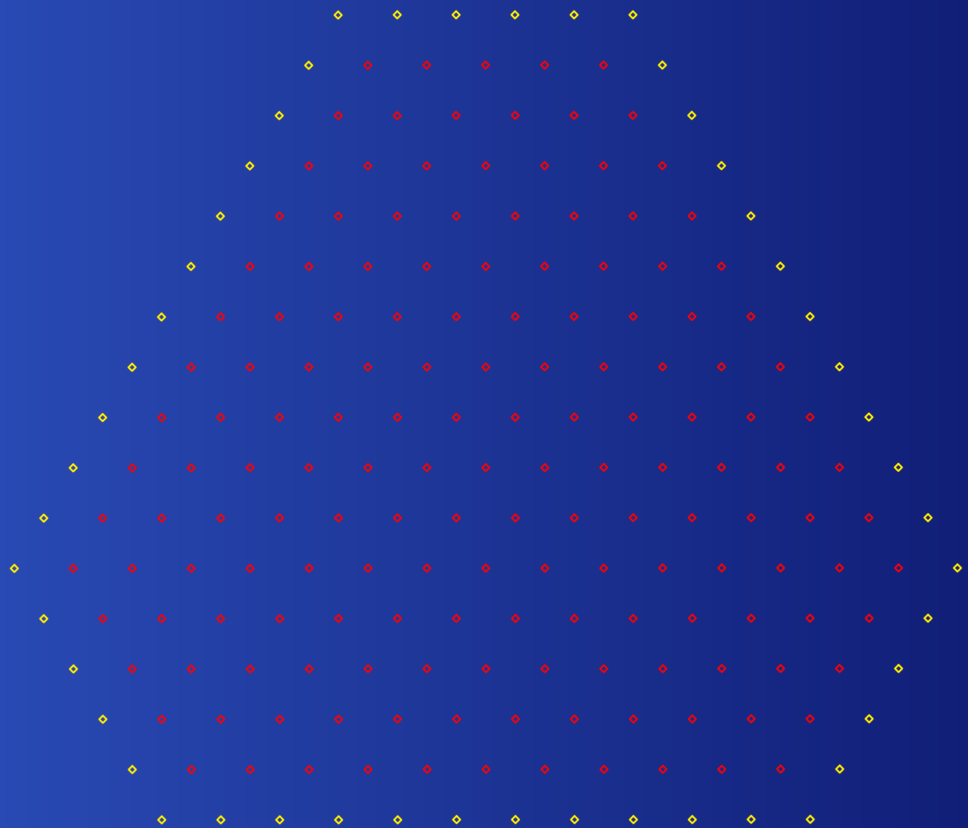
where $K_{\lambda\beta}$ is the number of ways of filling a SSYT of shape λ with integers distributed according to composition β.

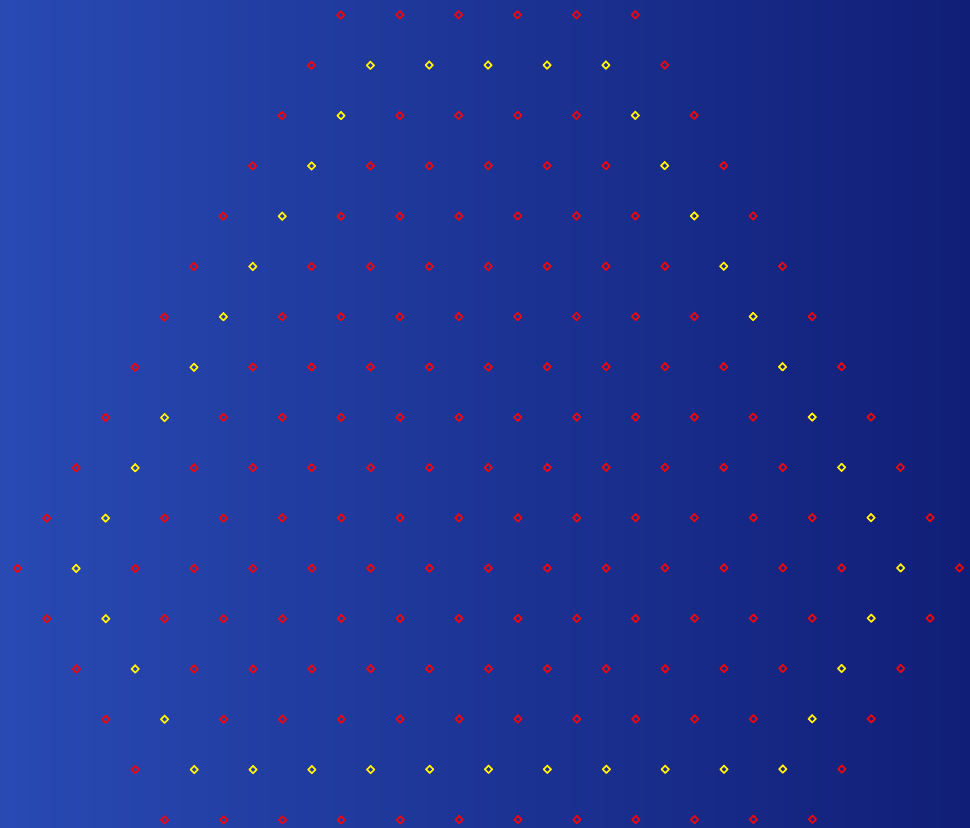
- The set of β's for which $K_{\lambda\beta} \neq 0$ consists of the lattice points inside the convex hull of the orbit of λ under $\mathfrak{S}_k$. This convex hull is a permutahedron.

$\lambda = (18,\ 7,\ 2)$



$$K_{\lambda\beta} = 1$$

$\lambda = (18,\ 7,\ 2)$



$$K_{\lambda\beta} = 2$$

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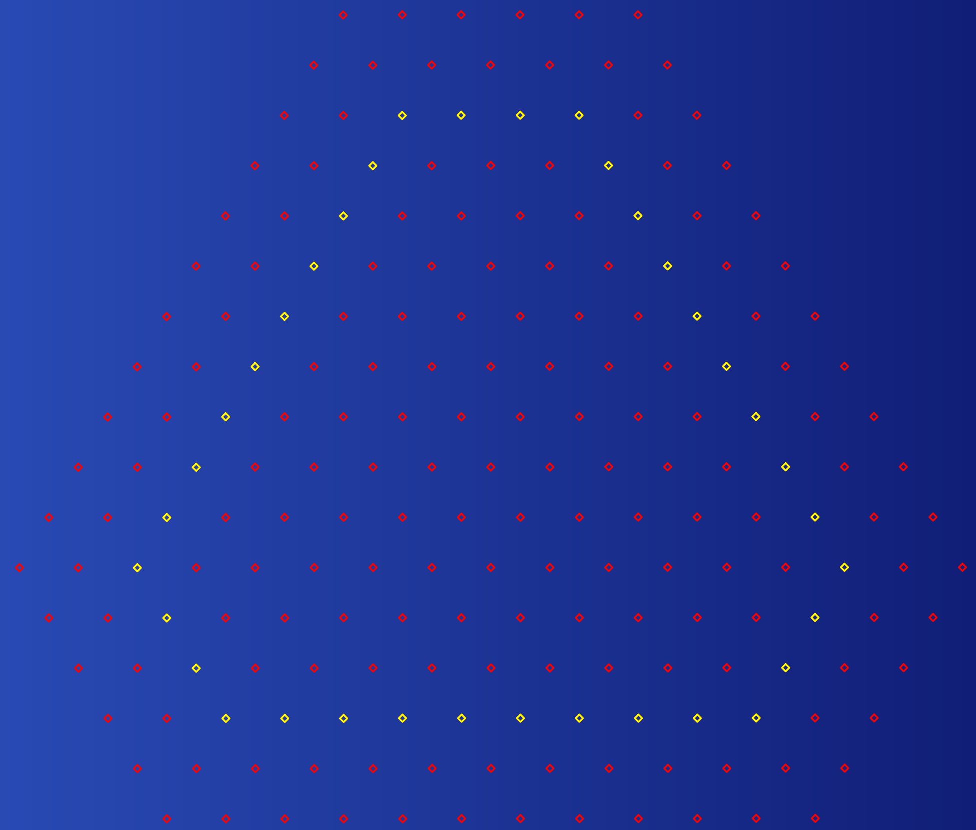

$$K_{\lambda\beta} = 3$$

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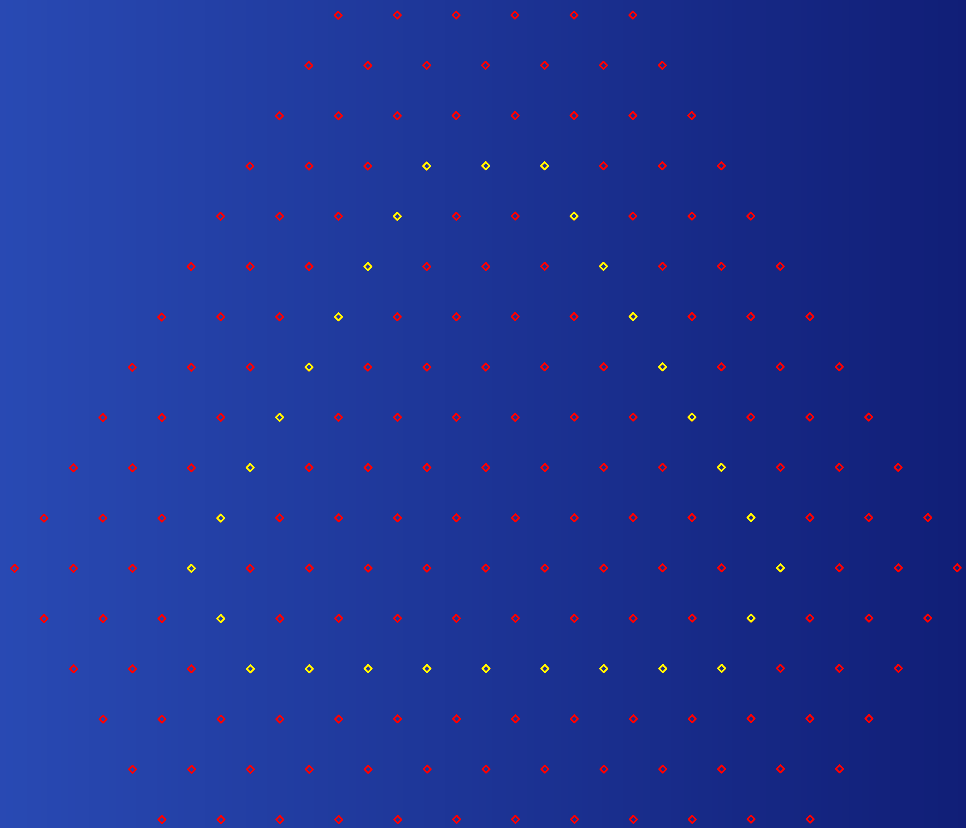

$$K_{\lambda\beta} = 4$$

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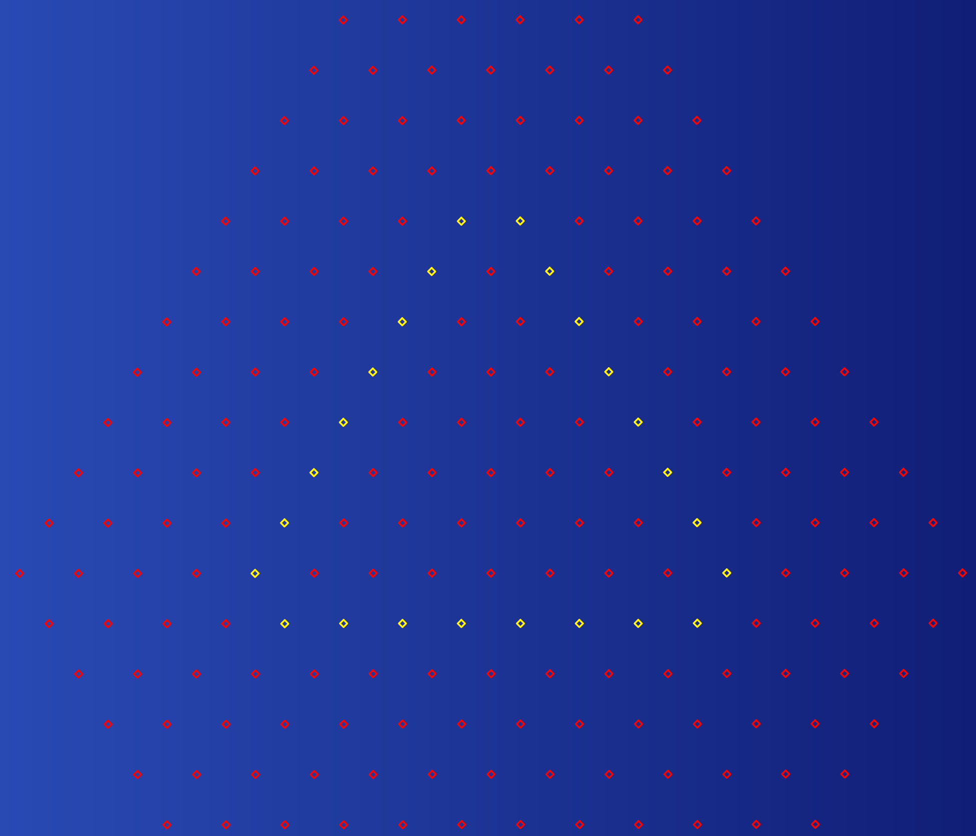

$$K_{\lambda\beta} = 5$$

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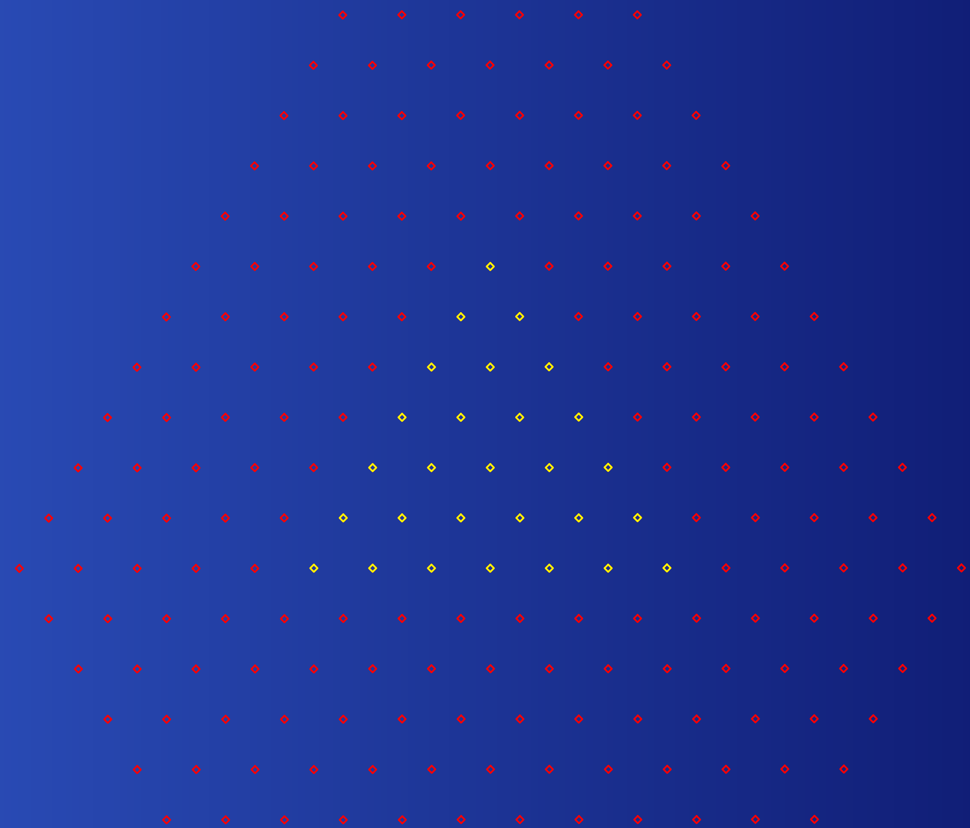
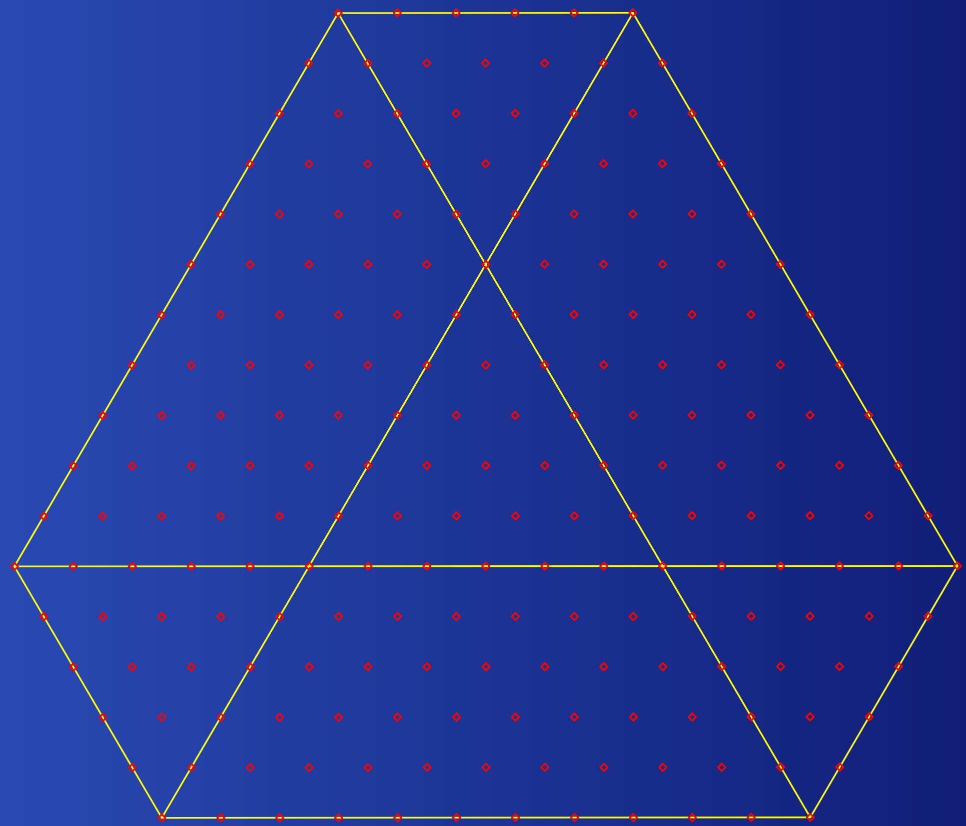


$$K_{\lambda\beta} = 6$$

$\lambda = (18, 7, 2)$



As λ varies

- Up to deformation: two “generic” cases

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- 8 polynomials suffice to describe all the Kostka numbers for partitions with at most three parts

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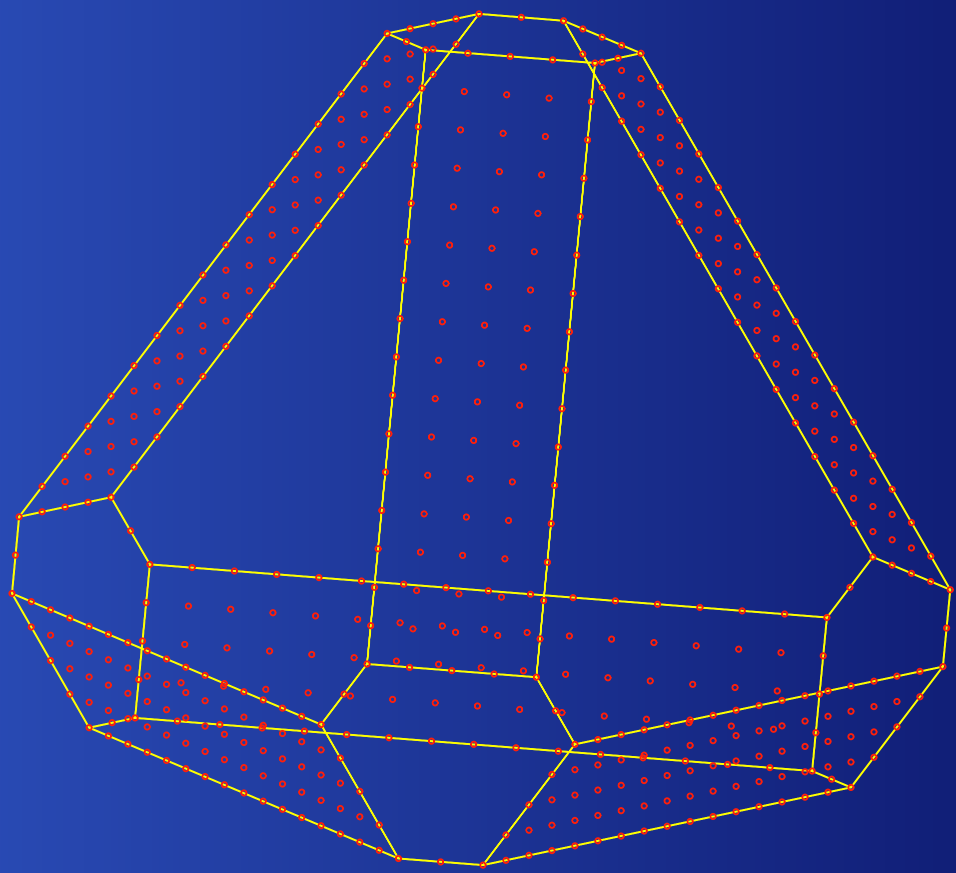
- 8 polynomials suffice to describe all the Kostka numbers for partitions with at most three parts
- Central region (*lacunary*) in which the Kostka numbers are constant

$\lambda = (23, 7, 5, 1)$



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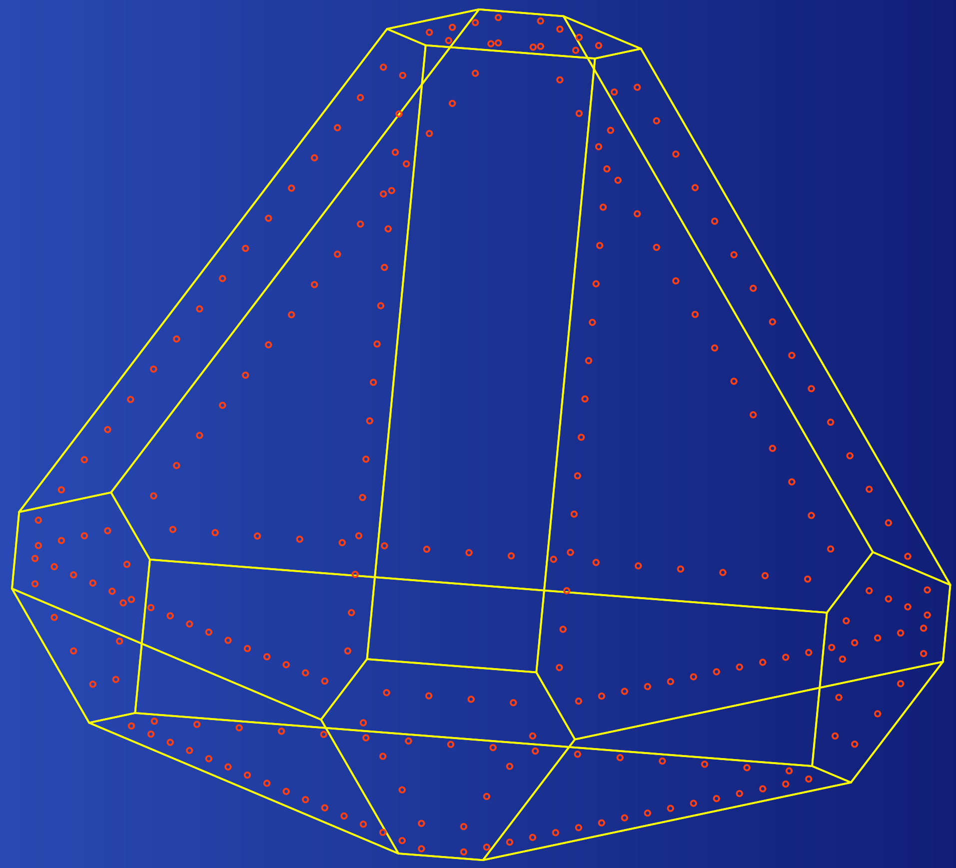

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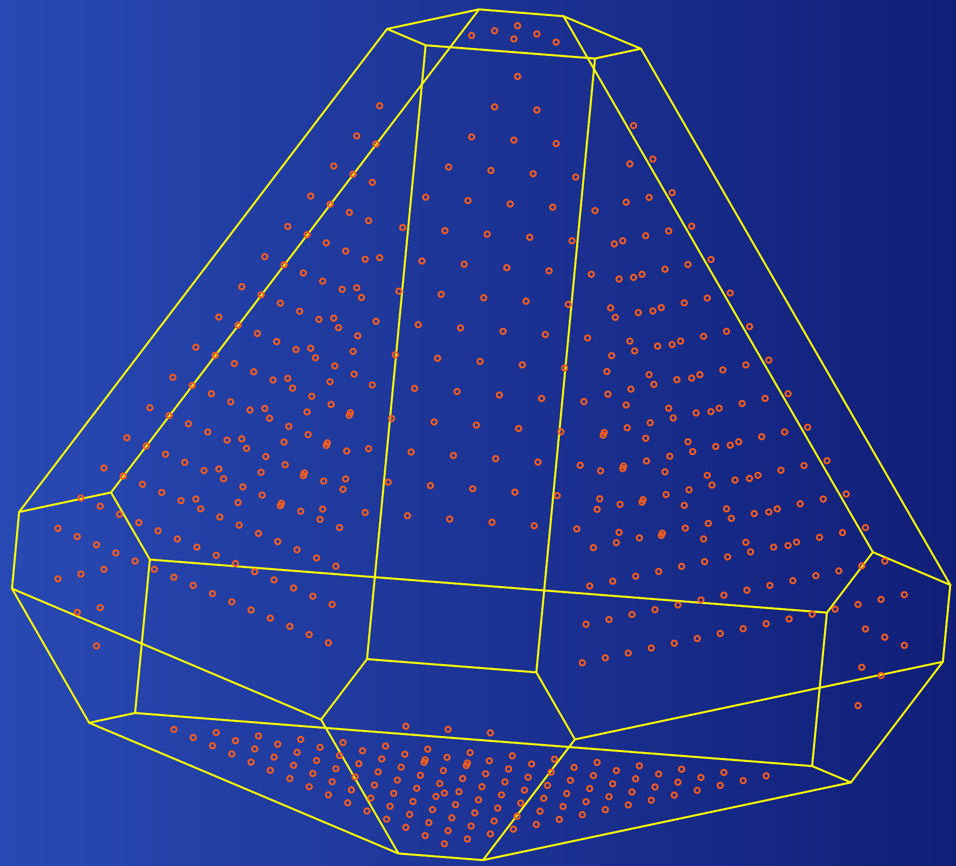

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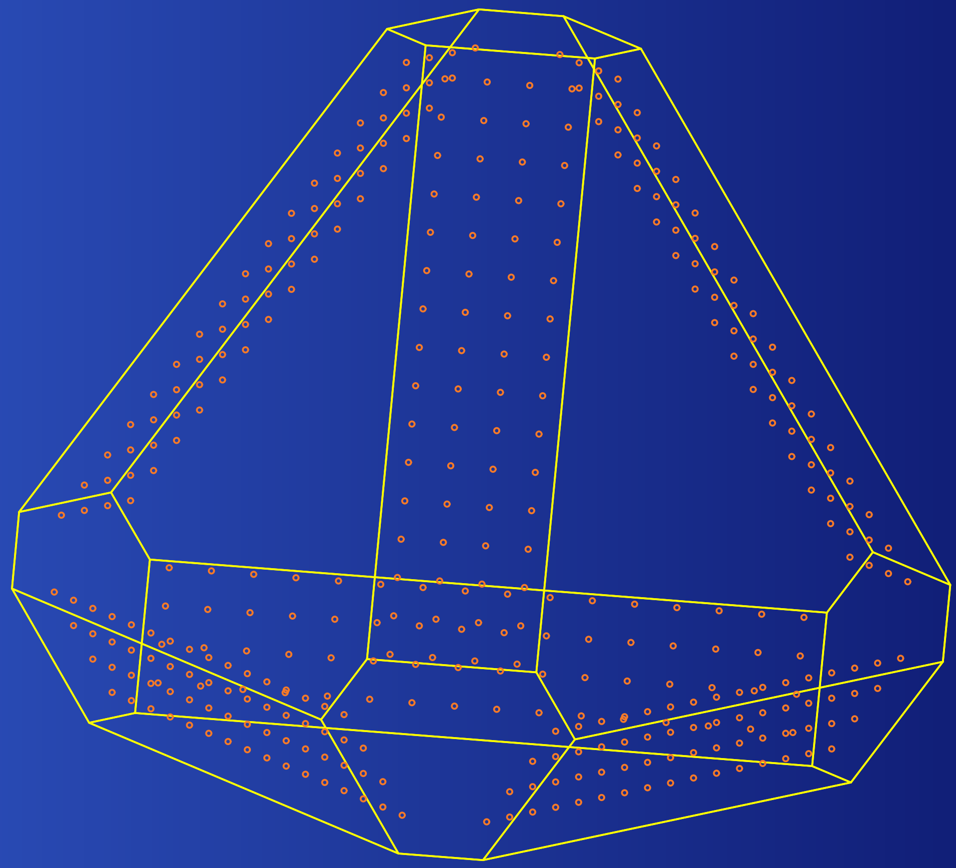

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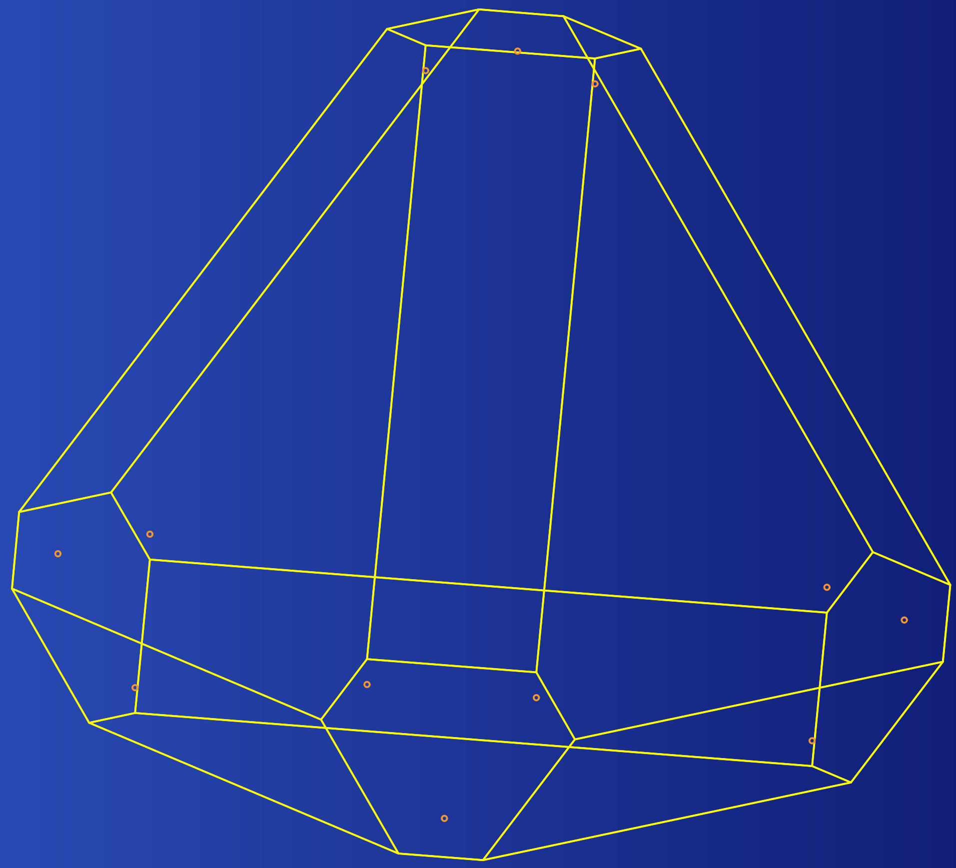

$$K_{\lambda\beta} = 5$$

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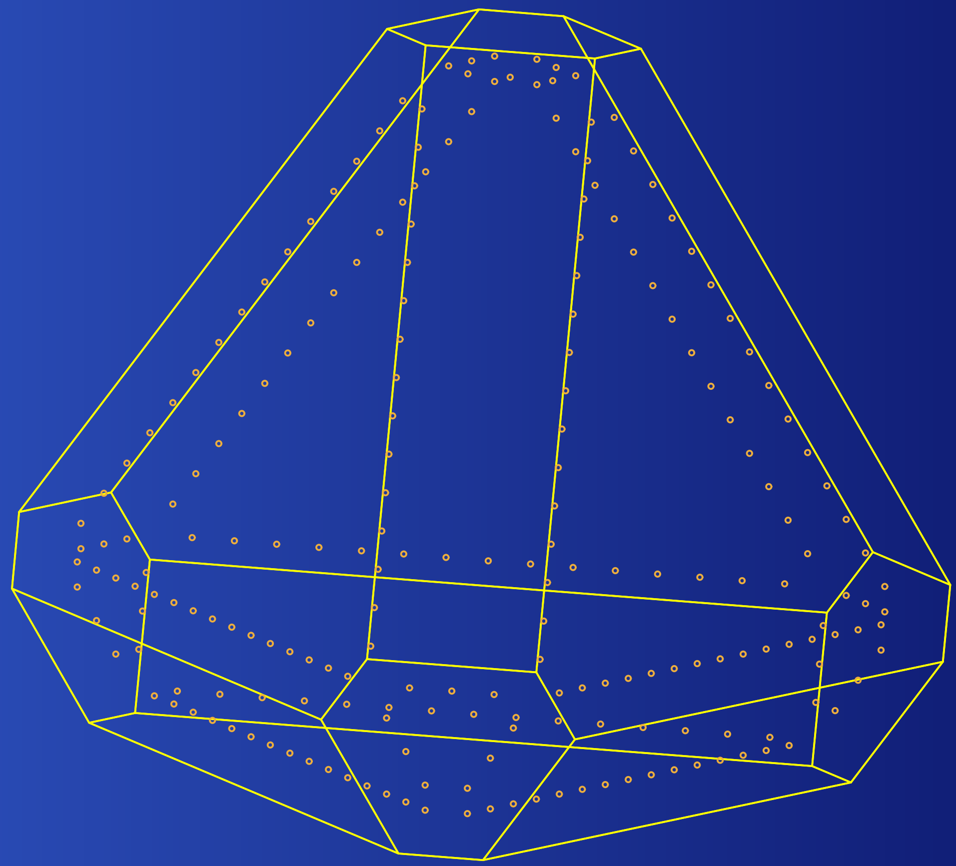

$$K_{\lambda\beta} = 7$$

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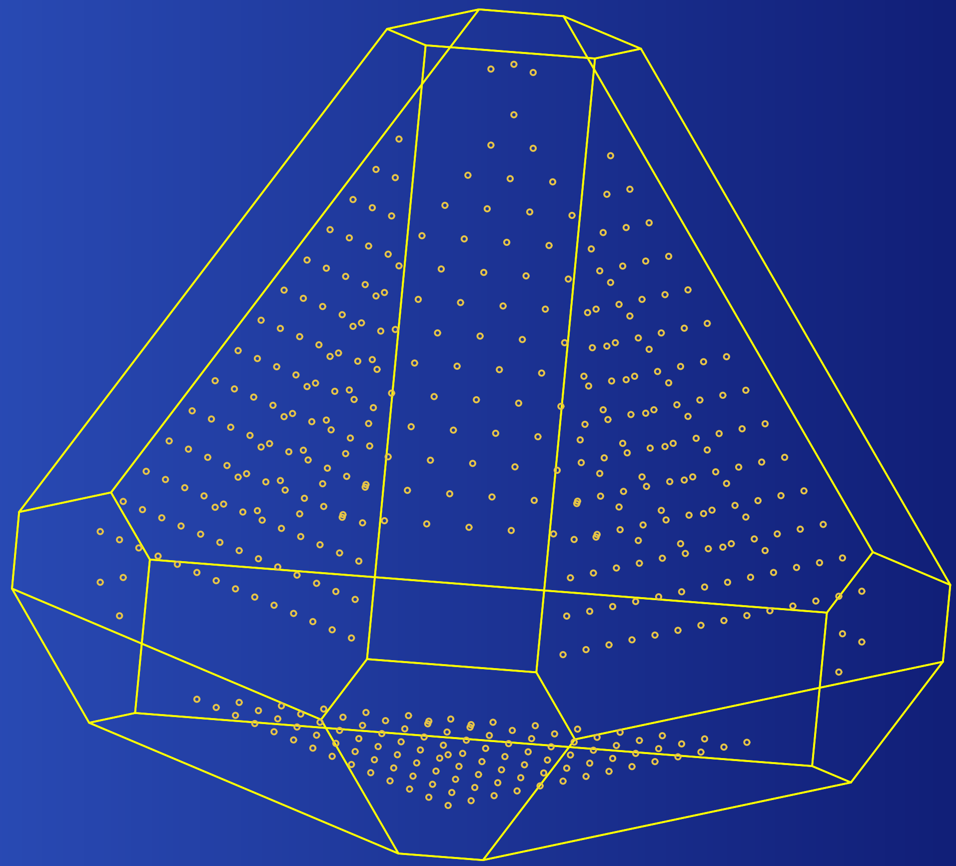

$$K_{\lambda\beta} = 9$$

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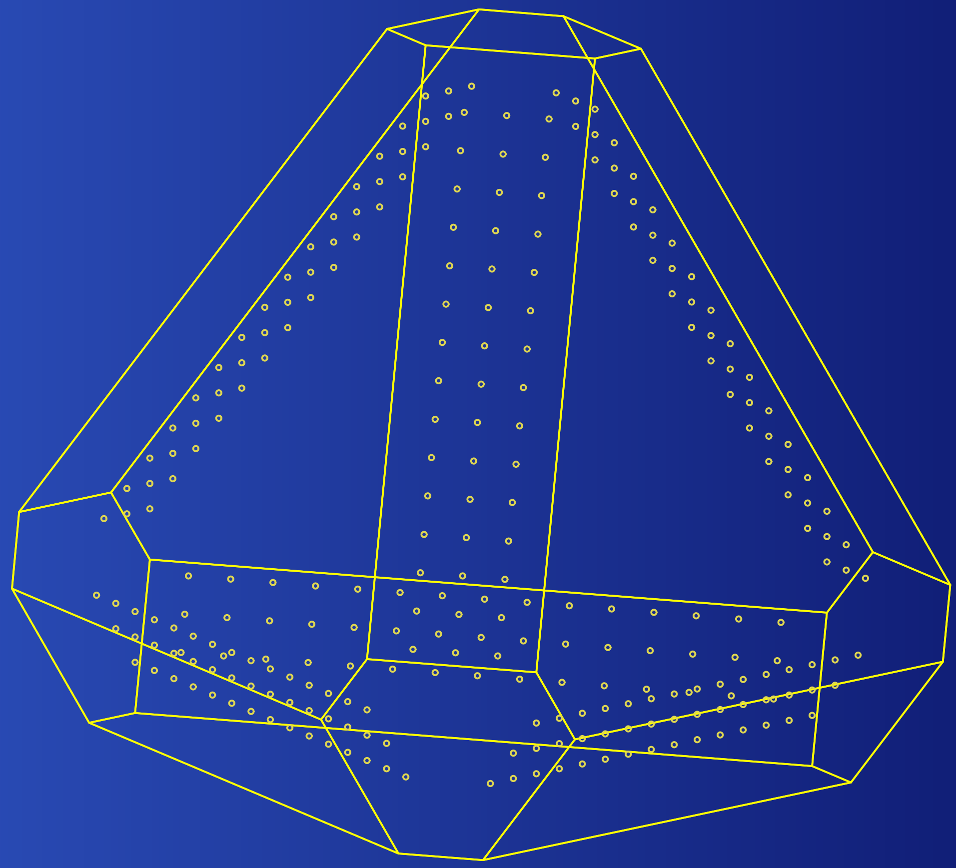

$$K_{\lambda\beta} = 10$$

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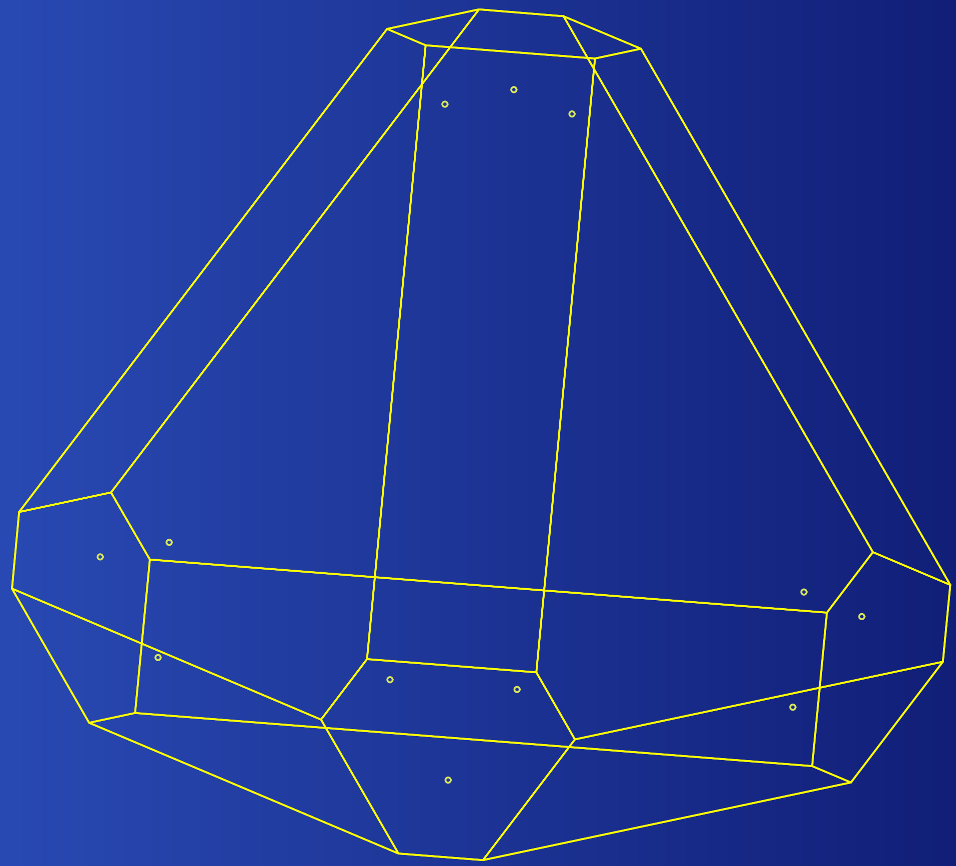
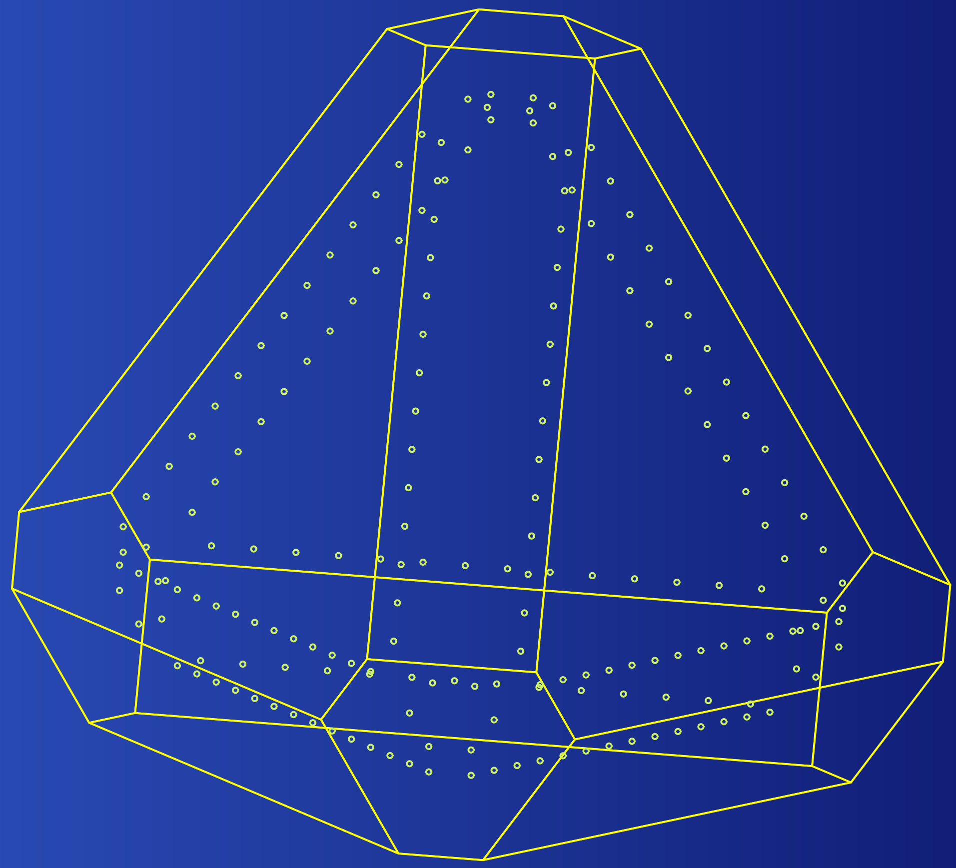


$$K_{\lambda\beta} = 12$$

$\lambda = (23,\ 7,\ 5,\ 1)$



$$K_{\lambda\beta} = 15$$

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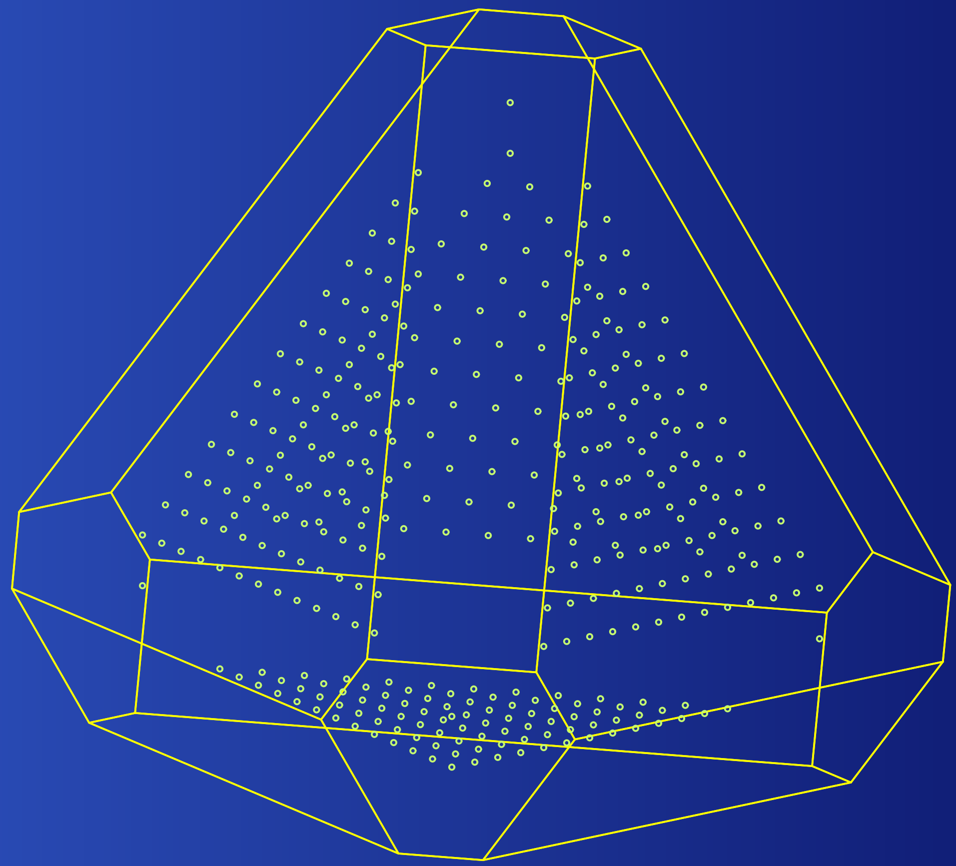

$$K_{\lambda\beta} = 18$$

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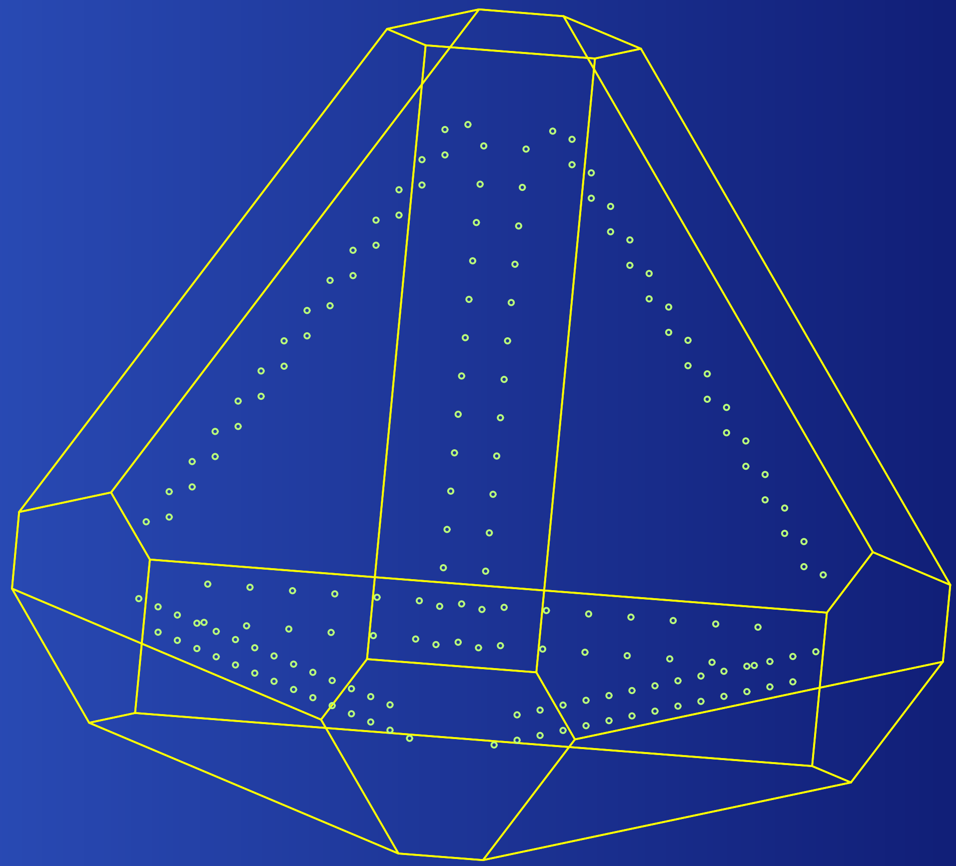

$$K_{\lambda\beta} = 19$$

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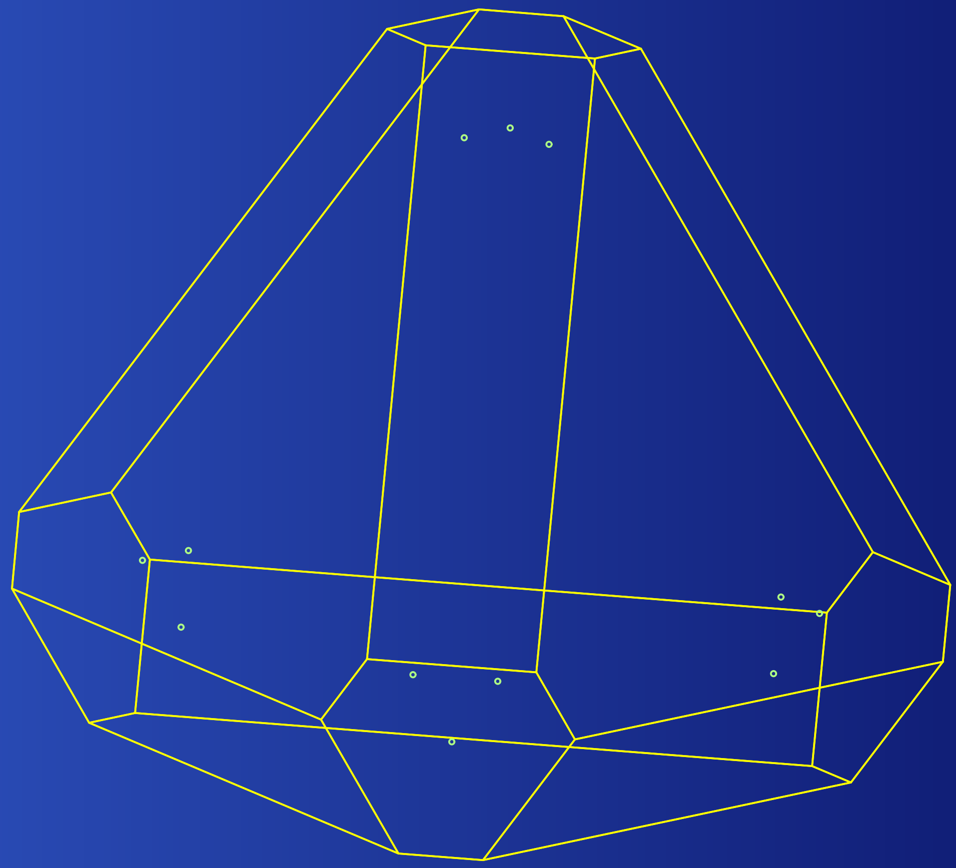

$$K_{\lambda\beta} = 22$$

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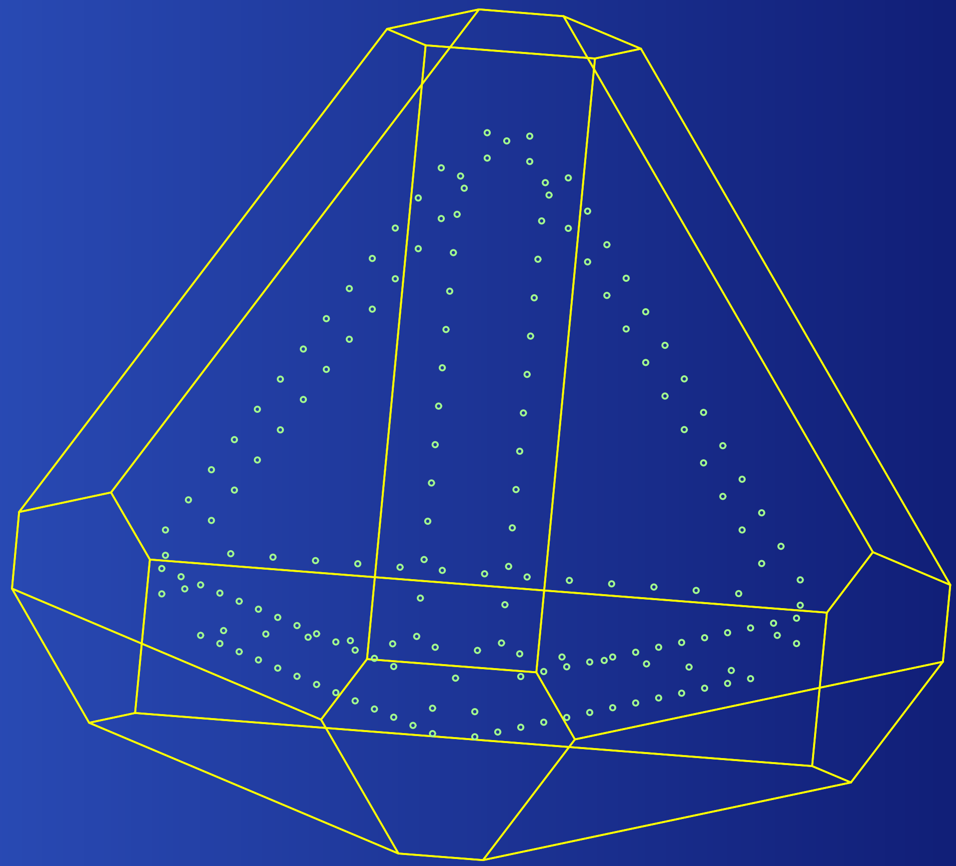

$$K_{\lambda\beta} = 26$$

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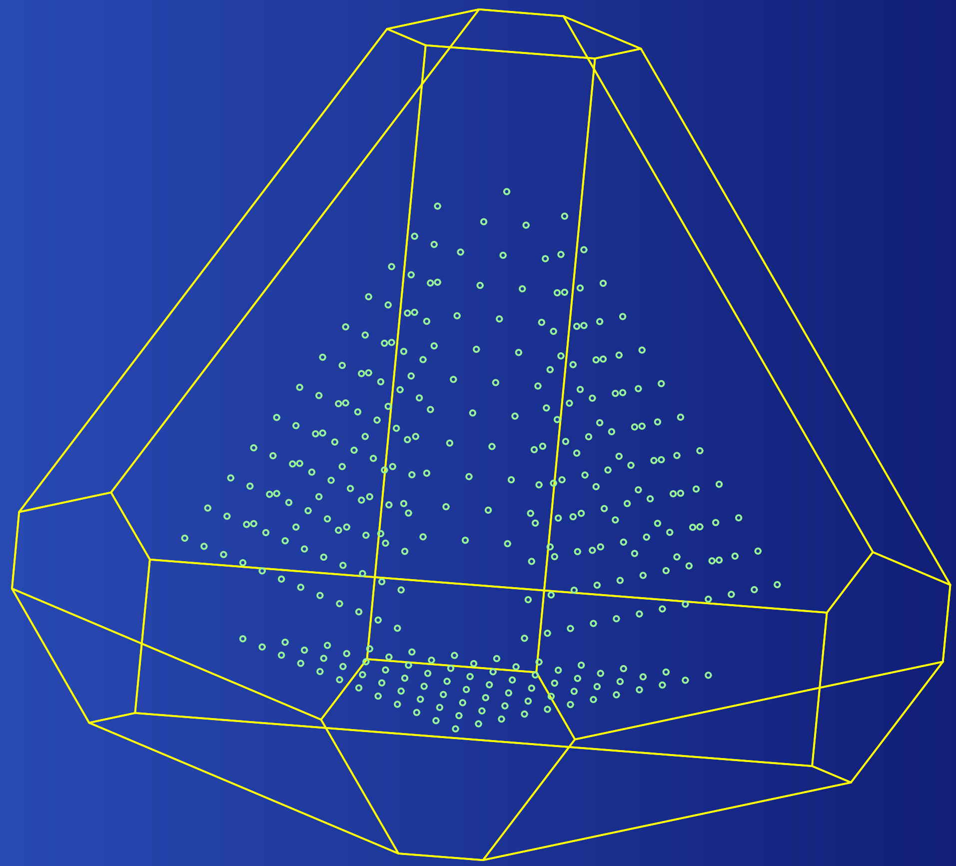

$$K_{\lambda\beta} = 30$$

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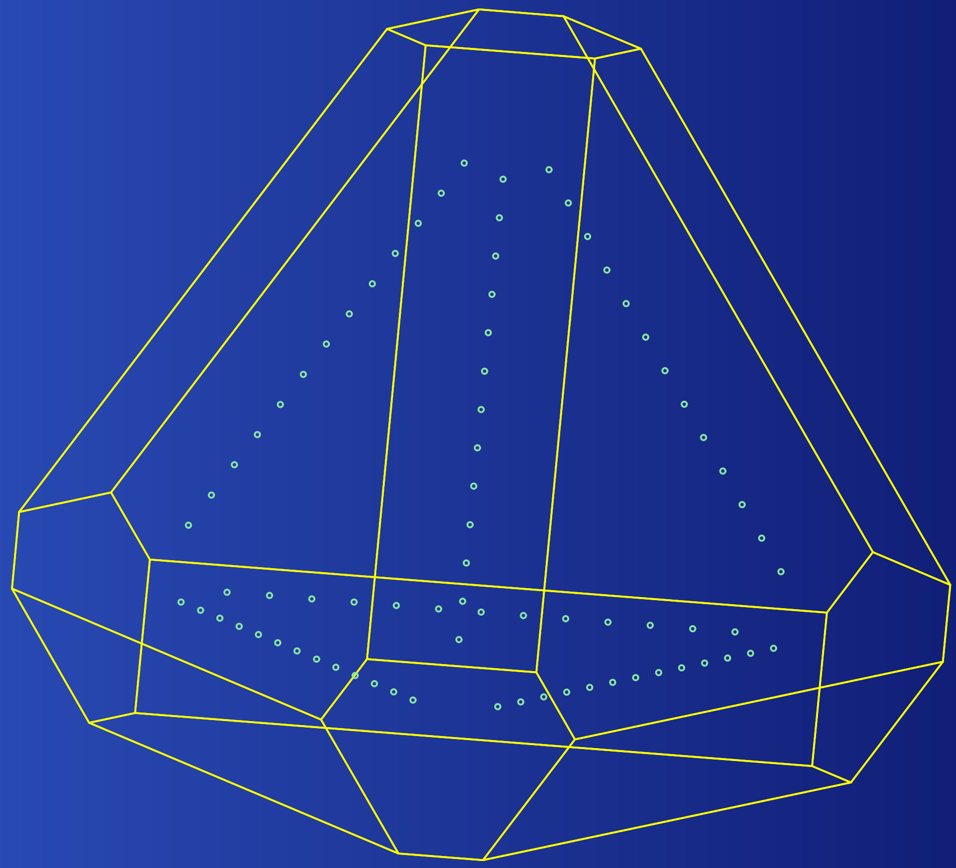

$$K_{\lambda\beta} = 31$$

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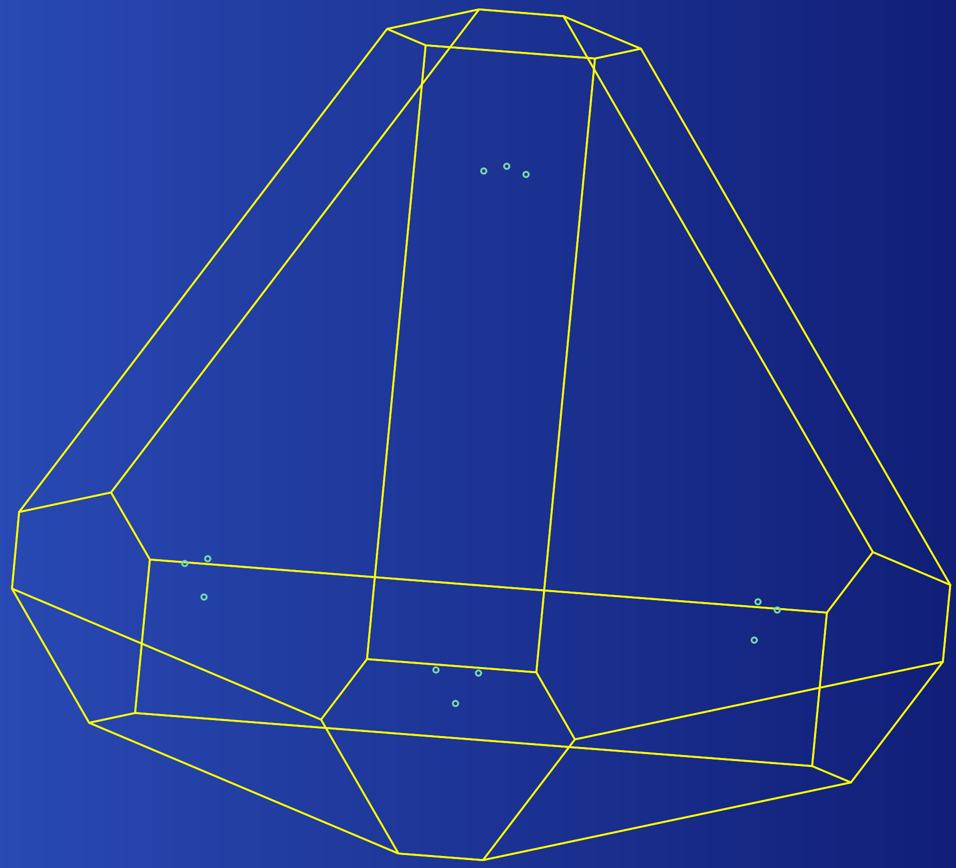

$$K_{\lambda\beta} = 35$$

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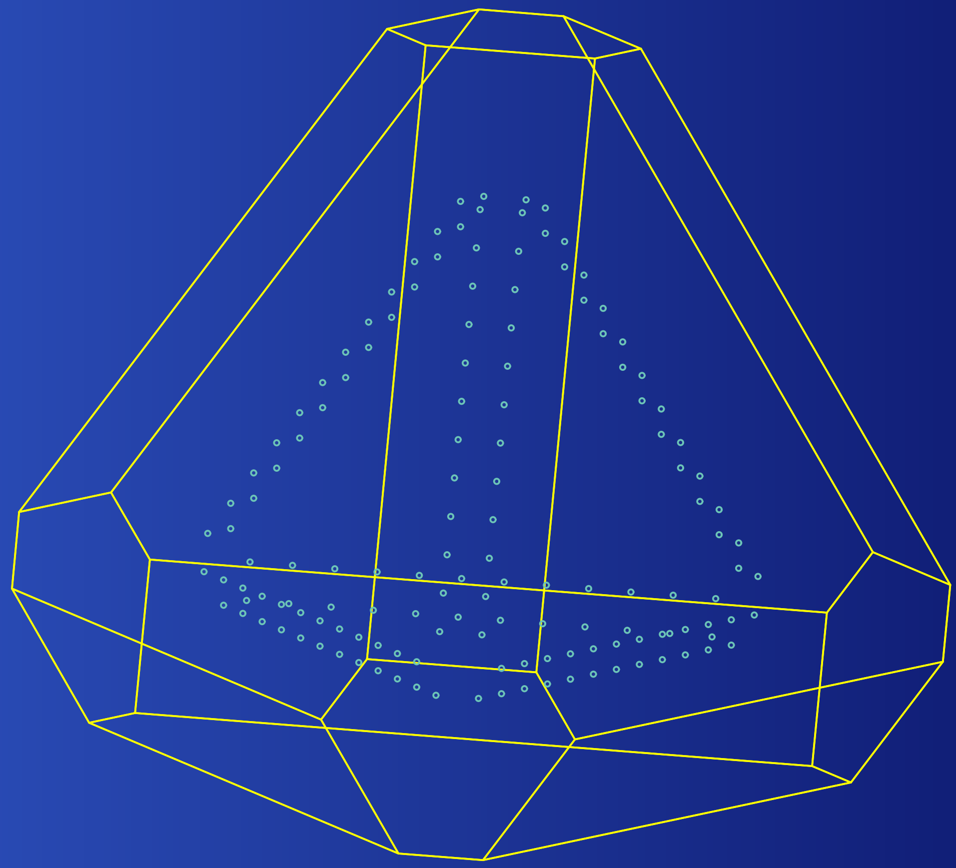

$$K_{\lambda\beta} = 40$$

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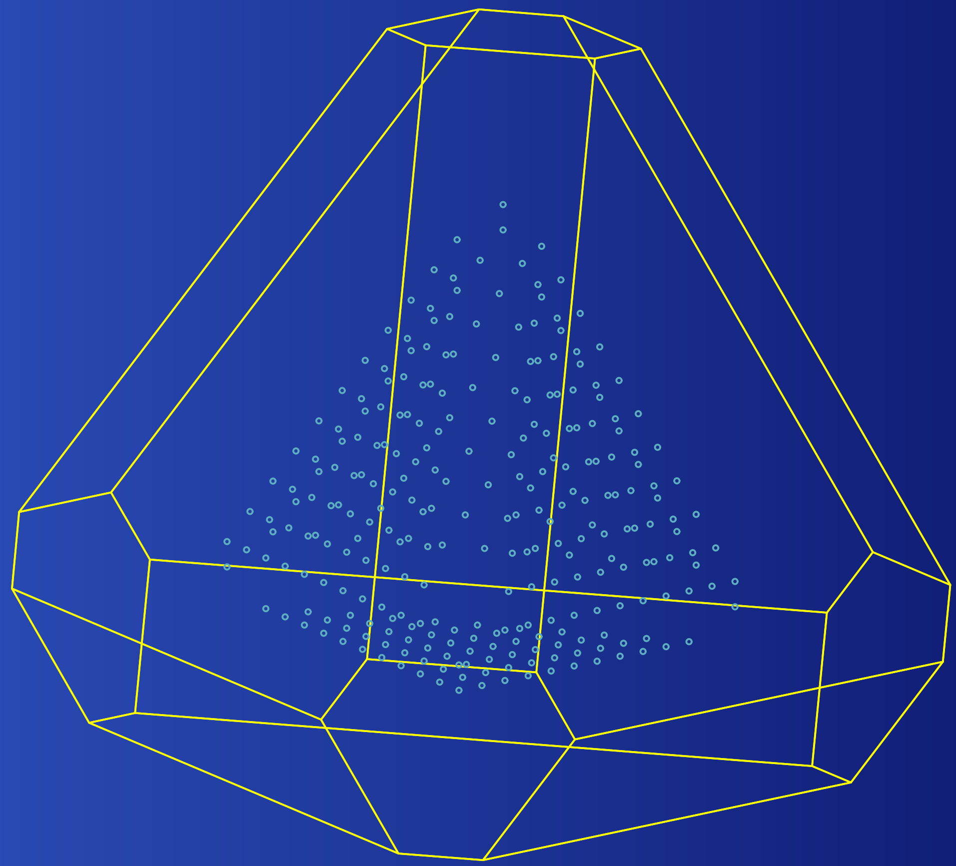
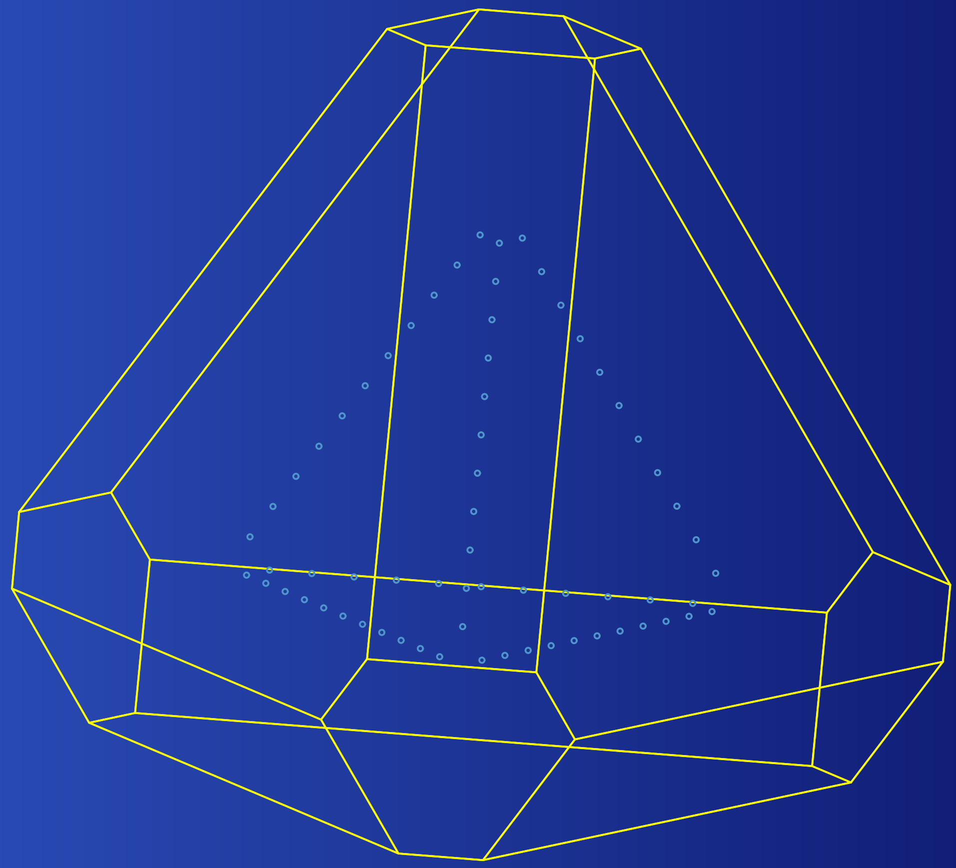


$$K_{\lambda\beta} = 45$$



$$K_{\lambda\beta} = 50$$

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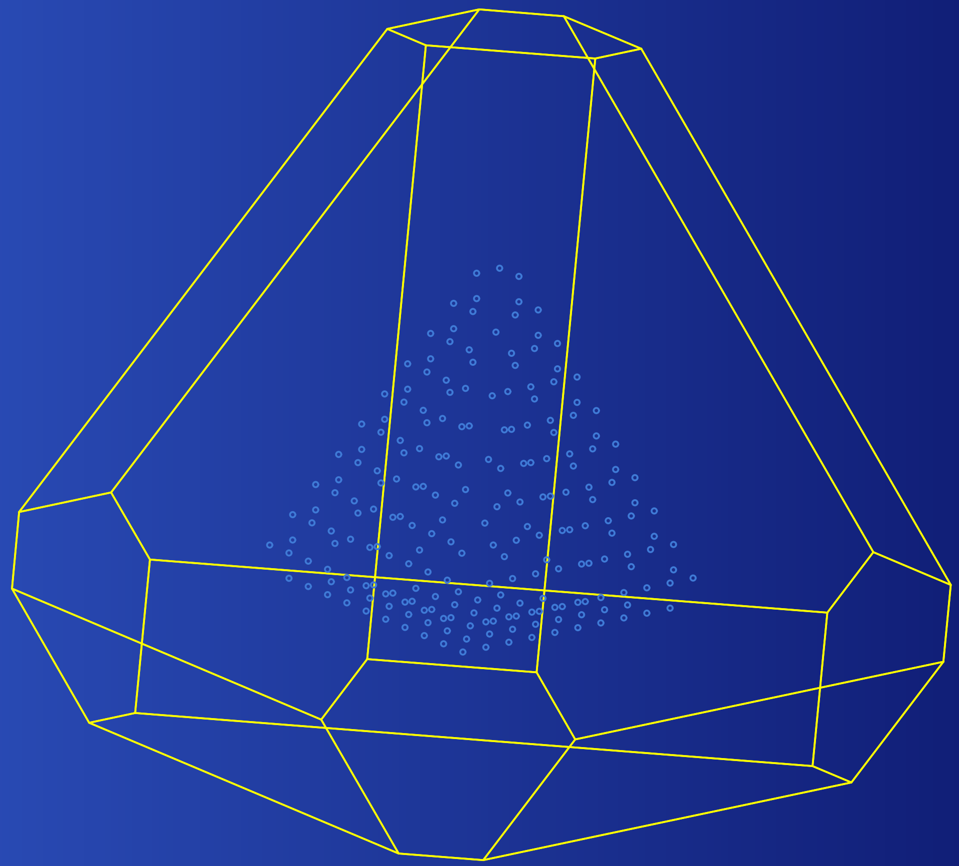

$$K_{\lambda\beta} = 55$$

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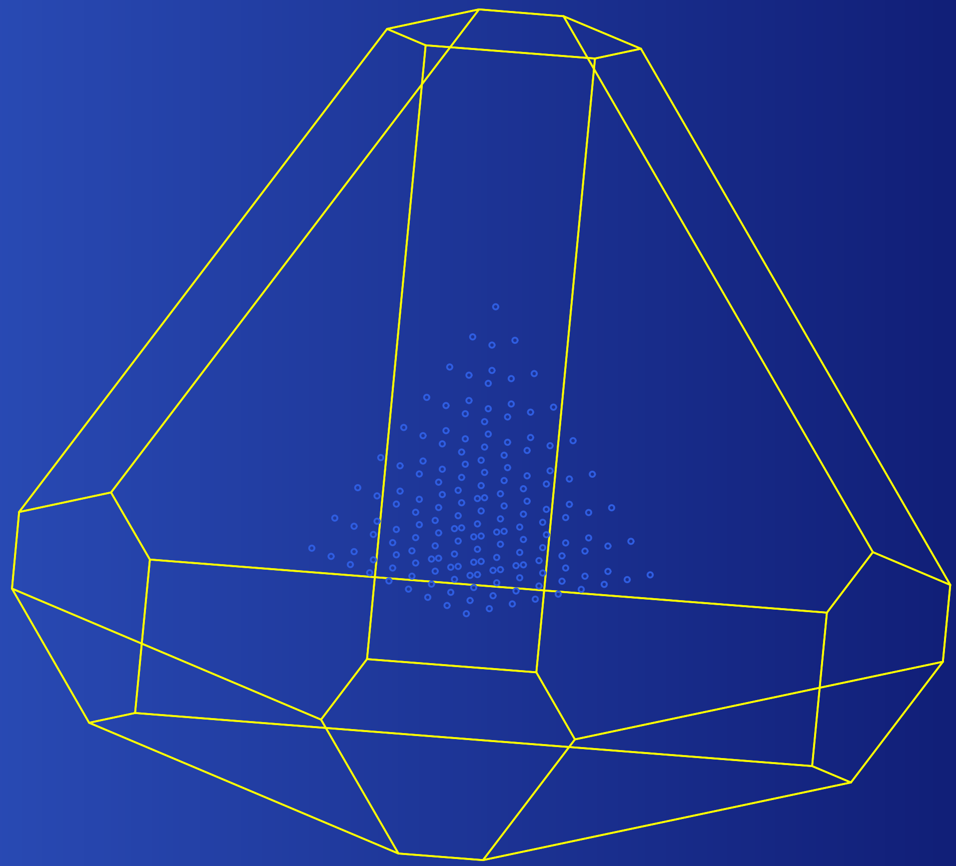

$$K_{\lambda\beta} = 60$$

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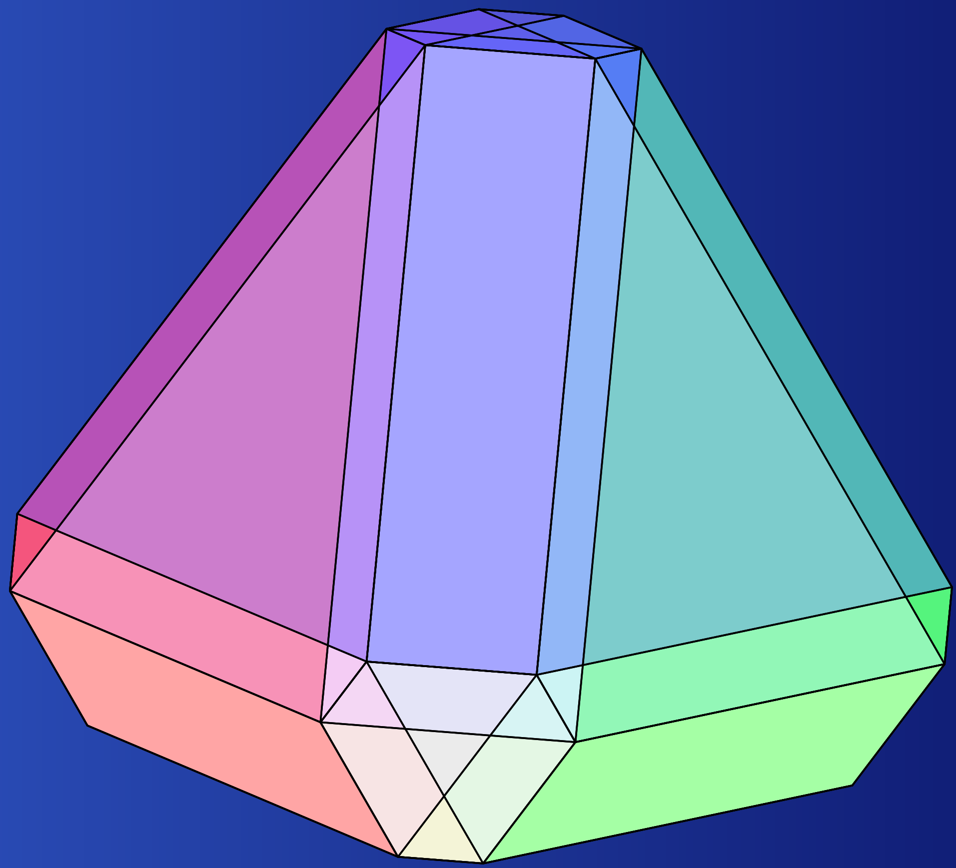

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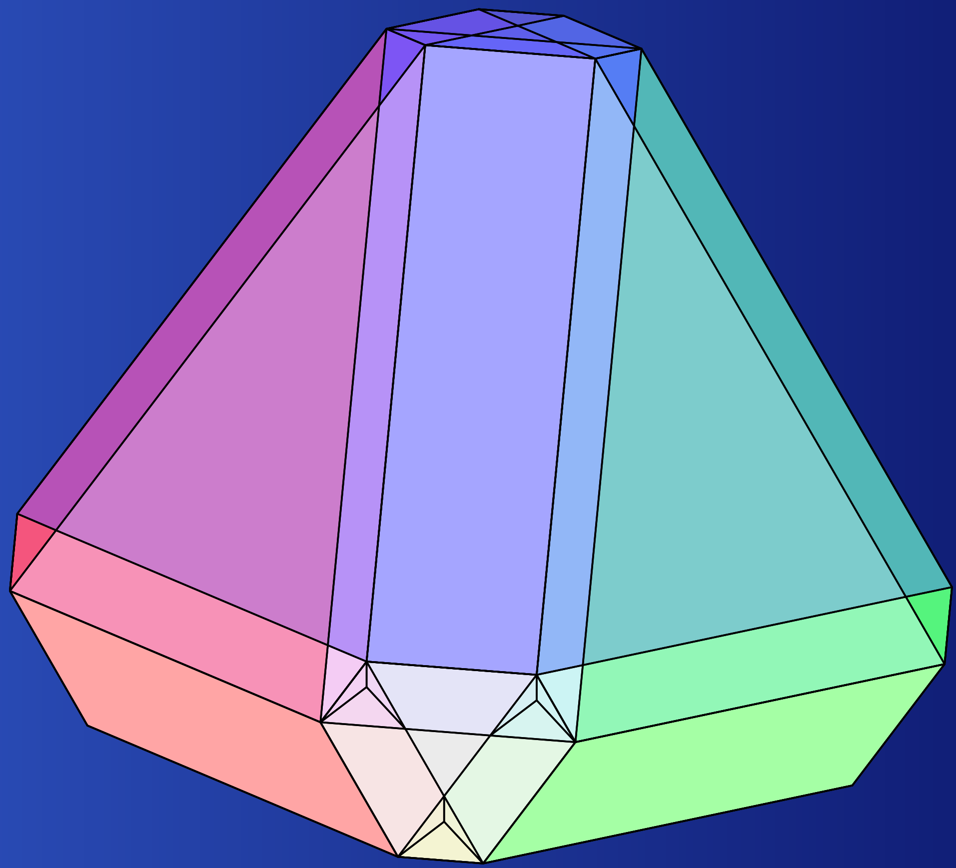

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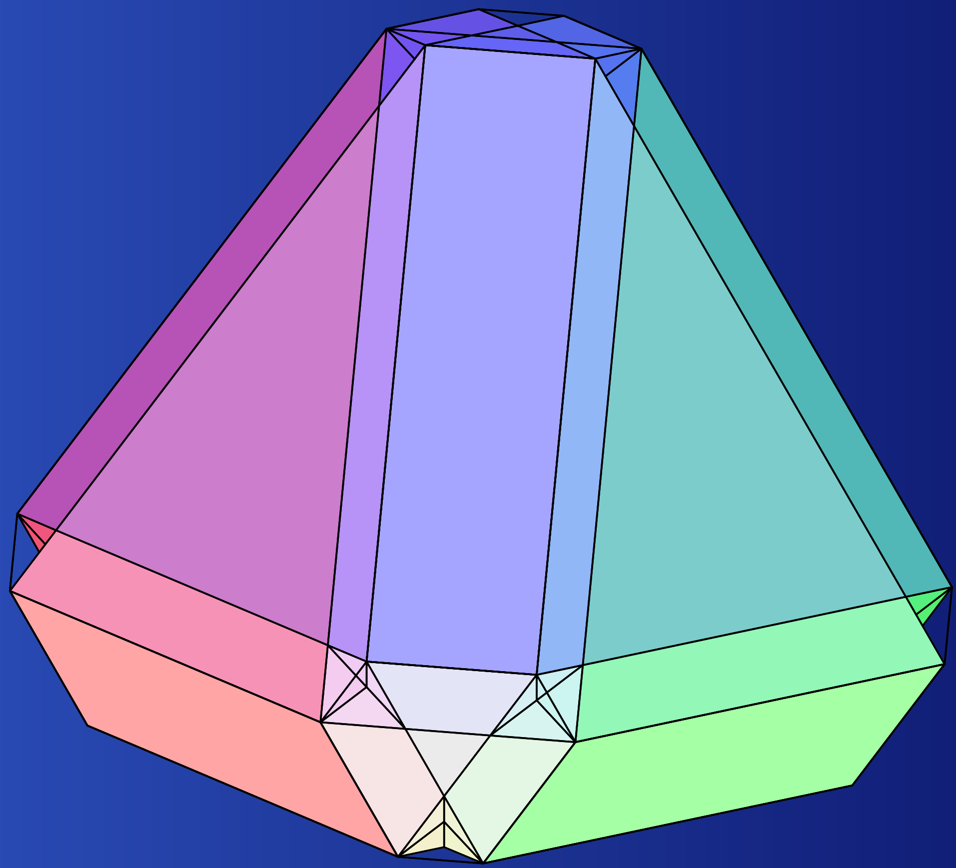

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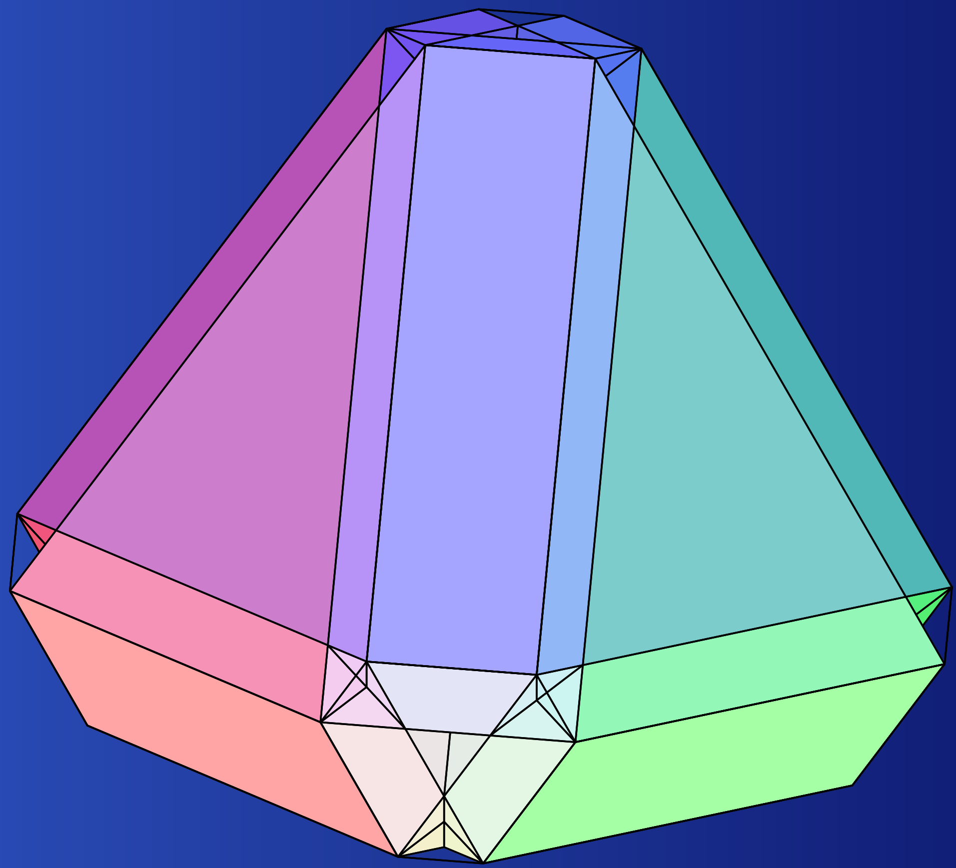

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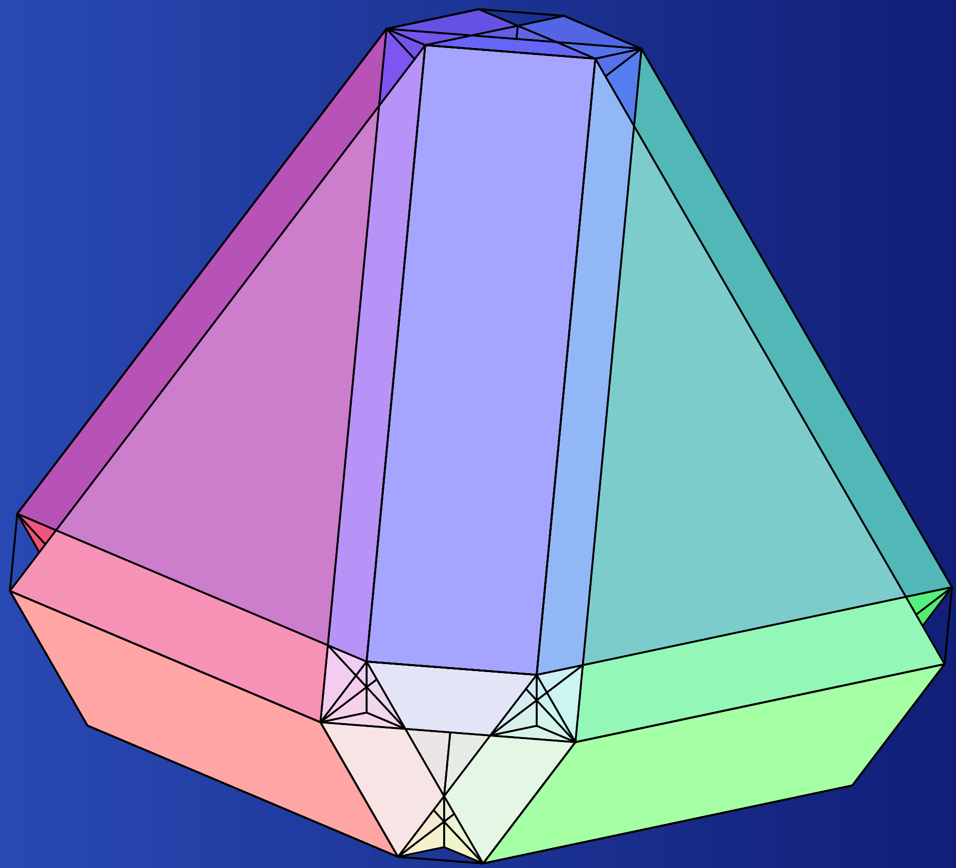

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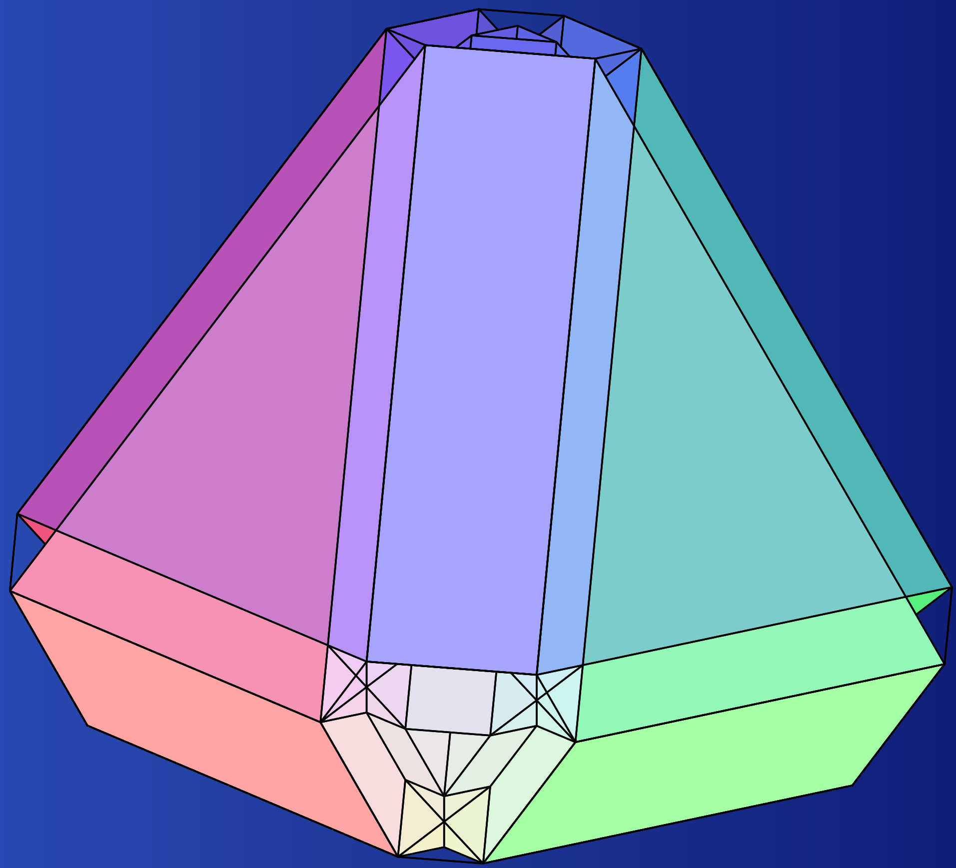

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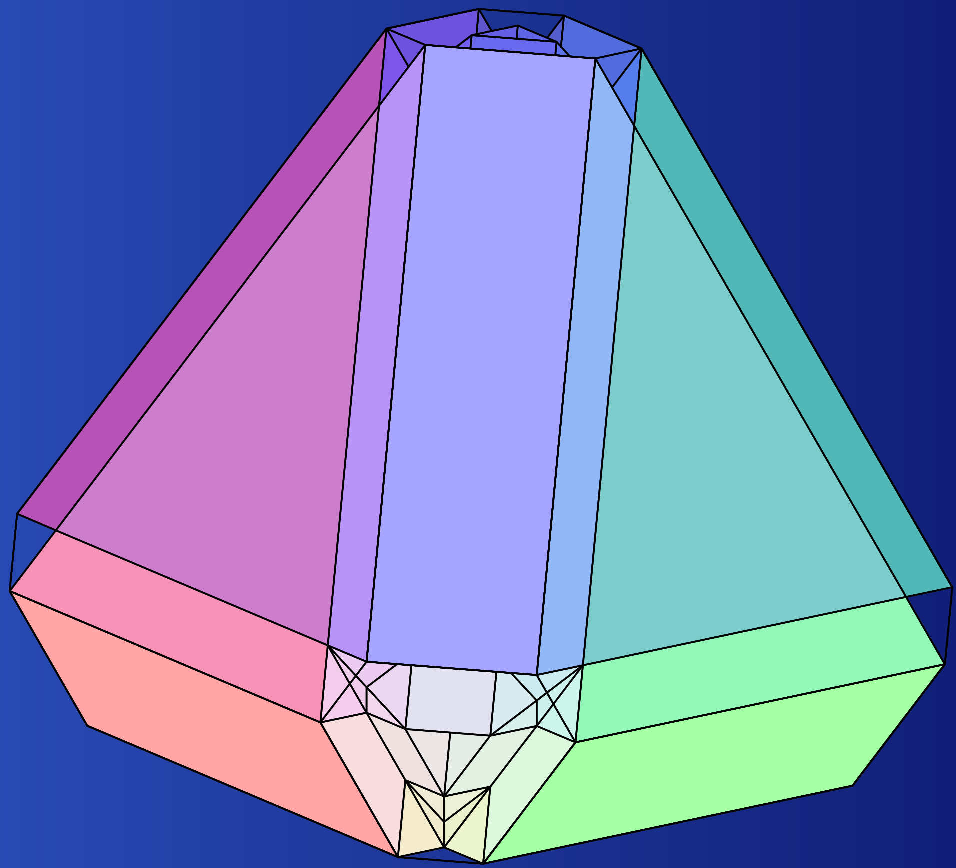

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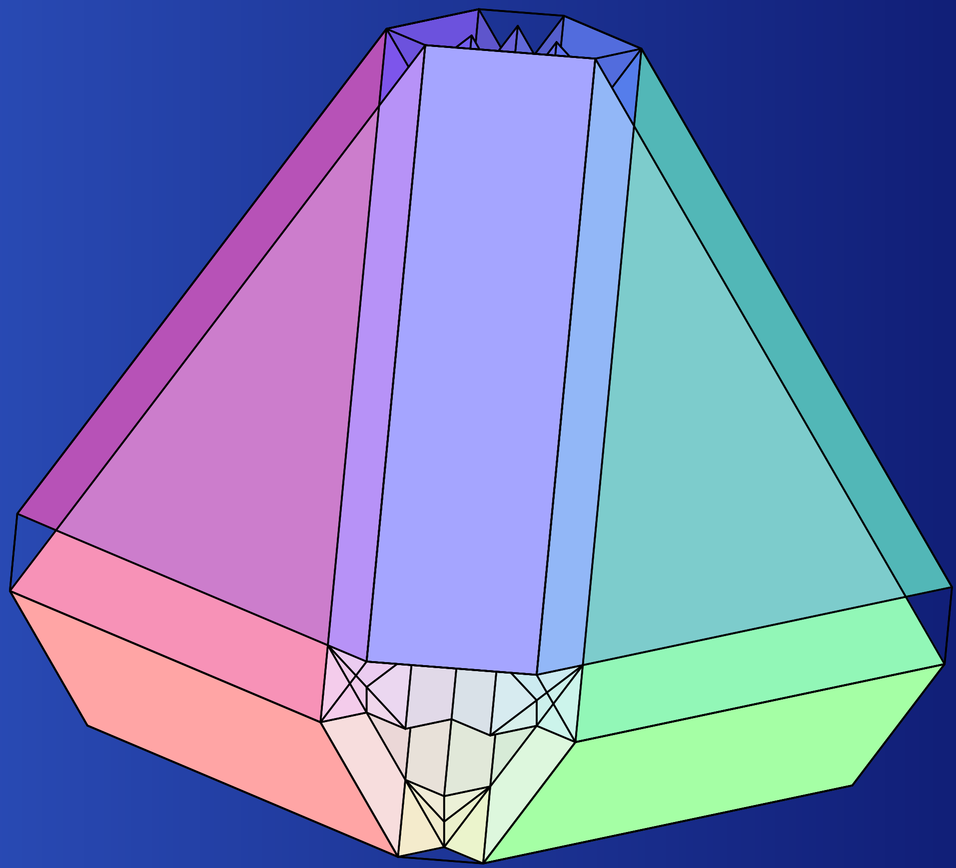

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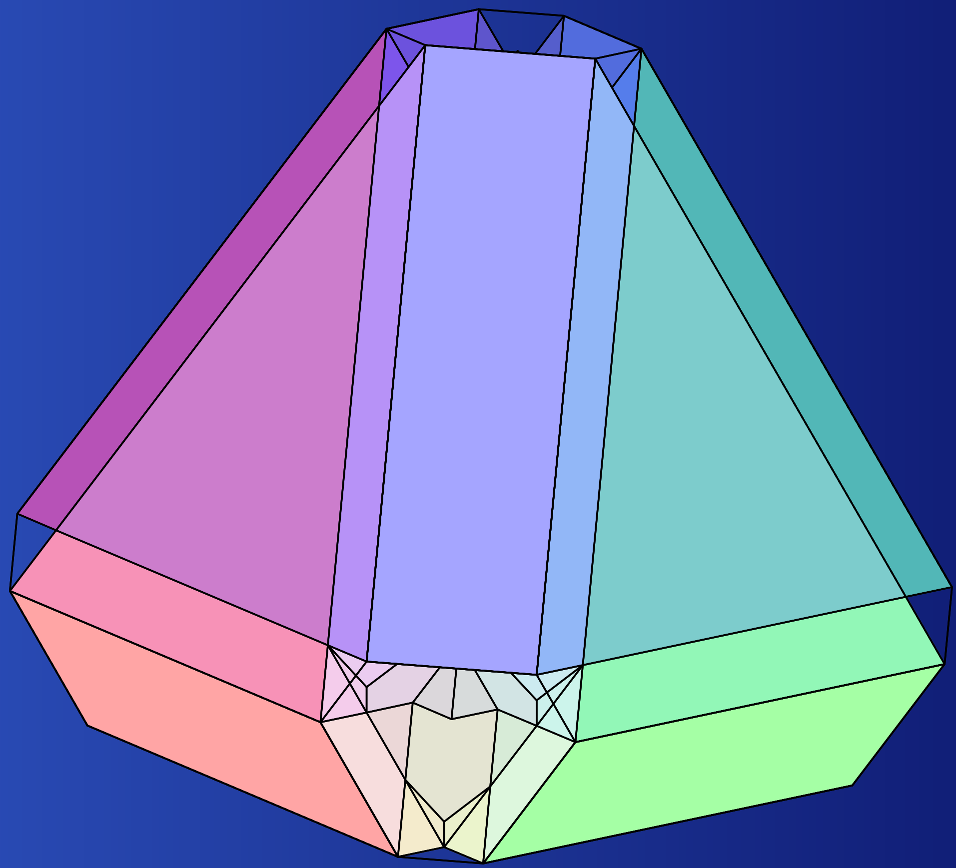

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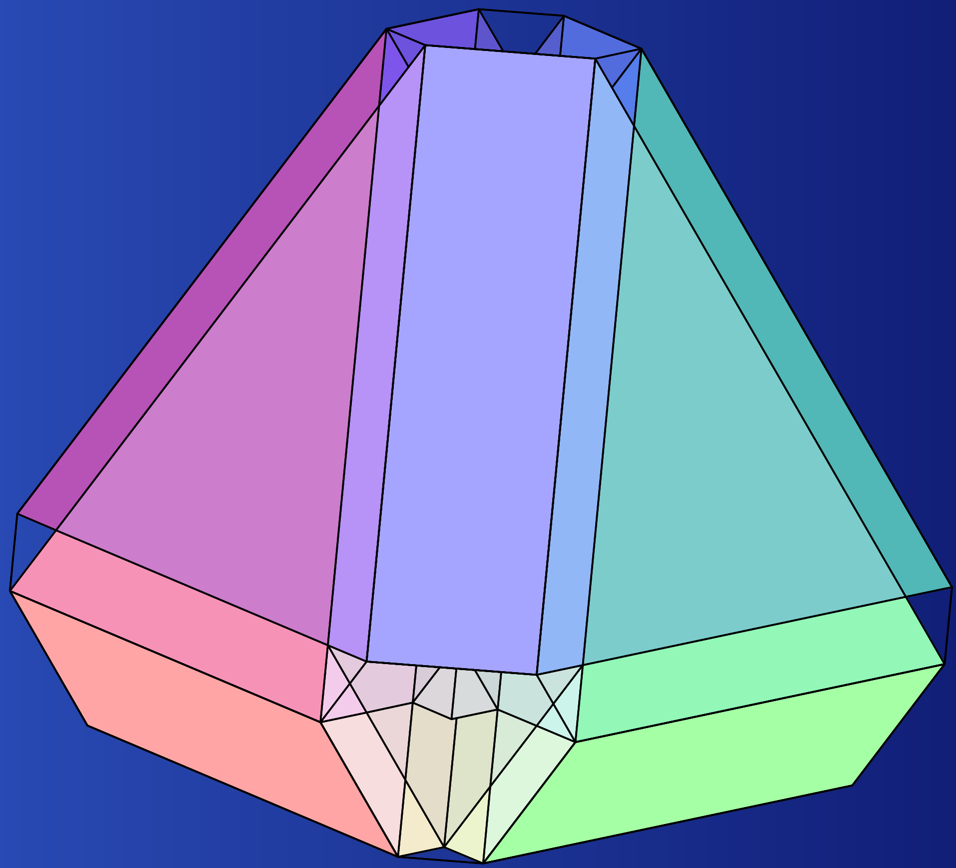

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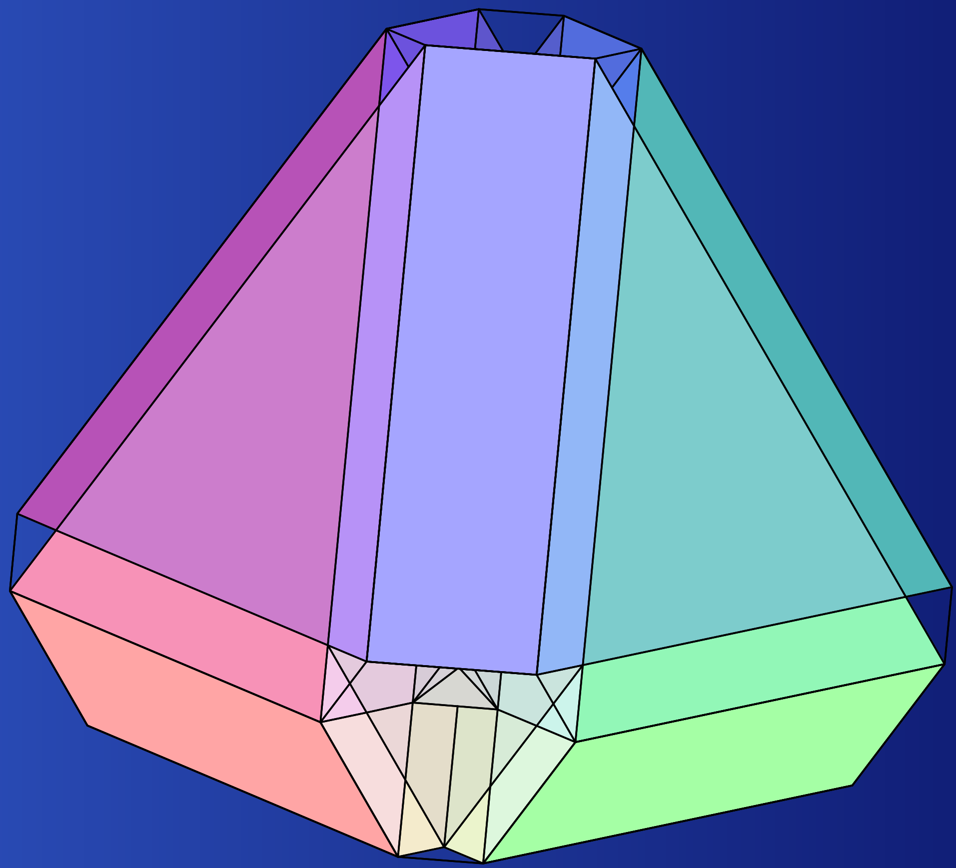

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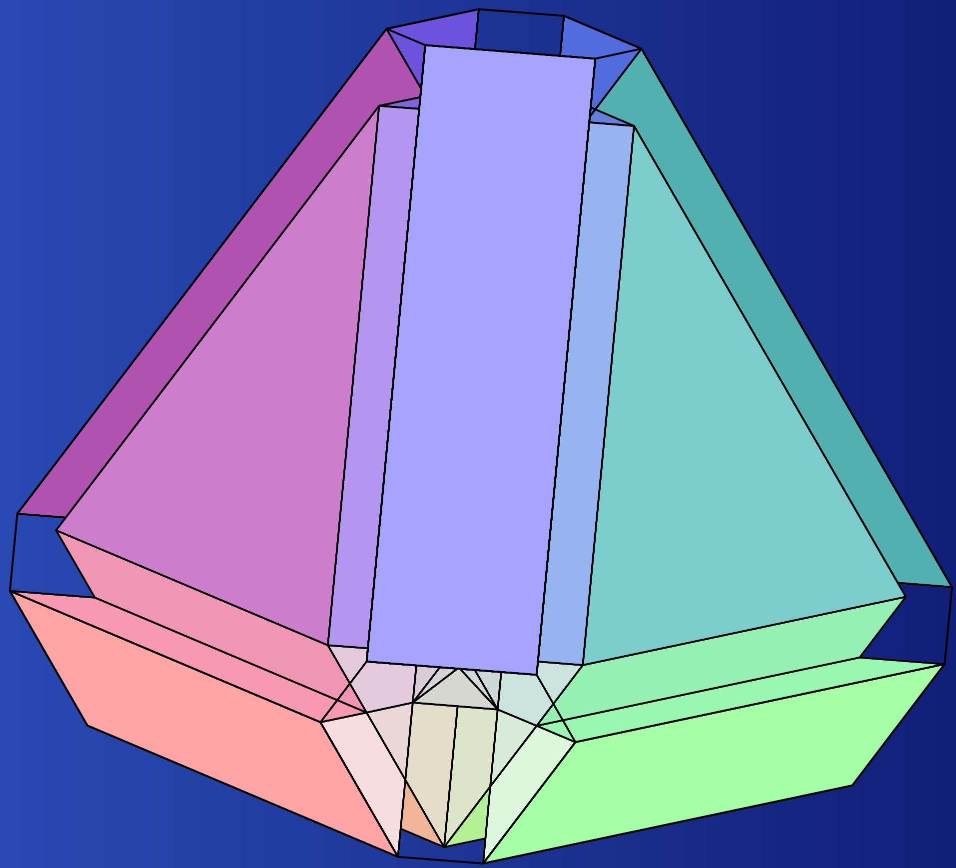

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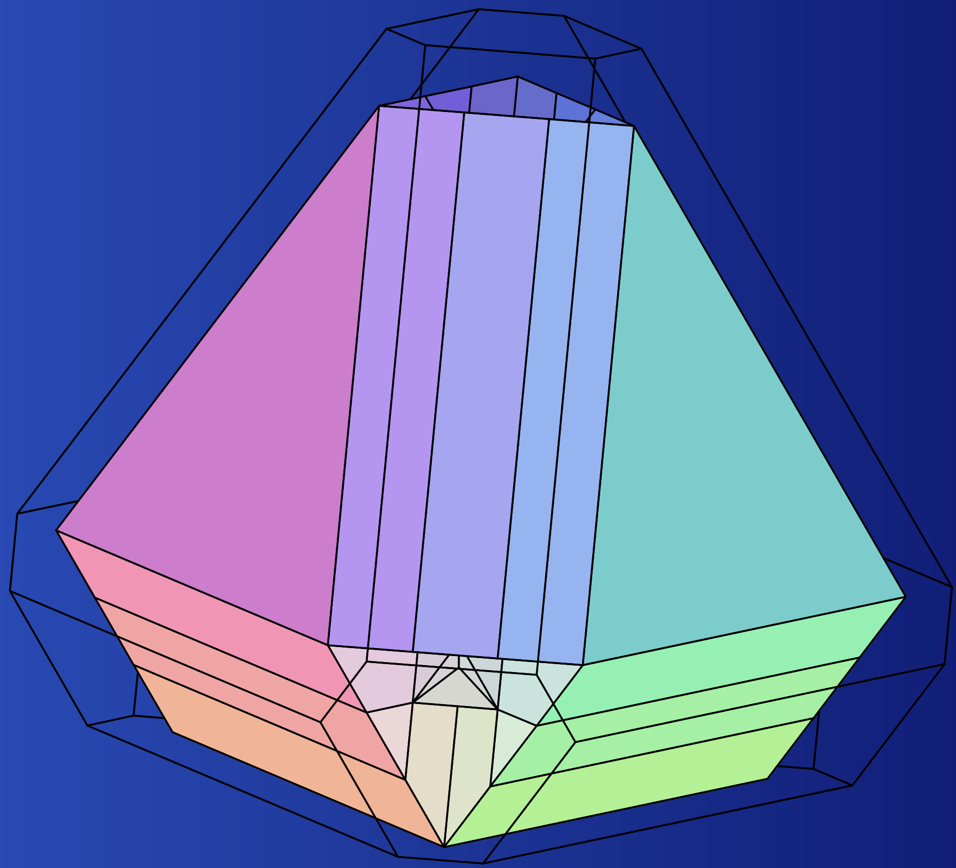

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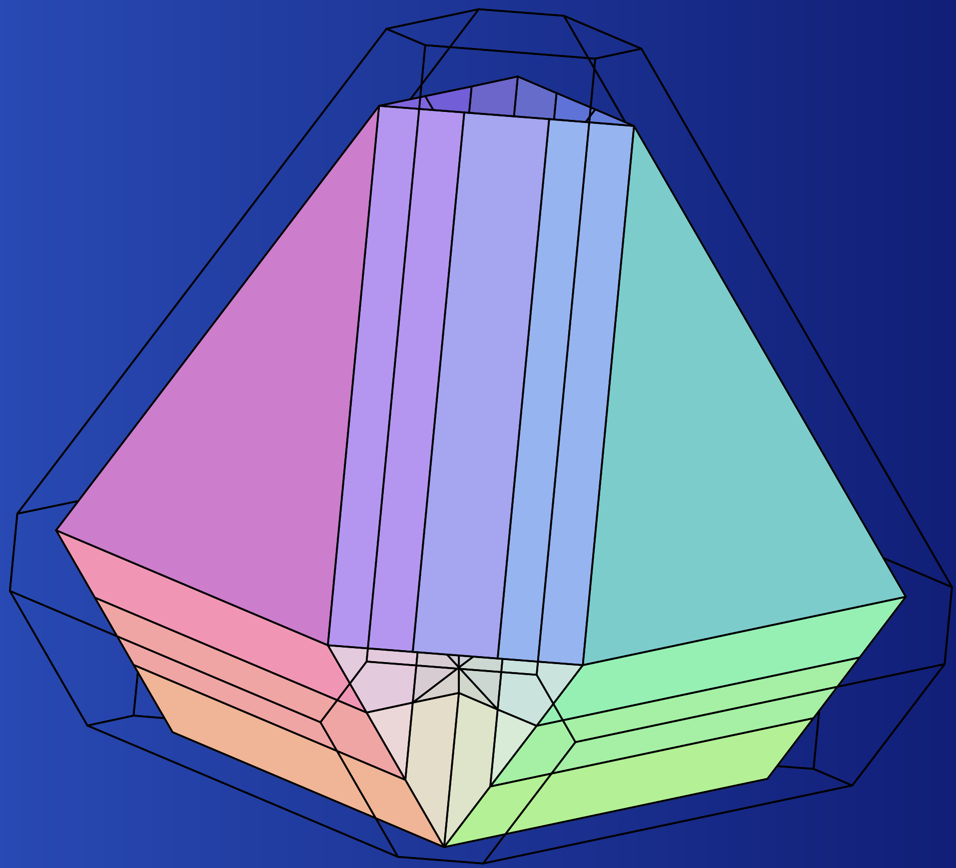

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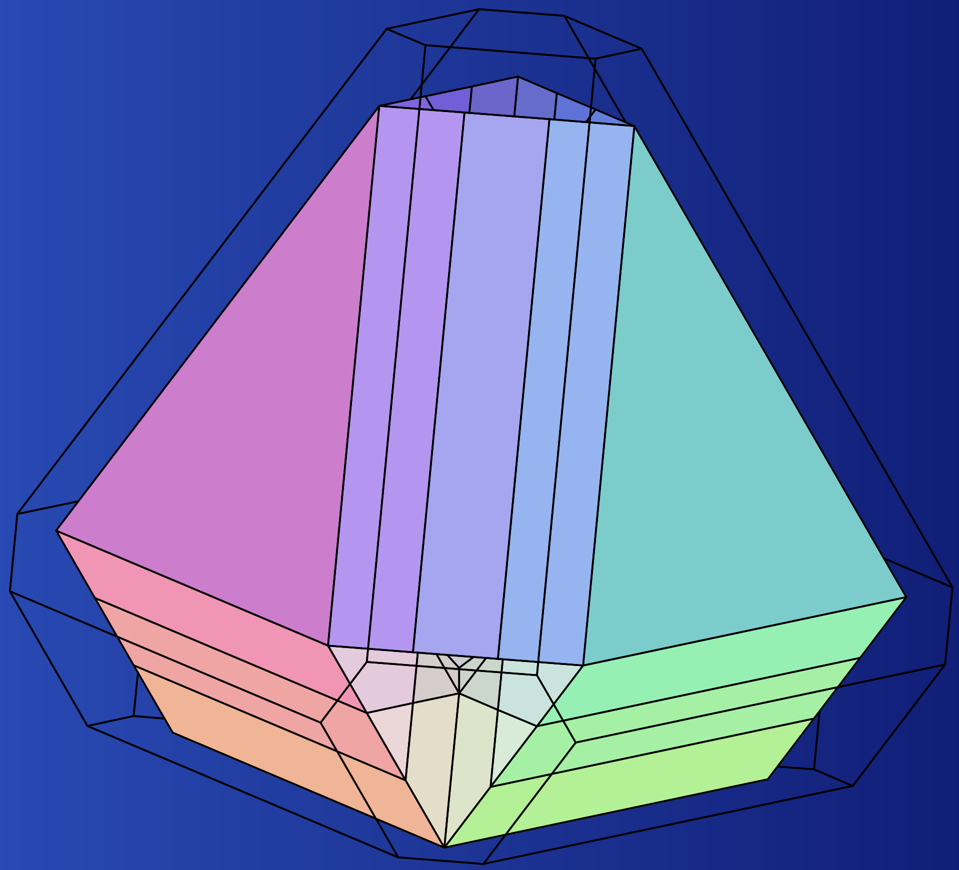

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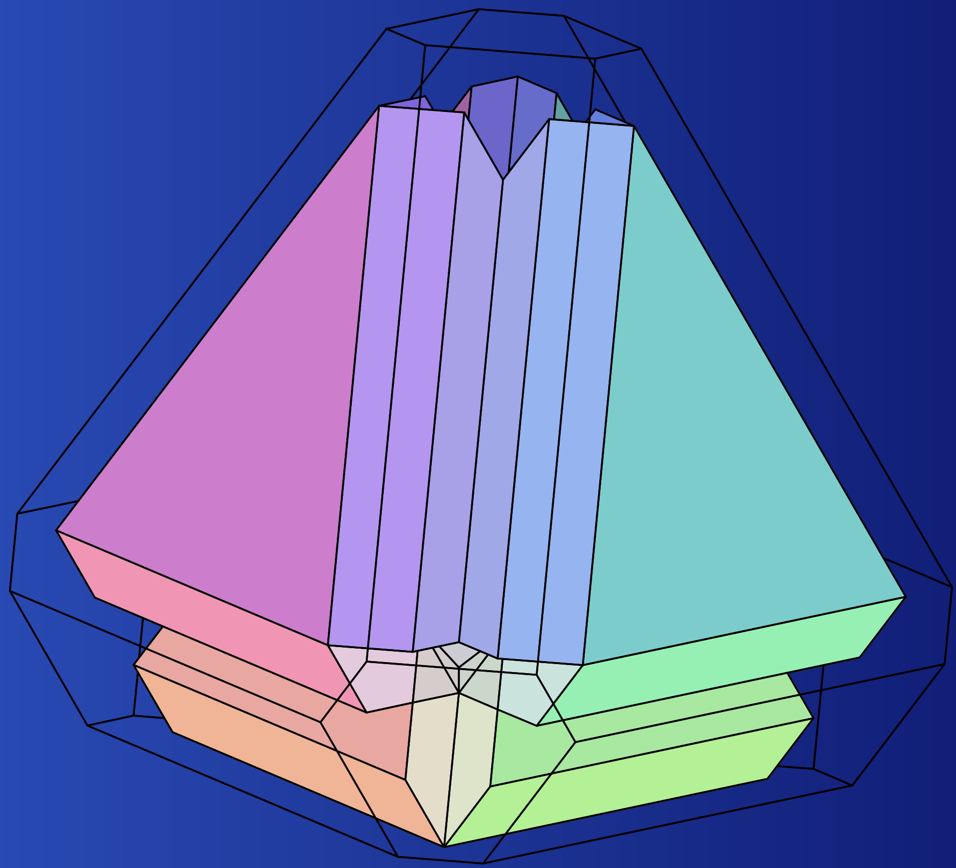

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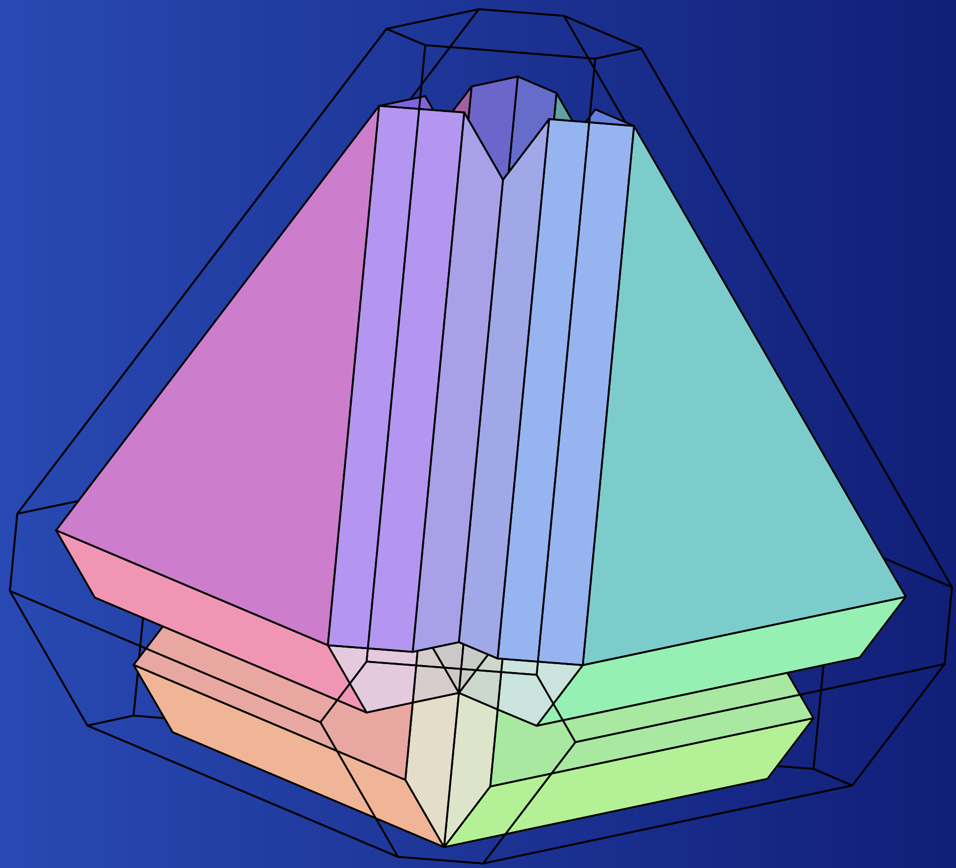

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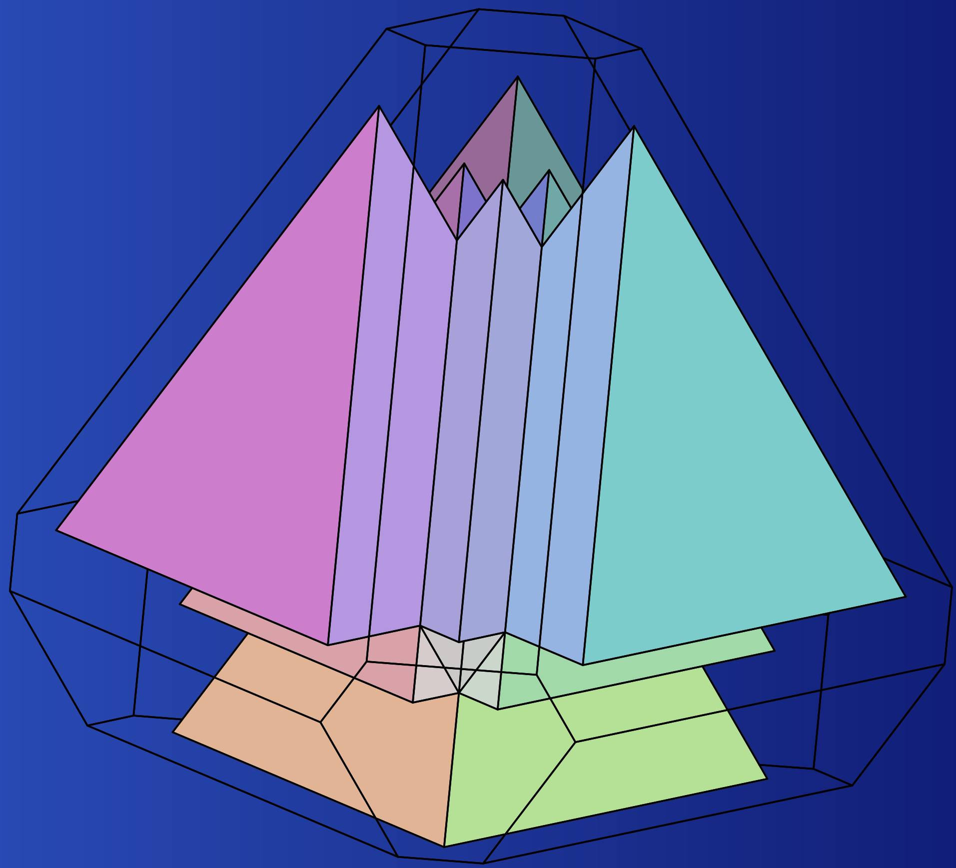

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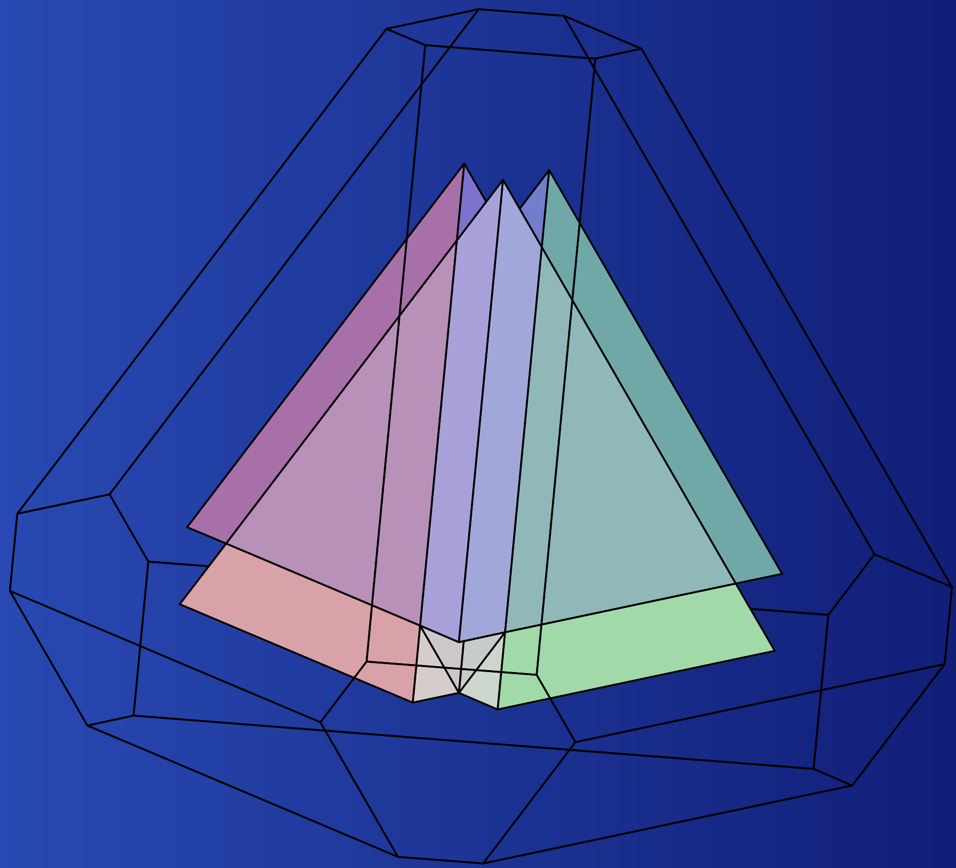

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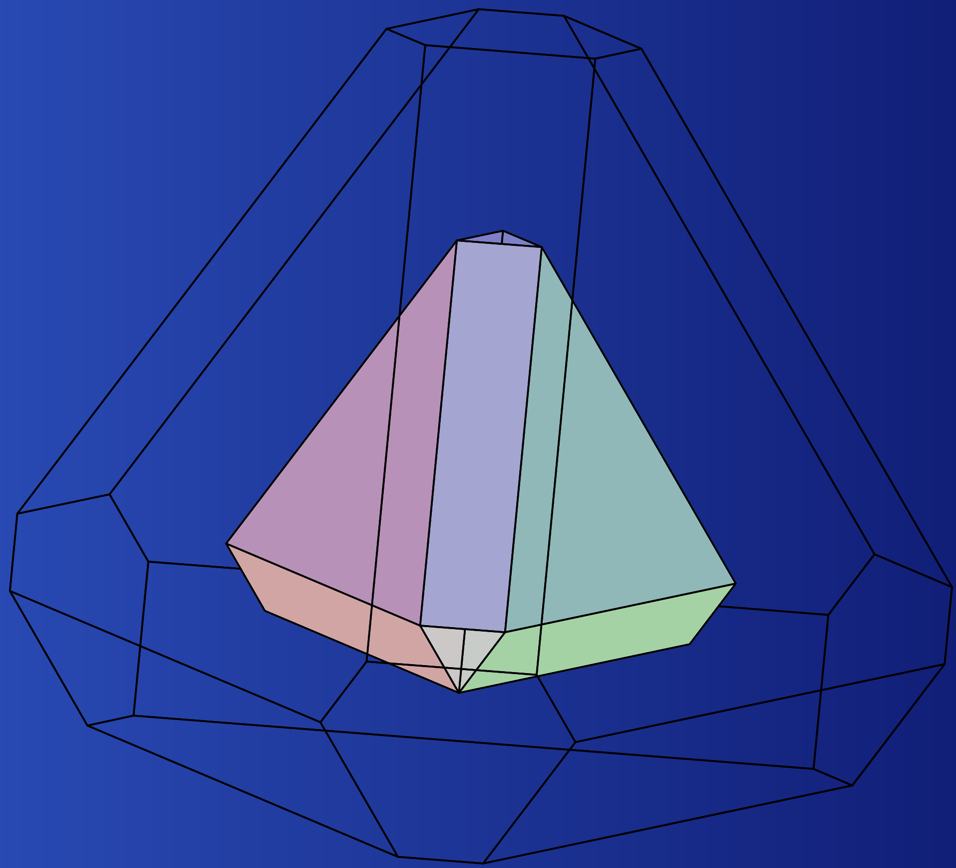

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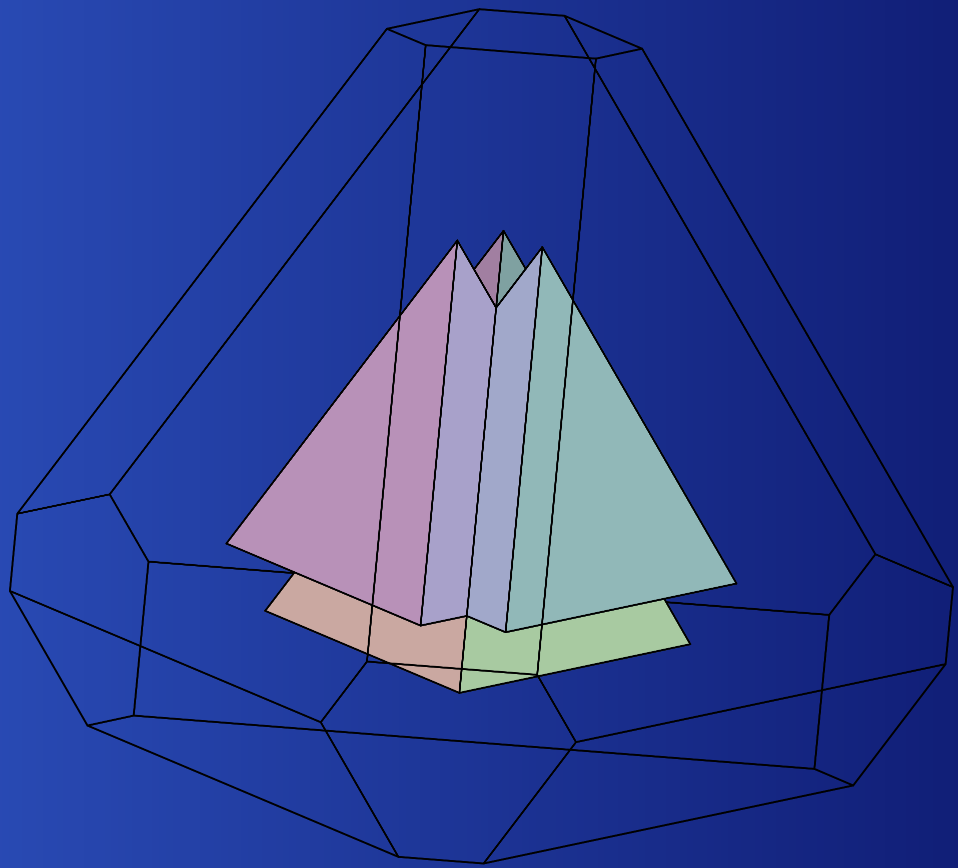

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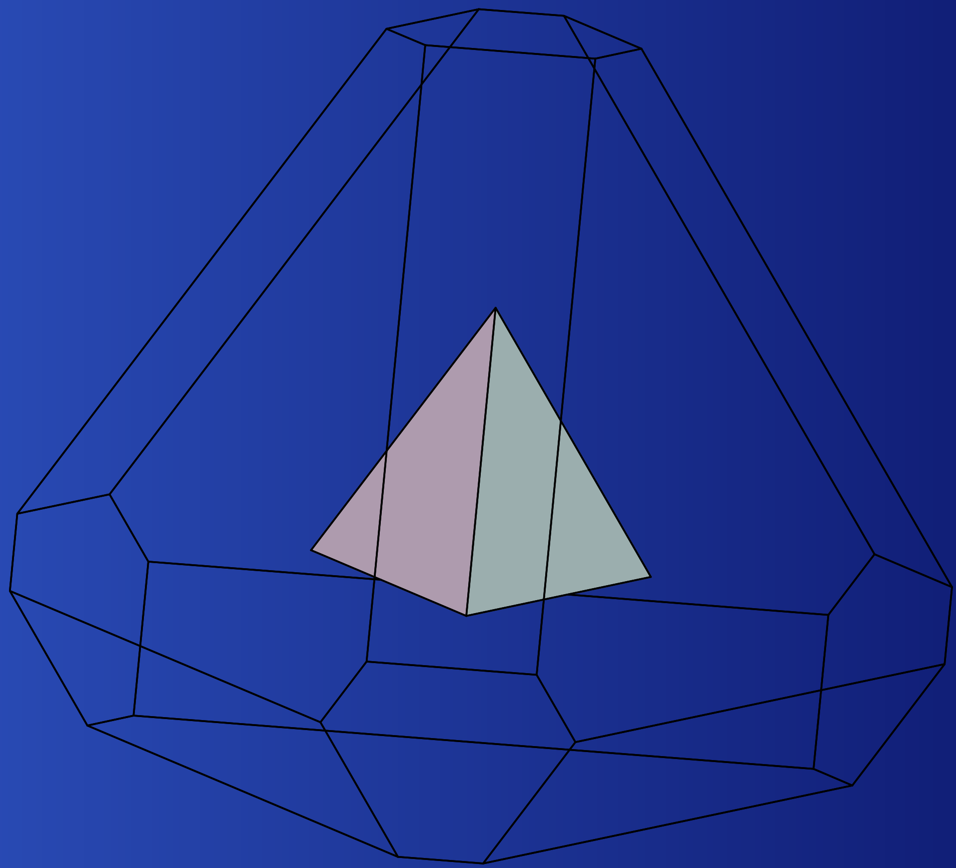

Roots and weights for A_{k-1}

- Roots

$$\Delta = \{e_i - e_j \ : \ 1 \leq i \neq j \leq k\}\,.$$

- Positive roots

$$\Delta_+ = \{e_i - e_j \ : \ 1 \leq i < j \leq k\}\,.$$

- Simple roots

$$\Pi = \{\underbrace{e_i - e_{i+1}}_{\alpha_i} \ : \ 1 \leq i \leq k-1\}\,.$$

- Fundamental weights : $\omega_1, \ldots, \omega_{k-1}$ defined by $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$.

$$\omega_i \equiv (\underbrace{1, 1, \ldots, 1}_{i \text{ times}}, \underbrace{0, 0, \ldots, 0}_{k-i \text{ times}})$$

- The normals to the facets of the permutahedron $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$ are the conjugates $\theta(\omega_i)$ of the fundamental weights.

$$\delta = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$$

Kostant's multiplicity formula

The Kostant partition function is the function

$$K(v) = \left| \left\{ (k_\alpha)_{\alpha \in \Delta_+} \in \mathbb{N}^{|\Delta_+|} \; : \; \sum_{\alpha \in \Delta_+} k_\alpha \alpha = v \right\} \right|,$$

i.e. $K(v)$ is the number of ways that v can be written as a sum of positive roots.

Kostant's multiplicity formula

The Kostant partition function is the function

$$K(v) = \left|\left\{(k_\alpha)_{\alpha\in\Delta_+} \in \mathbb{N}^{|\Delta_+|} \ : \ \sum_{\alpha\in\Delta_+} k_\alpha \alpha = v\right\}\right|,$$

i.e. $K(v)$ is the number of ways that v can be written as a sum of positive roots.

Kostant's multiplicity formula

$$K_{\lambda\beta} = \sum_{\sigma\in\mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma)} K(\sigma(\lambda+\delta) - (\beta+\delta)).$$

Gelfand-Tsetlin diagrams

A Gelfand-Tsetlin diagram is an array of integers of the form

$$\begin{array}{ccccccccc}
\lambda_1^{(k)} & & \lambda_2^{(k)} & & \cdots & & \lambda_{k-1}^{(k)} & & \lambda_k^{(k)} \\
& \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\
& & \ddots & & \vdots & & \iddots & & \\
& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}$$

such that

Gelfand-Tsetlin diagrams

$$
\begin{array}{ccccccccc}
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& & & & \lambda_1^{(1)} & & & &
\end{array}
$$

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& & \ddots & & \vdots & & \cdot^{\cdot^{\cdot}} & & \\
& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}$$

and

$$\begin{array}{ccc}
\lambda_j^{(i+1)} & & \lambda_{j+1}^{(i+1)} \\
\geq & & \geq \\
& \lambda_j^{(i)} &
\end{array}$$

for every such triangle in the diagram.

GT-diagrams and Kostka numbers

Lemma **(Gelfand-Tsetlin)**

The Kostka number $K_{\lambda\beta}$ is the number of Gelfand-Tsetlin diagrams with top row λ and row sums satisfying

$$\sum_{i=1}^{m} \lambda_i^{(m)} = \beta_1 + \cdots + \beta_m \qquad \text{for } 1 \leq m \leq k.$$

Gelfand-Tsetlin polytopes

$$
\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_{k-1} & & \lambda_k \\
& \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\
& & \ddots & & \vdots & & \iddots & & \\
& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}
$$

GT_λ $\mathrm{GT}_{\lambda\beta}$

GT-diagrams and SSYTs

$$\begin{array}{ccccccc} 7 & & 5 & & 4 & & 1 \\ & 6 & & 5 & & 2 & \\ & & 5 & & 3 & & \\ & & & 3 & & & \end{array}$$

$$\beta_1 + \beta_2 + \beta_3 + \beta_4 = 17$$
$$\beta_1 + \beta_2 + \beta_3 = 13$$
$$\beta_1 + \beta_2 = 8$$
$$\beta_1 = 3$$

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1	1	1

(3)

GT-diagrams and SSYTs

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1	1	1	2	2
2	2	2		

$$(5, 3)$$

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1	1	1	2	2	3
2	2	2	3	3	
3	3				

$$(6, 5, 2)$$

GT-diagrams and SSYTs

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$$\beta_1 = 3$$

1	1	1	2	2	3	4
2	2	2	3	3		
3	3	4	4			
4						

$$(7, 5, 4, 1)$$

Vector partition functions

Let M be a $d \times n$ matrix over the integers. The vector partition function associated to M is the function

$$\begin{array}{rccc} \phi_M : & \mathbb{Z}^d & \longrightarrow & \mathbb{N} \\ & b & \mapsto & |\{x \in \mathbb{N}^n \ : \ Mx = b\}| \end{array}$$

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Example

If $M = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and $b = \binom{3}{1}$ then $\phi_M(b) = 3$

since $b = \binom{1}{0} + \binom{0}{1} + \binom{2}{0} = \binom{1}{0} + \binom{2}{1} = 3\binom{1}{0} + \binom{0}{1}$.

Polytopes and partition functions

- If M is such that $\ker M \cap \mathbb{R}^n_{\geq 0} = 0$, then

$$P_b = \{x \in \mathbb{R}^n_{\geq 0} \ : \ Mx = b\}$$

is a polytope.

$\phi_M(b)$ is the number of integral points in P_b.

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$\phi_M(b)$ is the number of integral points in P_b .

- ϕ_M vanishes outside of $\mathrm{pos}(M)$.

The structure of partition functions

- ϕ_M is piecewise quasipolynomial of degree $n - \mathrm{rank}(M)$. (Sturmfels)

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The structure of partition functions

- ϕ_M is piecewise quasipolynomial of degree $n - \mathrm{rank}(M)$. (Sturmfels)
- The domains of quasipolynomiality form a complex of convex polyhedral cones, the chamber complex of ϕ_M .
- Alekseevskaya, Gelfand and Zelevinsky described how to determine the chamber complex of a partition function from its matrix.

Determining the chamber complex

We can assume without loss of generality that M has full rank d .

- Find all the $d \times d$ nonsingular submatrices M_σ of M .

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- Find all the $d \times d$ nonsingular submatrices M_σ of M.
- Determine the cone $\tau_\sigma = \mathrm{pos}(M_\sigma)$ spanned by the columns of M_σ.
- The chamber complex of ϕ_M is the common refinement of the τ_σ.

The Kostant partition function for A_3

$$\Delta_+^{(A_3)} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

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$K(v) = \phi_{M_{A_3}}(v)$ for

$$M_{A_3} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

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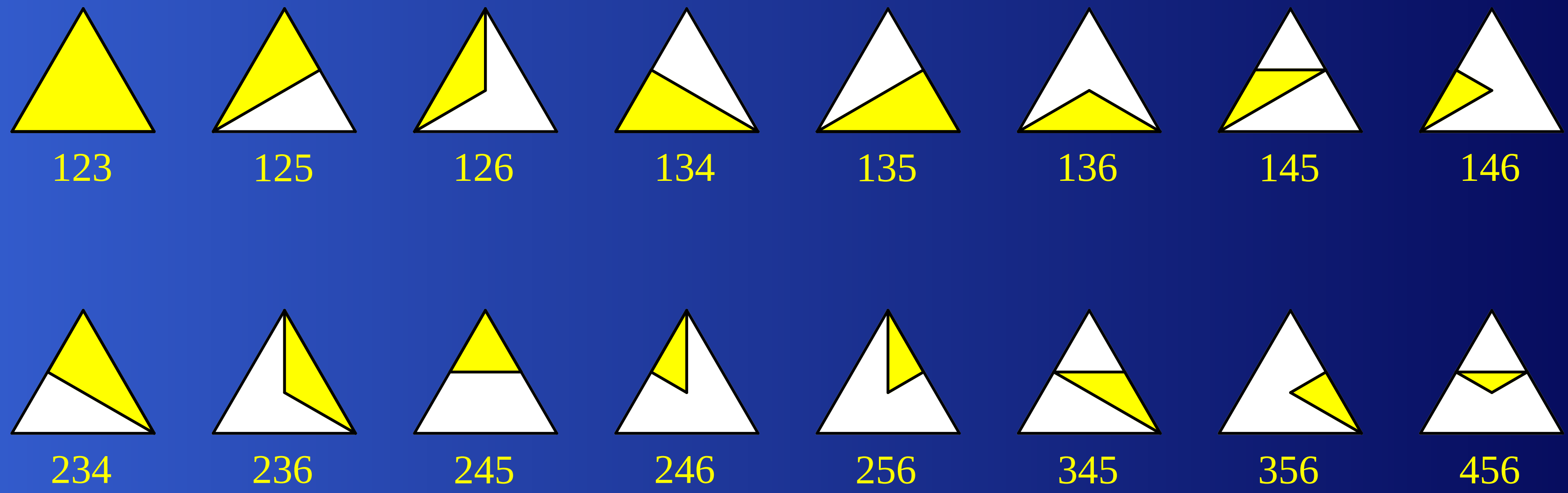
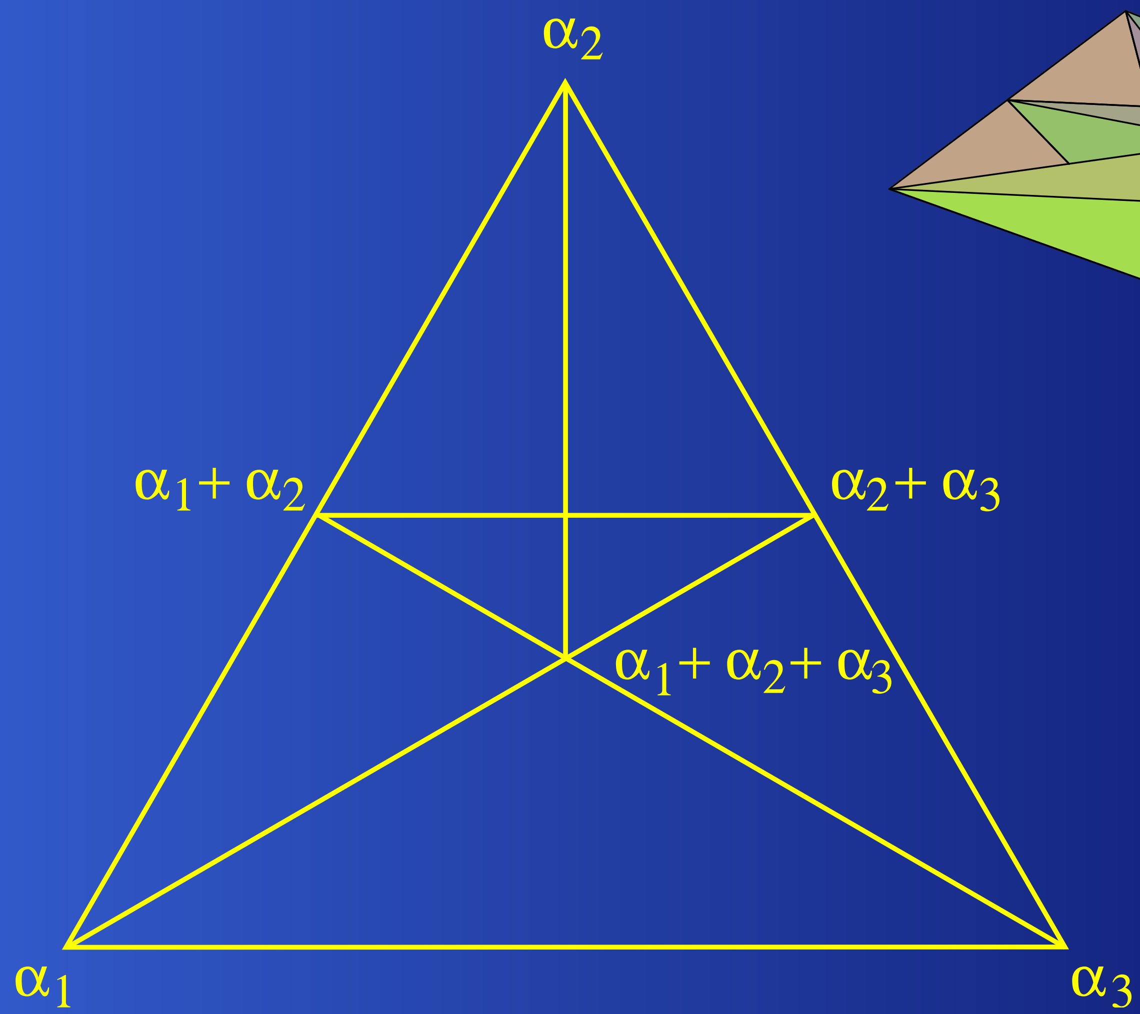
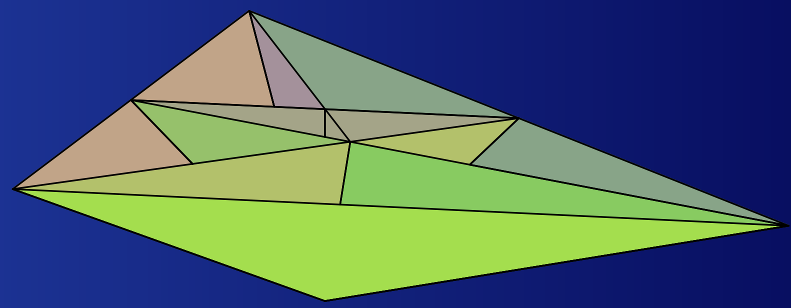
$$M_{A_3} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} \mathcal{B} &= \{123, 125, 126, 134, 135, 136, 145, 146, \\ &\quad 234, 236, 245, 246, 256, 345, 356, 456\}\,. \end{aligned}$$


123
125
126
134
135
136
145
146
234
236
245
246
256
345
356
456


α_2
$\alpha_1 + \alpha_2$
$\alpha_2 + \alpha_3$
$\alpha_1 + \alpha_2 + \alpha_3$
α_1
α_3



Unimodularity

A $d \times n$ matrix of full rank d is unimodular if all its $d \times d$ submatrices have determinant 0 or ± 1.

Vector partitions functions of unimodular matrices are polynomial over the cones of their chamber complexes. (Sturmfels)

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Lemma **(well-known)** *The matrix M_{A_n} is unimodular for all n.*

Corollary *The Kostant partition function for A_{k-1} is polynomial of degree $\binom{k-1}{2}$ over the cones of its chamber complex.*

A partition function for the $K_{\lambda\beta}$

Theorem A

For every k, we can find integer matrices E_k and B_k such that the Kostka numbers for partitions with at most k parts can be written as

$$K_{\lambda\beta} = \phi_{E_k}\left(B_k \begin{pmatrix} \lambda \\ \beta \end{pmatrix} \right) .$$

Example: A_2

Gelfand-Tsetlin diagrams for A_2 have the form

$$\begin{matrix} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \mu_1 & & \mu_2 & \\ & & \nu & & \end{matrix}$$

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Gelfand-Tsetlin diagrams for A_2 have the form

$$\begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \mu_1 & & \mu_2 & \\ & & \nu & & \end{array}$$

Row sums:

$$\begin{aligned} \nu &= \beta_1 \\ \mu_1 + \mu_2 &= \beta_1 + \beta_2 \\ \lambda_1 + \lambda_2 + \lambda_3 &= \beta_1 + \beta_2 + \beta_3\,. \end{aligned}$$

$$
\begin{aligned}
\mu_1 &\leq \lambda_1 \\
-\mu_1 &\leq -\lambda_2 \\
-\mu_1 &\leq \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 &\leq \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 &\leq -\beta_1 \\
-\mu_1 &\leq -\beta_2 \,.
\end{aligned}
$$

$$
\begin{aligned}
\mu_1 &+ s_1 &= \lambda_1 \\
-\mu_1 &+ s_2 &= -\lambda_2 \\
-\mu_1 &+ s_3 &= \lambda_2 - \beta_1 - \beta_2 \\
\mu_1 &+ s_4 &= \beta_1 + \beta_2 + \lambda_1 + \lambda_2 \\
-\mu_1 &+ s_5 &= -\beta_1 \\
-\mu_1 &+ s_6 &= -\beta_2 \,.
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- The s_i are constrained to be nonnegative.

$$
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\end{aligned}
$$

- The s_i are constrained to be nonnegative.
- Finally we can use $\mu_1 = \lambda_1 - s_1$ to get rid of μ_1.

$$
\begin{aligned}
s_1 + s_2 &= \lambda_1 - \lambda_2 \\
-s_2 + s_3 &= 2\lambda_2 - \beta_1 - \beta_2 \\
s_2 + s_4 &= \beta_1 + \beta_2 + \lambda_1 \\
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$$

- Solving for $s_i \geq 0 \;\; \forall\, i$.

$$
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-s_2 + s_6 &= \lambda_2 - \beta_2
\end{aligned}
$$

- Solving for $s_i \geq 0 \ \forall\, i$.
- Requiring the s_i's to be integers yields all integer solutions to the Gelfand-Tsetlin constraints.

So we are solving

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}}_{E_2} \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_1 - \lambda_2 \\ 2\lambda_2 - \beta_1 - \beta_2 \\ \beta_1 + \beta_2 + \lambda_1 \\ \lambda_2 - \beta_1 \\ \lambda_2 - \beta_2 \end{pmatrix}}_{B_2\binom{\lambda}{\beta}}$$

for $\vec{s} \in \mathbb{N}^6$.

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for $\vec{s} \in \mathbb{N}^6$. Hence $$K_{\lambda\beta} = \phi_{E_2}\left(B_2\binom{\lambda}{\beta}\right).$$

A chamber complex for the $K_{\lambda\beta}$

- Theorem A implies that the Kostka numbers are given by quasipolynomials over the cells of a chamber complex $\mathcal{C}^{(k)}$.

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A chamber complex for the $K_{\lambda\beta}$

- Theorem A implies that the Kostka numbers are given by quasipolynomials over the cells of a chamber complex $\mathcal{C}^{(k)}$.

- The vector partition function ϕ_{E_k} puts λ and β on an equal footing: $\mathcal{C}^{(k)}$ is a complex in (λ, β)-space.

- By intersecting $\mathcal{C}^{(k)}$ with the affine subspace corresponding to fixing λ, we get the domains of quasipolynomiality for $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$.

The Duistermaat-Heckman function

- For every λ there is a function, the Duistermaat-Heckman function, that is piecewise polynomial on $\operatorname{conv}(\mathfrak{S}_k \cdot \lambda)$.

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$$K_{\lambda\beta} = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma)} K(\sigma(\lambda+\delta) - (\beta+\delta)) .$$

$$f_\lambda^{\mathrm{DH}}(\beta) = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma)} \tilde{K}(\sigma(\lambda) - \beta) .$$

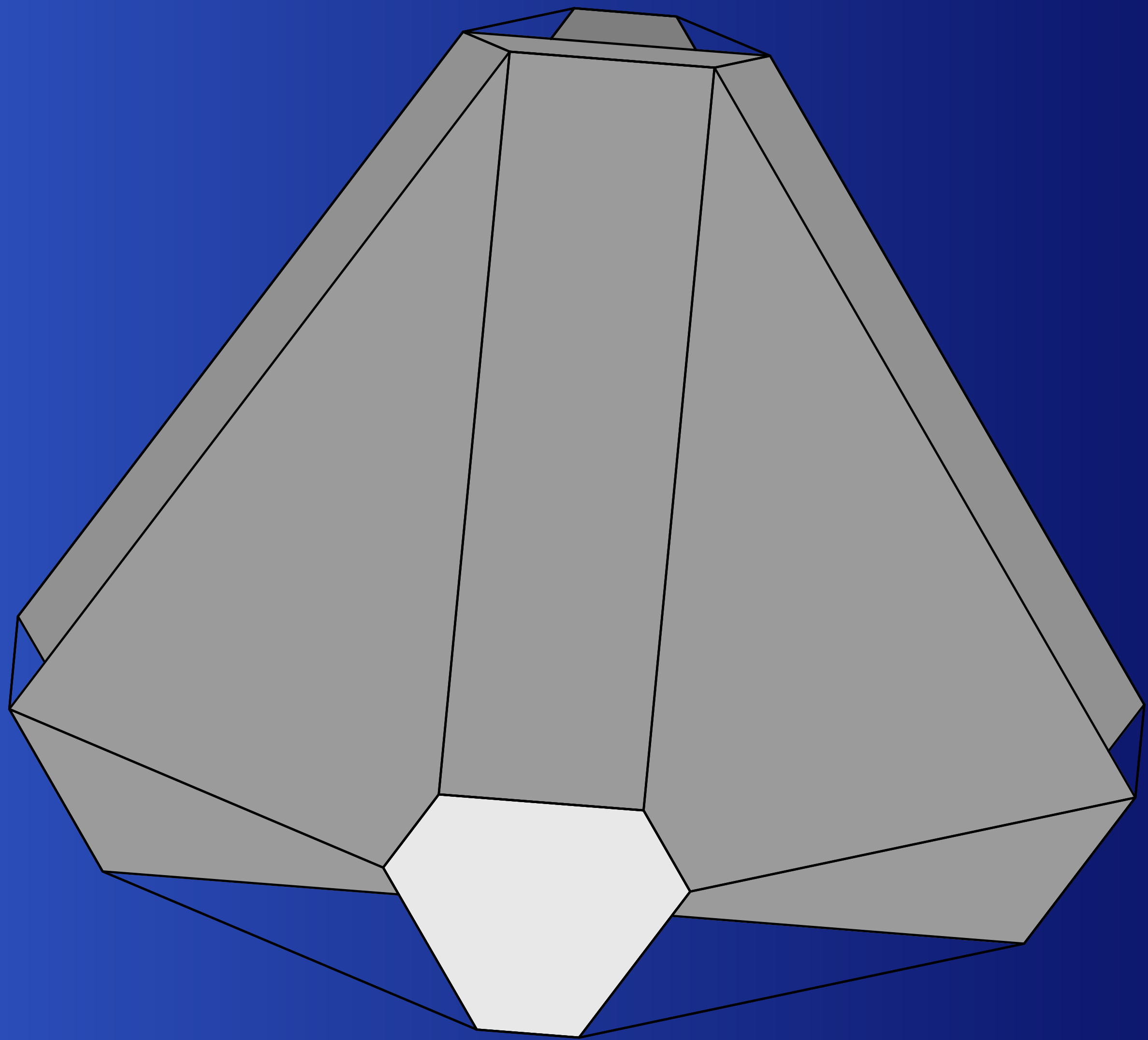
Theorem **(Heckman, Guillemin-Lerman-Sternberg)**

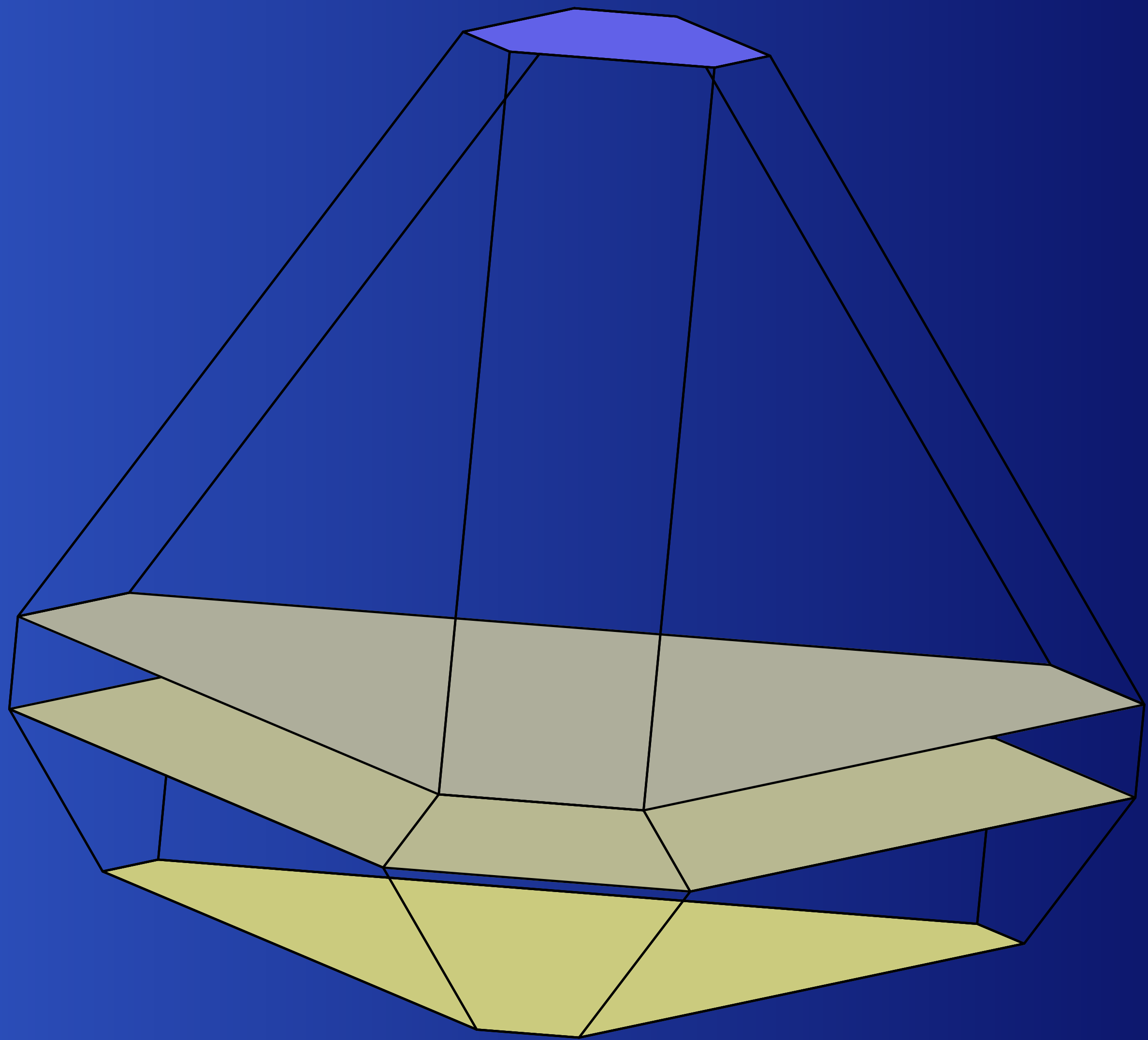
Consider the convex polytopes

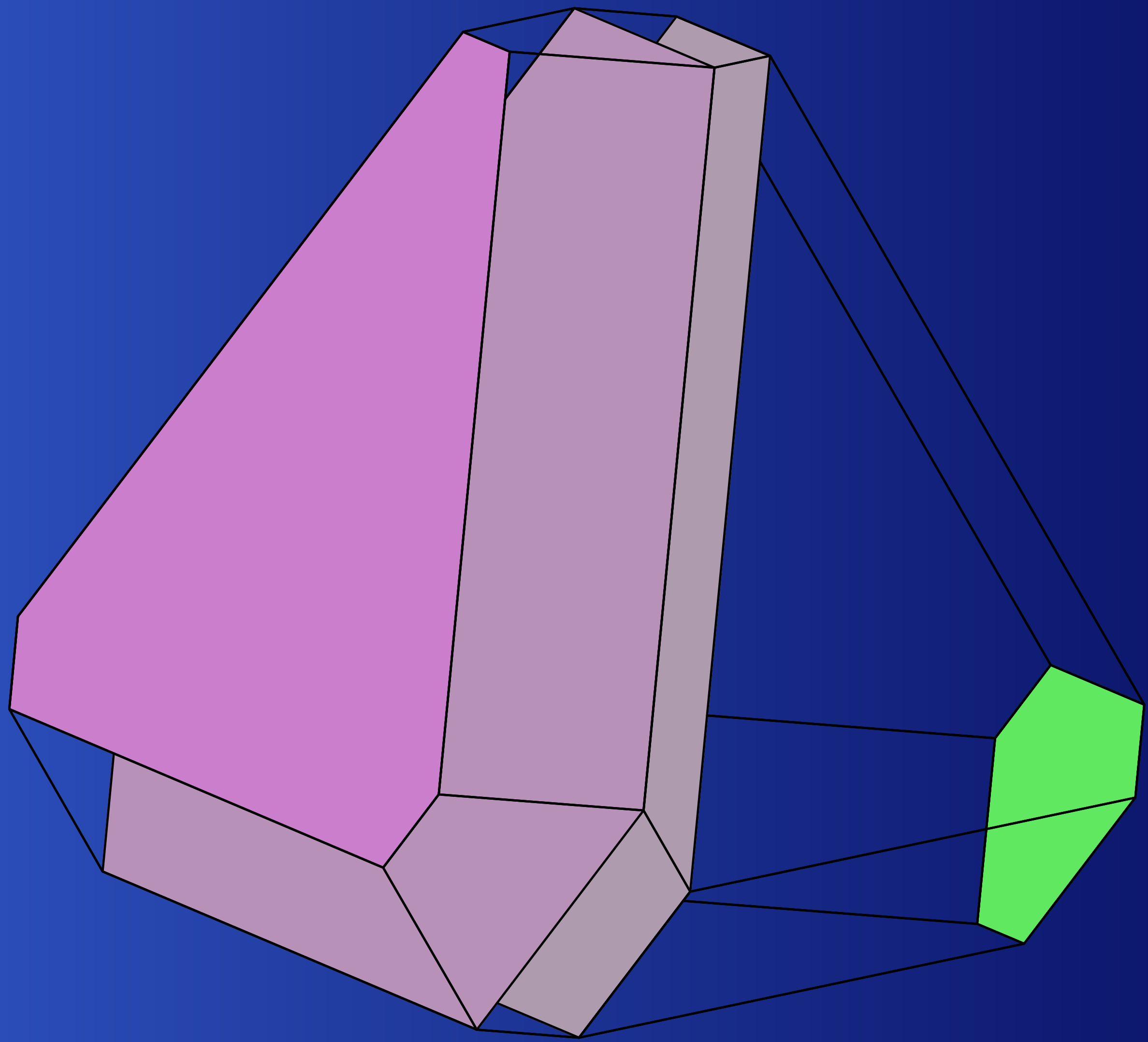
$$\mathrm{conv}(W \cdot \sigma(\lambda))$$

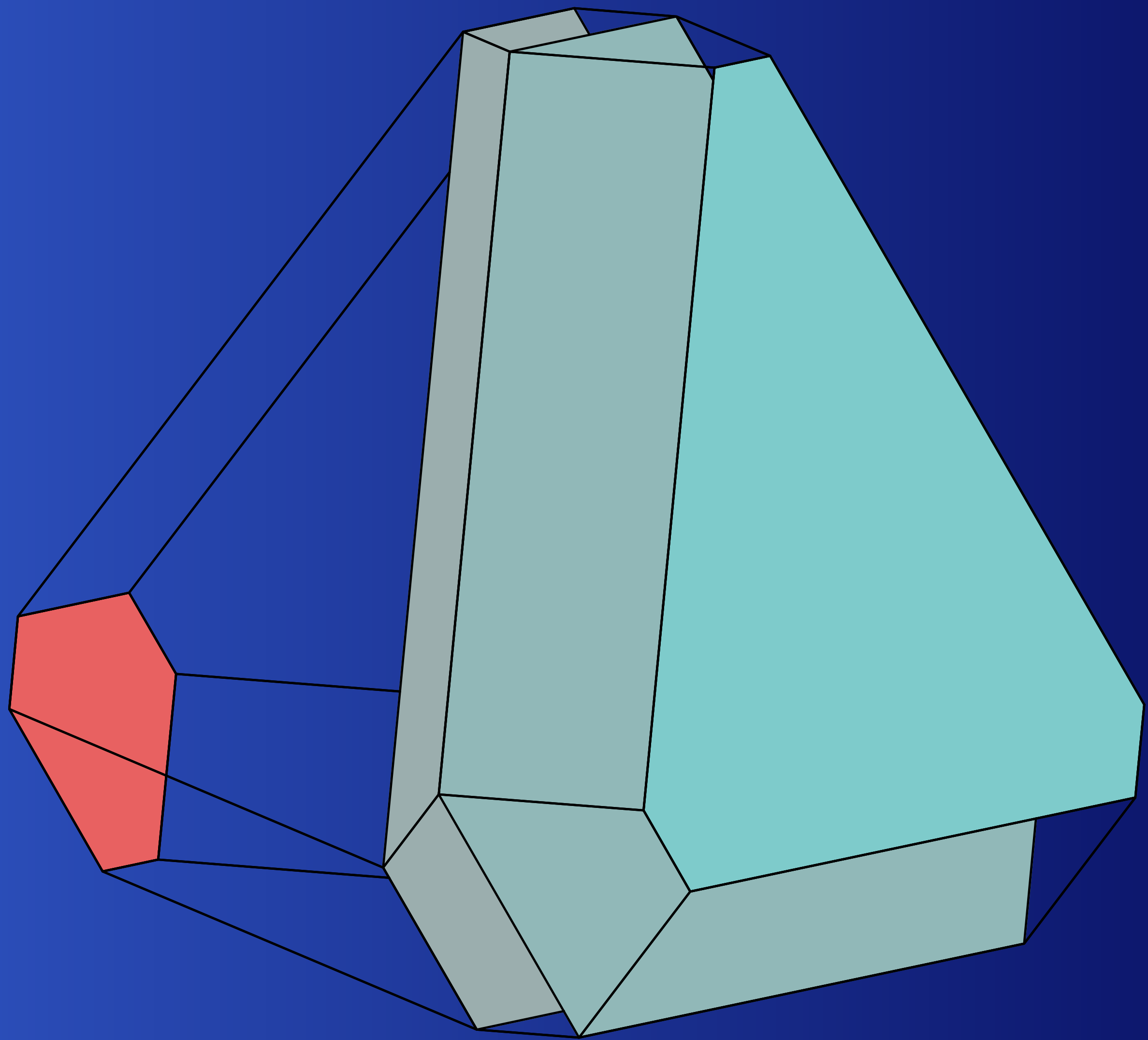
where $\sigma \in \mathfrak{S}_k$ *and* W *is the stabilizer of a facet of* $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$.

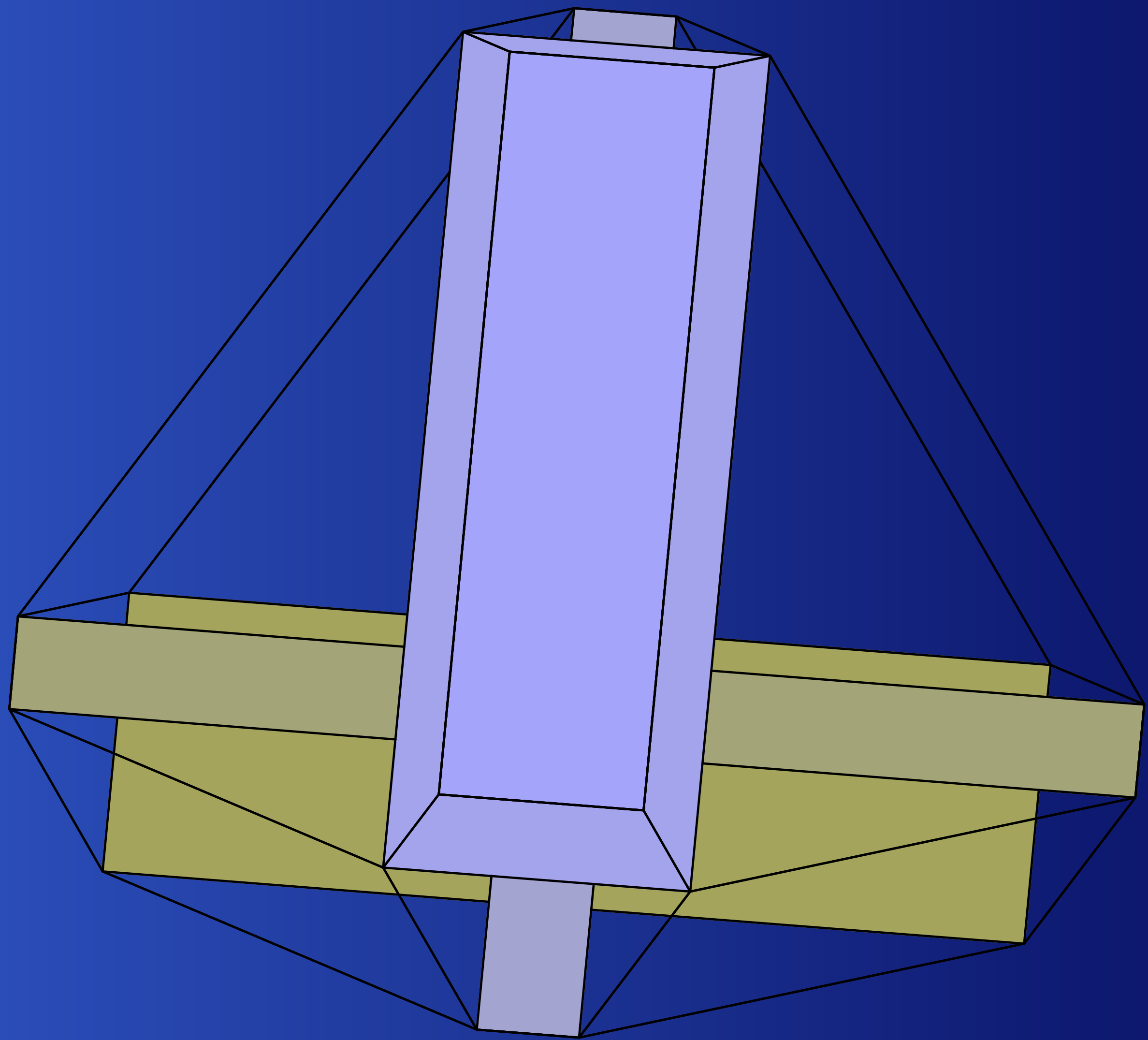
These polytopes are walls that partition $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$ *into convex subpolytopes over which the Duistermaat-Heckman function is polynomial.*

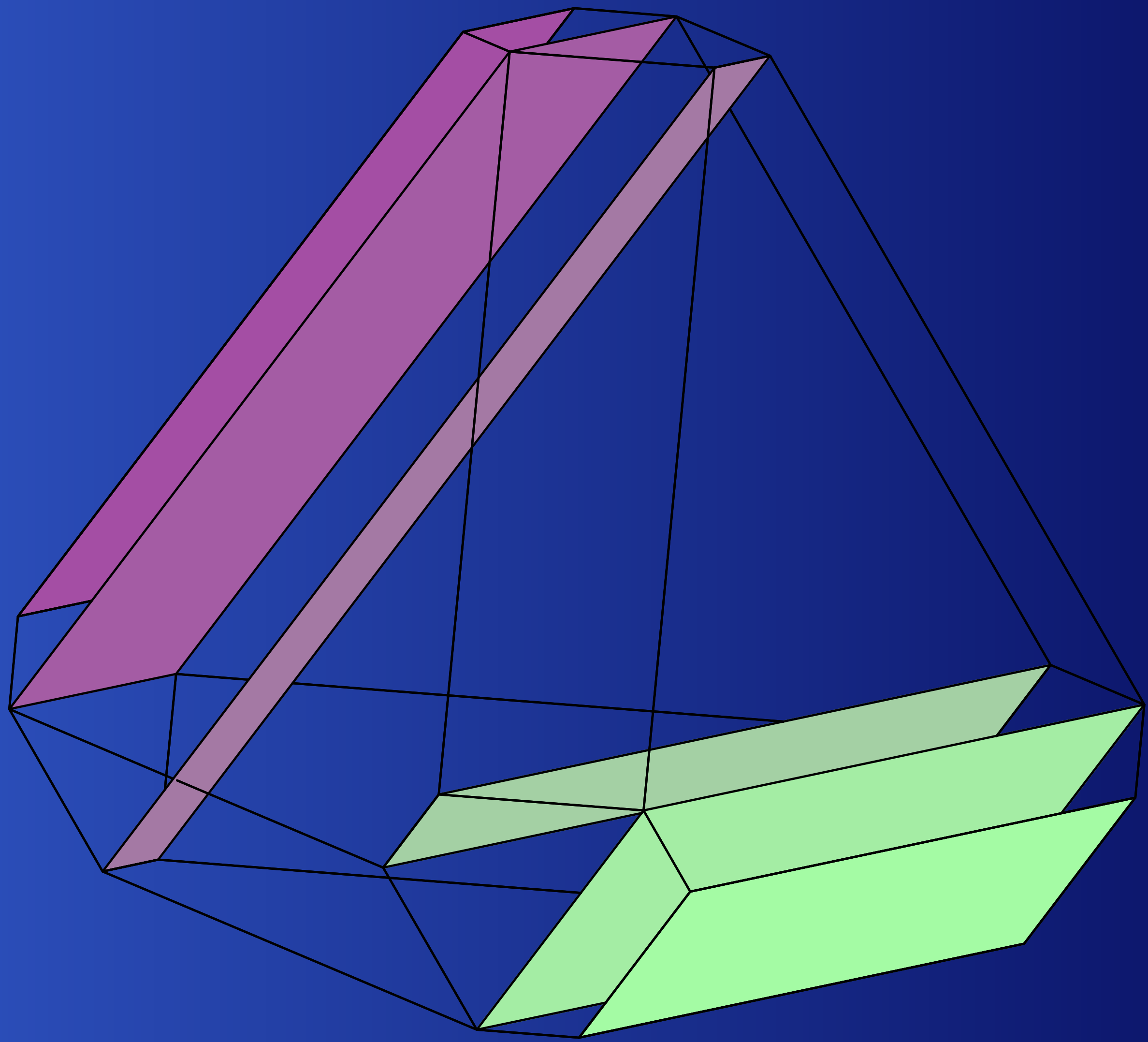


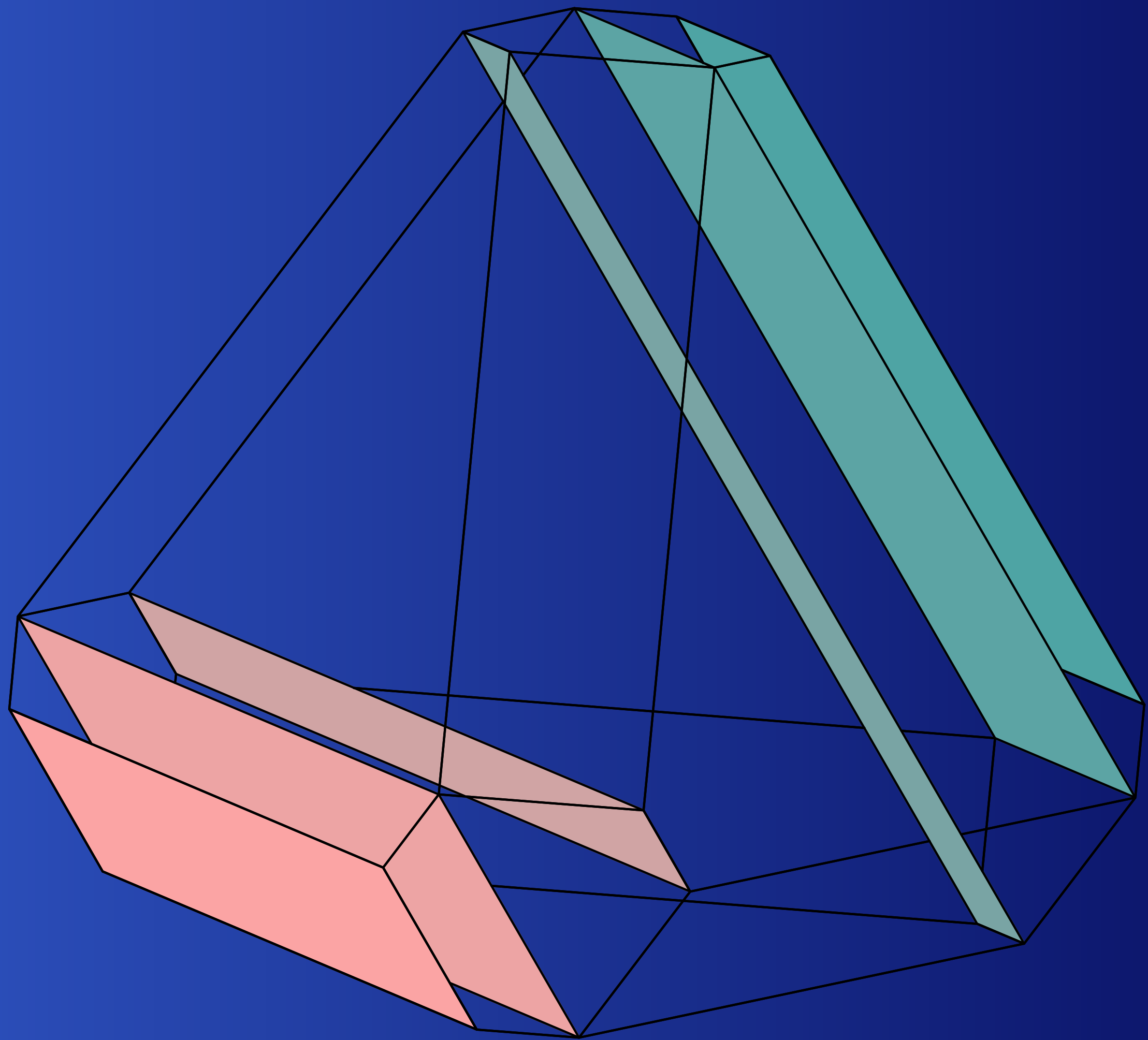


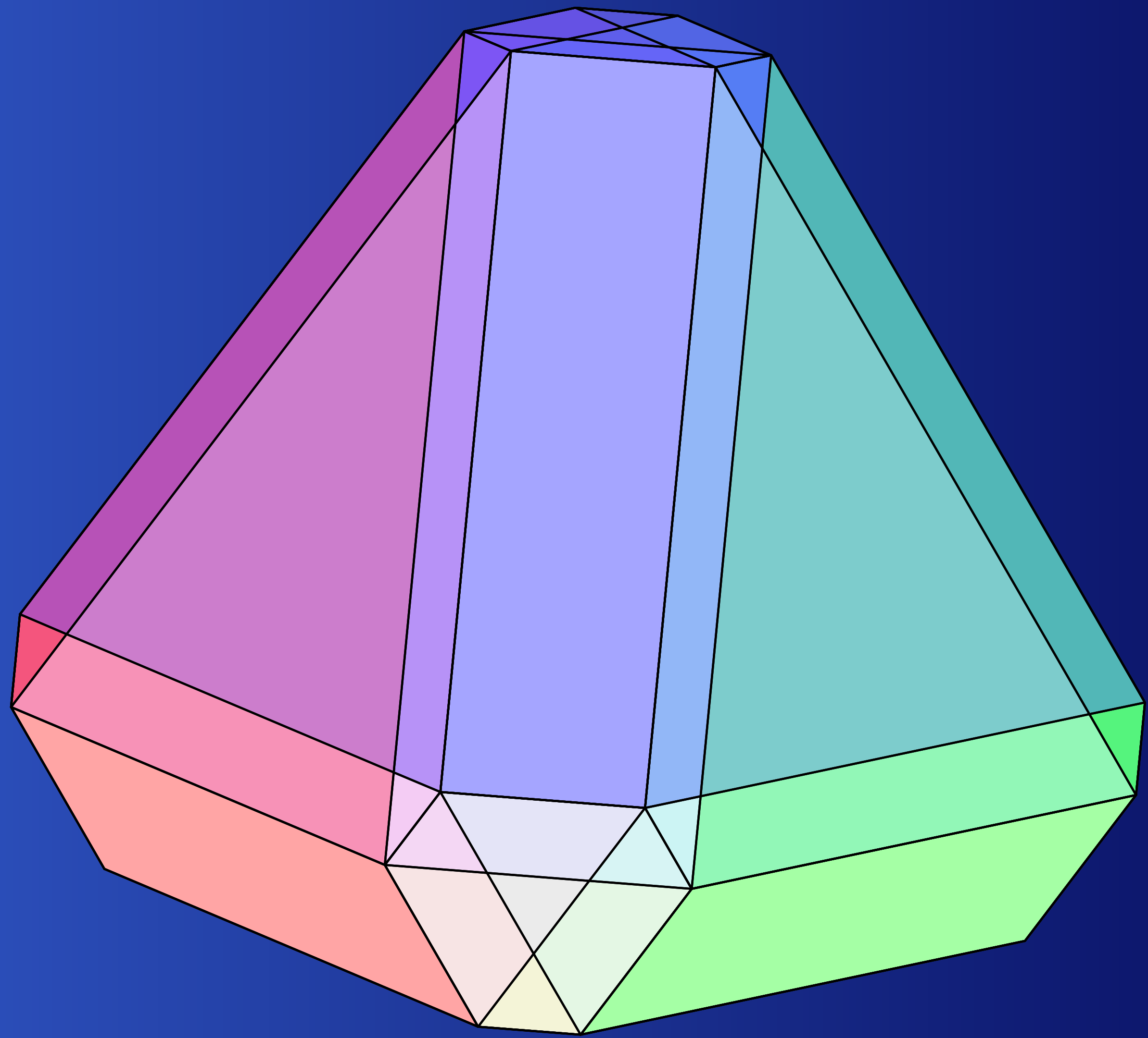










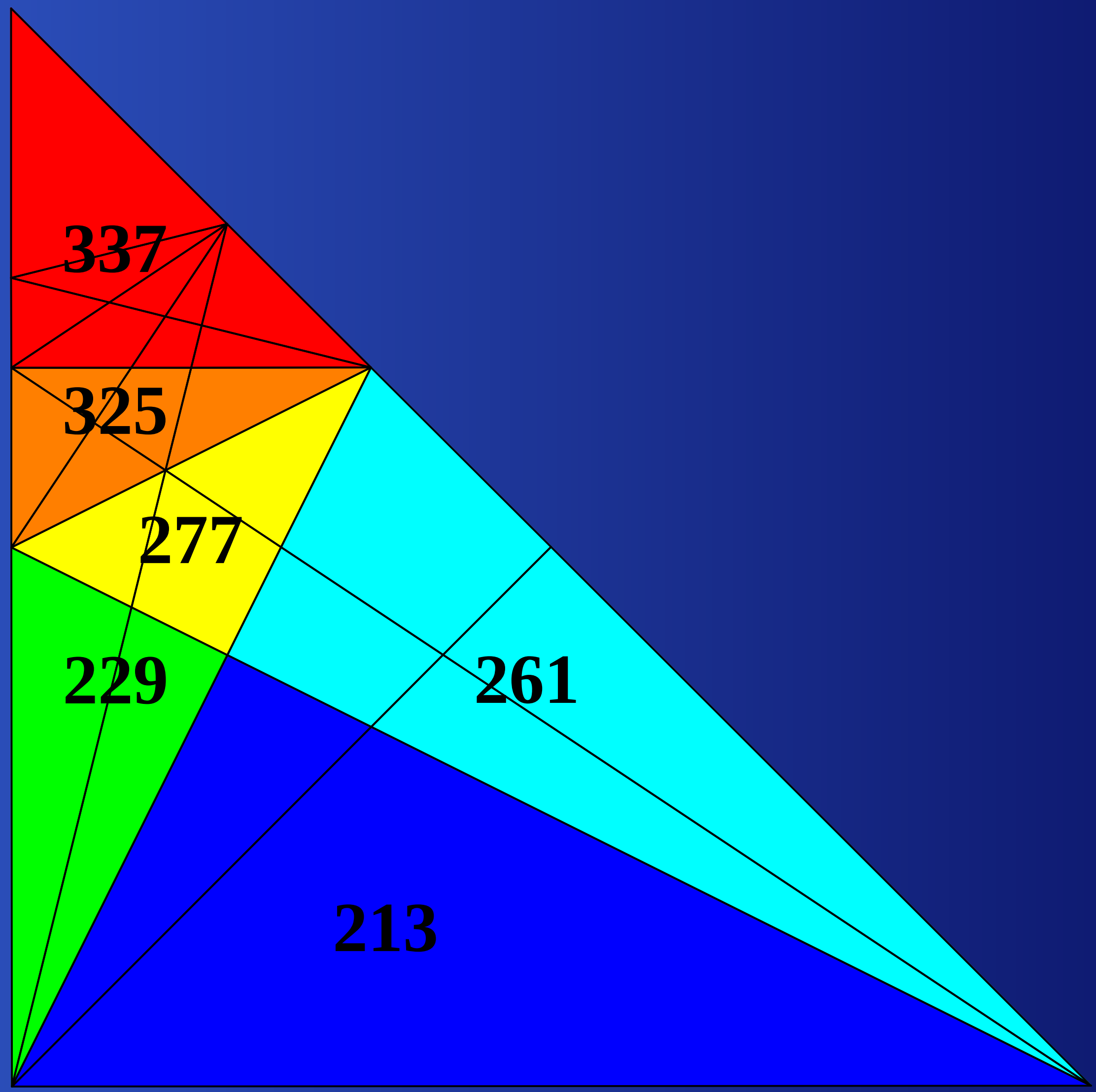


Theorem B

The partitions of the permutahedron into its domains of polynomiality for the Kostka numbers and for the Duistermaat-Heckman function are the same.

Namely, the domains are the regions determined by the theorem of Heckman.

A_3



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- a uniform combinatorial description for the walls partitioning the permutahedron into its domains of quasipolynomiality for the Kostka numbers;

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- a uniform combinatorial description for the walls partitioning the permutahedron into its domains of quasipolynomiality for the Kostka numbers;
- that these domains are actually domains of polynomiality.

The Kostant arrangements

The Kostant arrangements will be the main tool to

- complete the proof that the Kostka numbers are given by polynomials on the cones of a chamber complex;

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The Kostant arrangements will be the main tool to

- complete the proof that the Kostka numbers are given by polynomials on the cones of a chamber complex;
- find interesting factorization patterns in the polynomials giving the Kostka numbers.

- Kostant's multiplicity formula:

$$K_{\lambda\beta} = \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma)} K(\sigma(\lambda+\delta) - (\beta+\delta))\,.$$

Kostant partition function is piecewise polynomial

$\Downarrow$

Kostka numbers are locally polynomial

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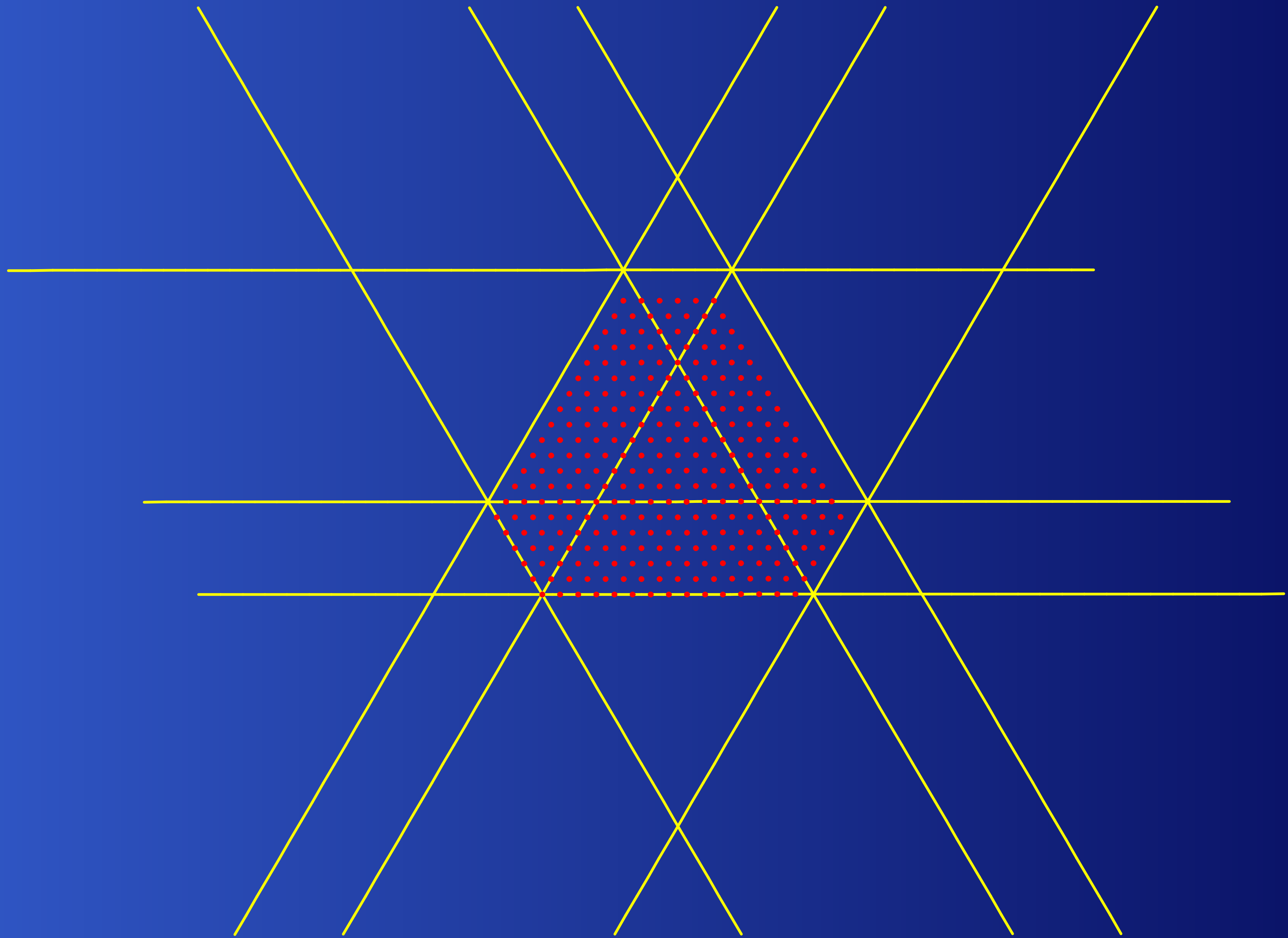
Kostant partition function is piecewise polynomial

$$\Downarrow$$

Kostka numbers are locally polynomial

- We will find a family of hyperplane arrangements over whose regions the Kostka numbers are given by polynomials.

Example: $\lambda = (21,\ 7,\ 2)$



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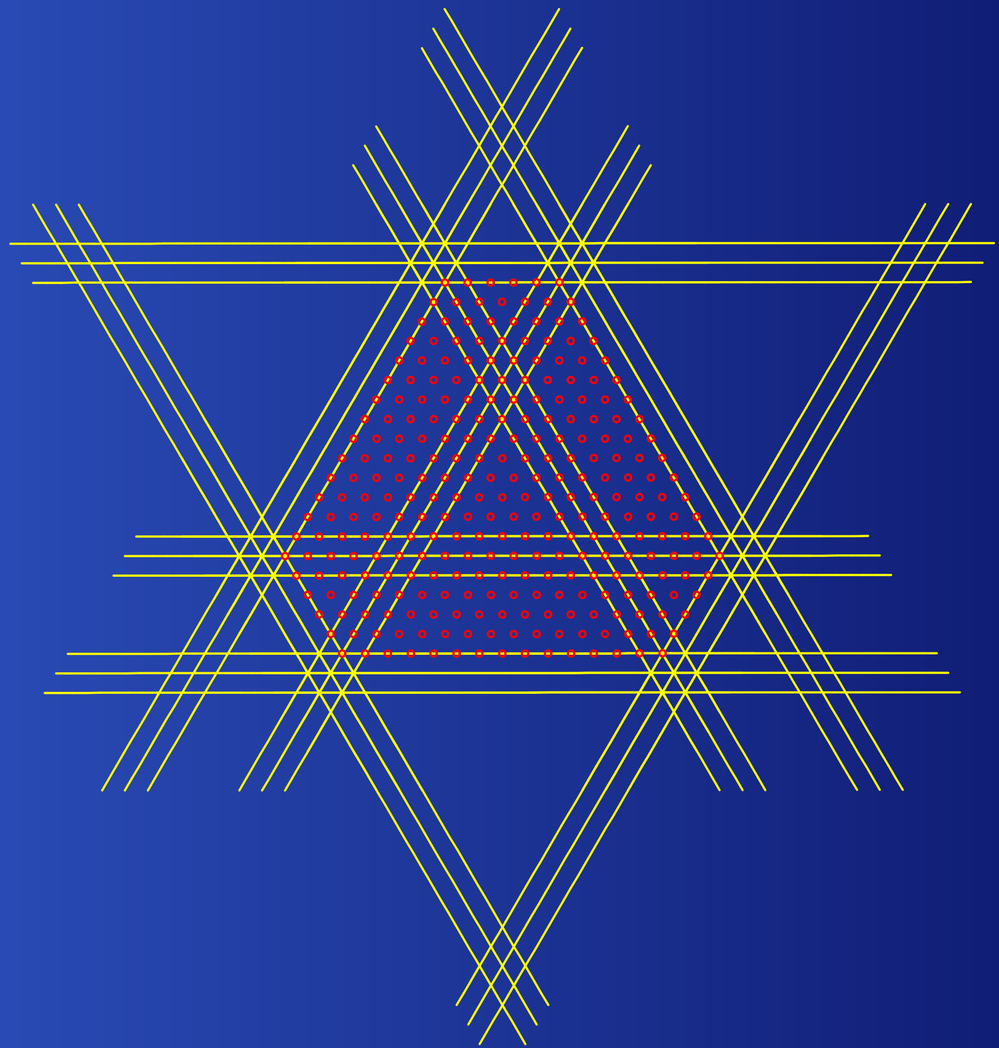

Walls of the permutahedron

- Walls supporting the facets of the top-dimensional domains of the permutahedron (partition for the Duistermaat-Heckman function):

$$\langle\, \sigma(\lambda) - \psi(\beta),\ \theta(\omega_j)\,\rangle = 0\,.$$

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Polynomiality in the chamber complex

Theorem C

The quasipolynomials giving the Kostka numbers in the cones of $\mathcal{C}^{(k)}$ are polynomials of degree $\binom{k-1}{2}$ in the β_i, with coefficients of degree $\binom{k-1}{2}$ in the λ_j.

Lemma

For each cone C of the chamber complex for the Kostka numbers, we can find a region R of any of the Kostant arrangements such that $C \cap R$ contains an arbitrarily large ball.

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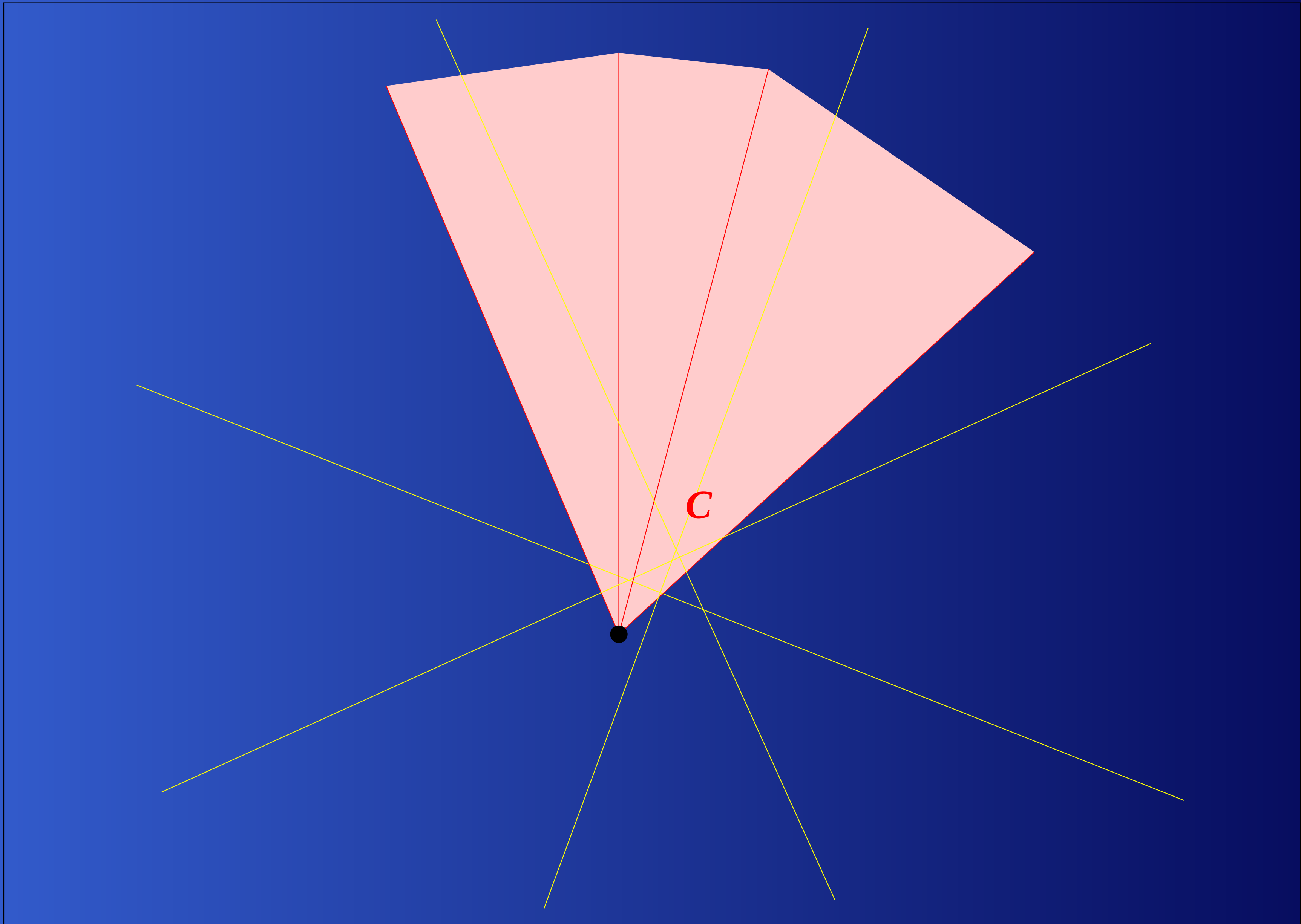
- Then the polynomial on R and the quasipolynomial on C agree on all the lattice points (λ, β) in that ball.
- The degree bounds follow from the degree bounds on the Kostant partition function.

Idea of proof



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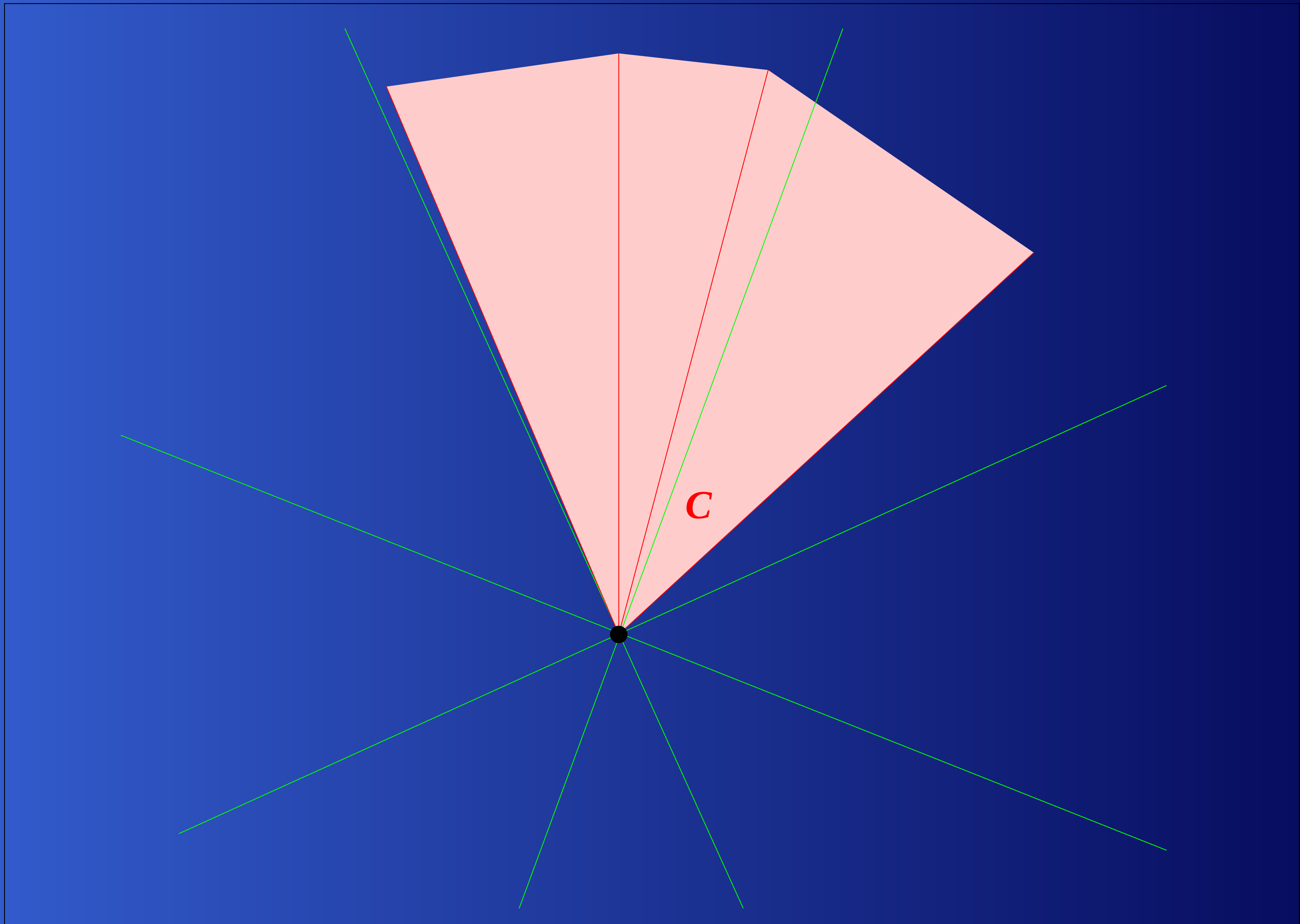

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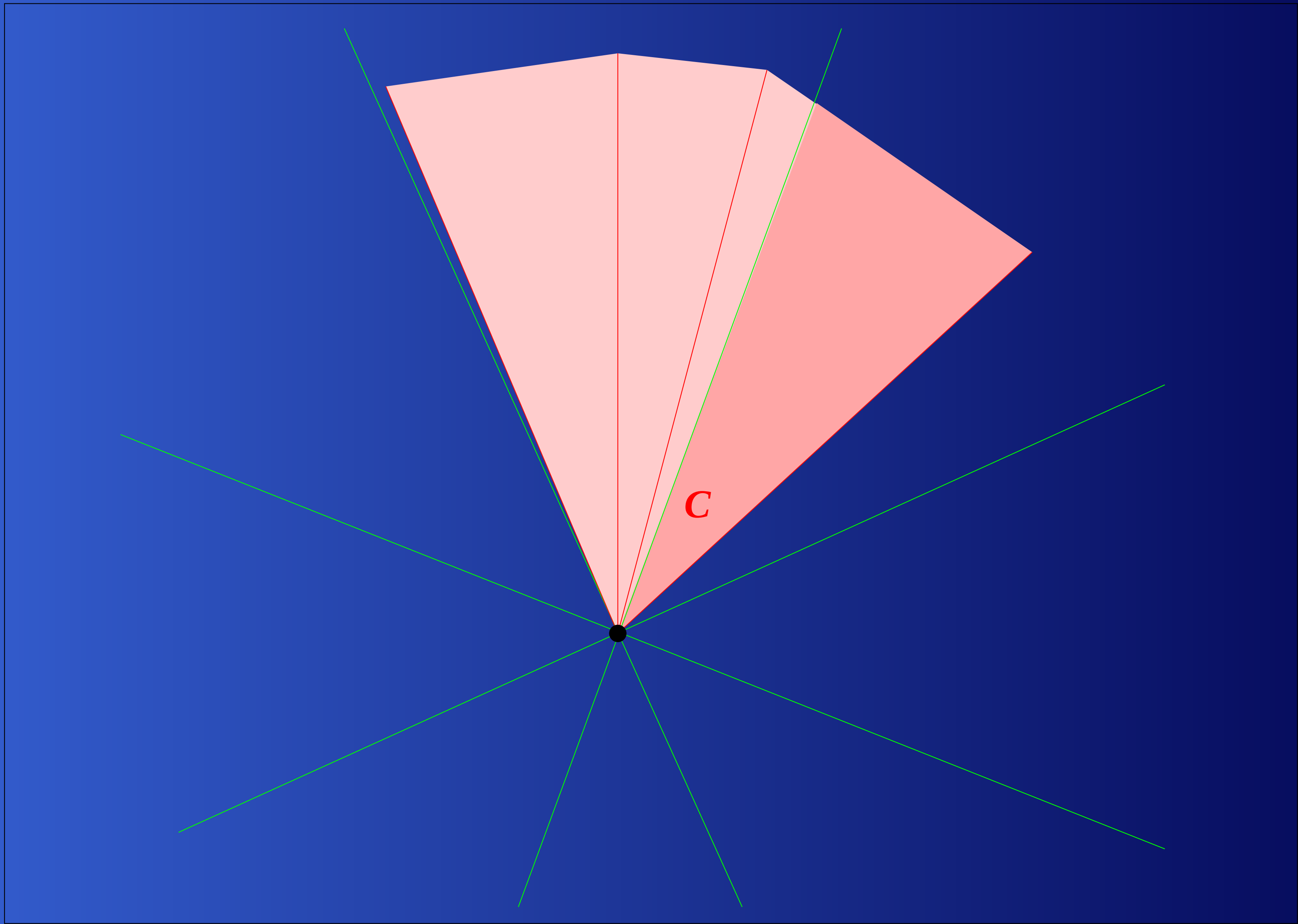

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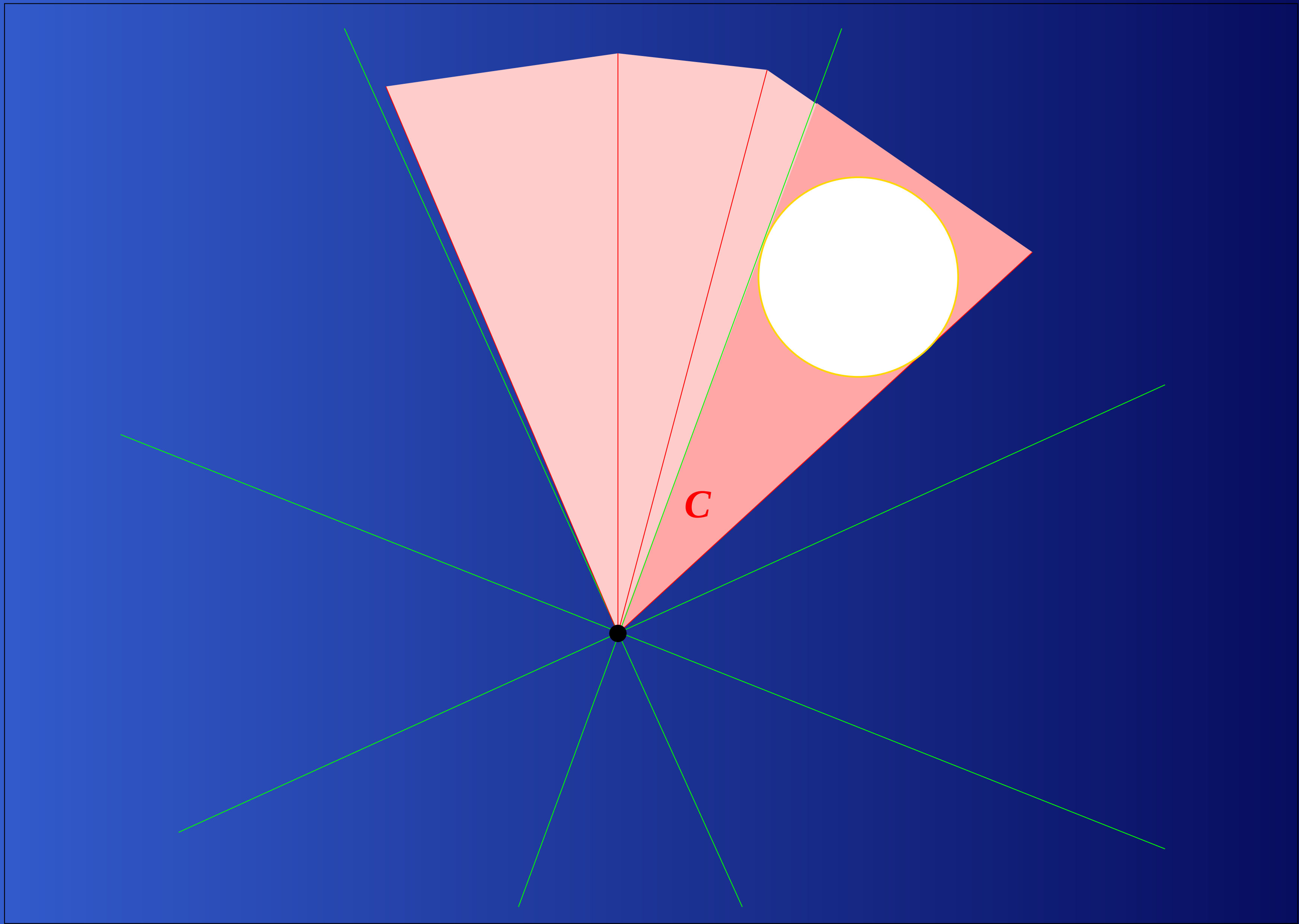

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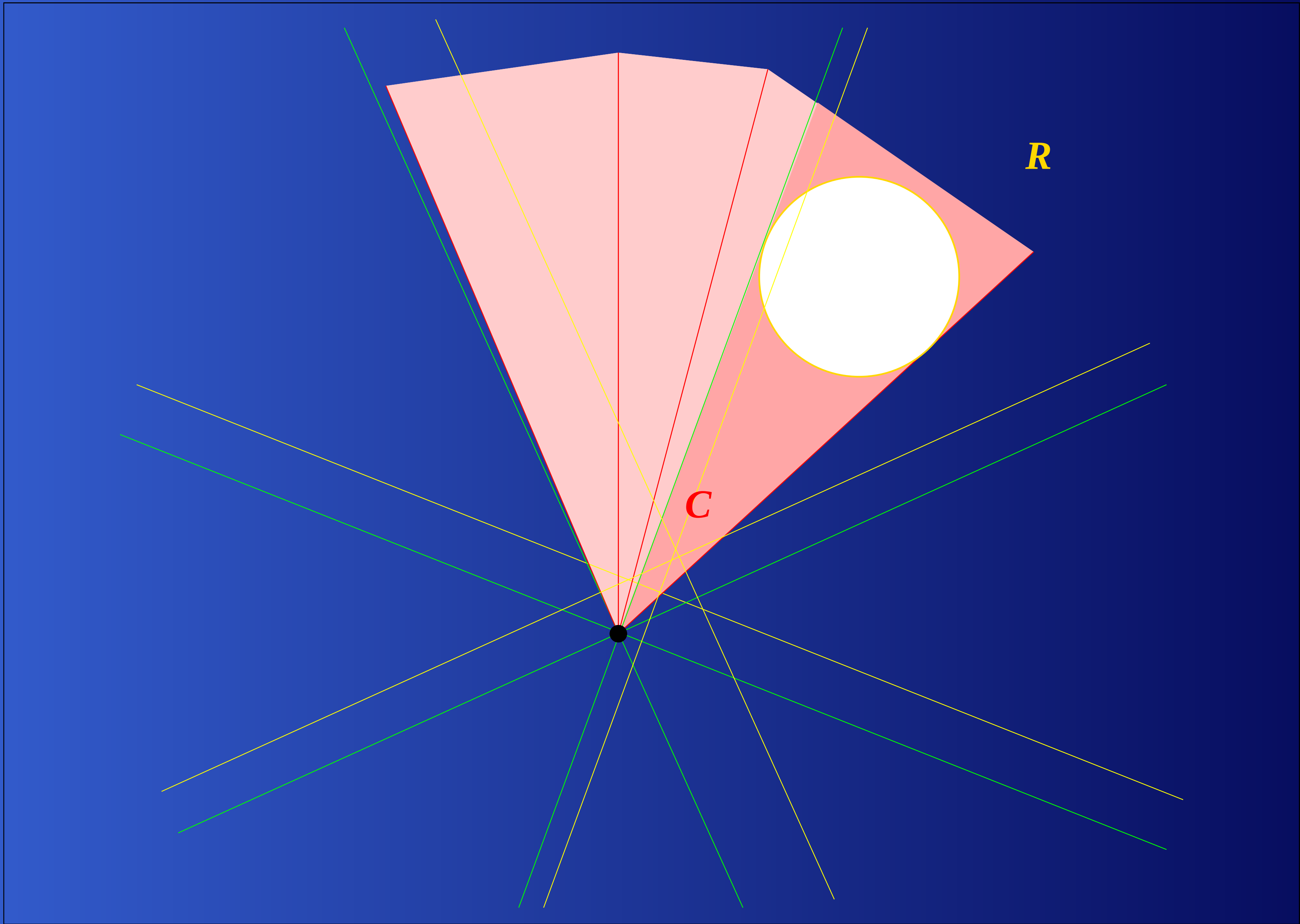

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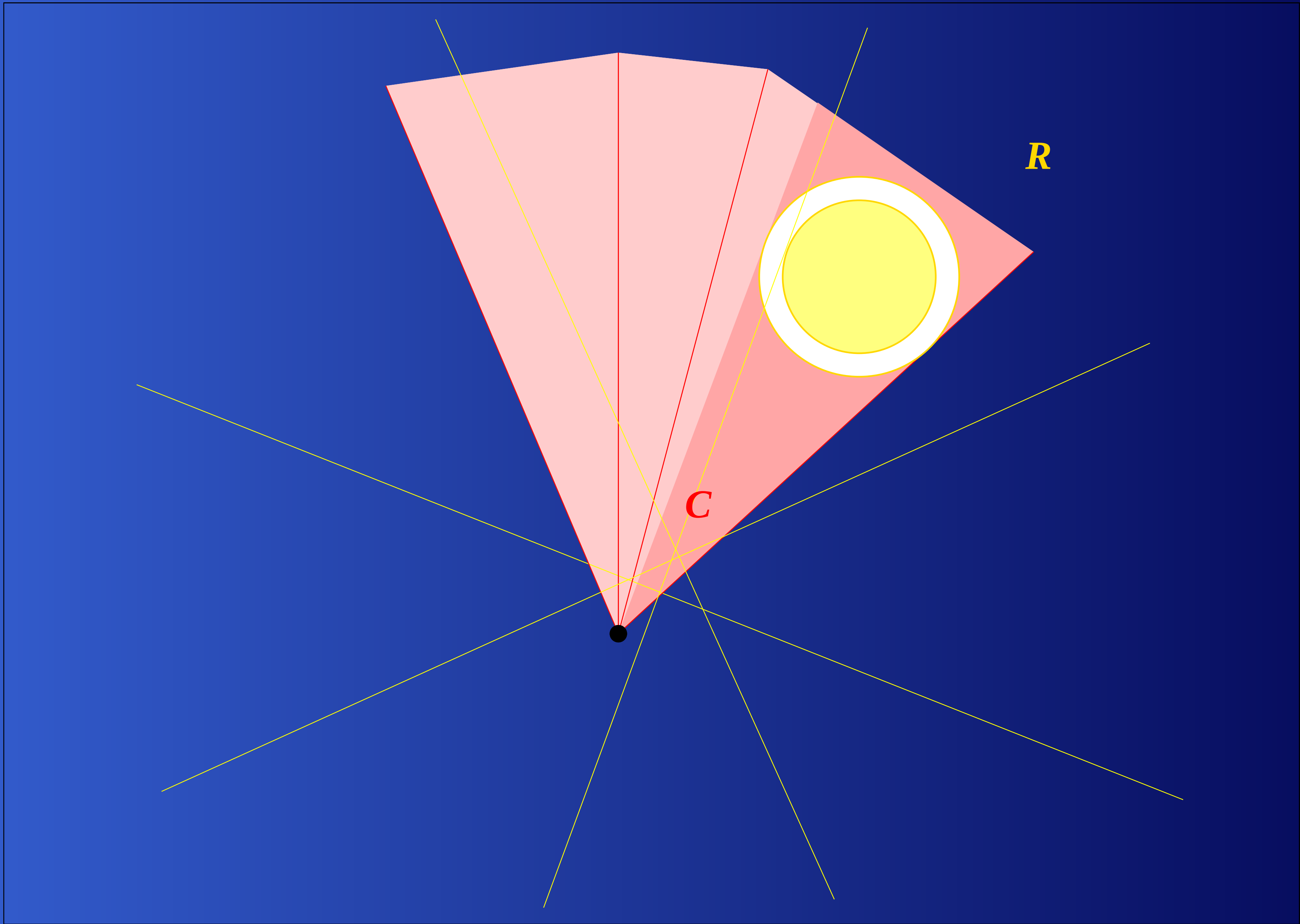

Scaling (or stretching)

Corollary

For any $\lambda, \beta \in \Lambda_W$ *with* $\lambda - \beta \in \Lambda_R$*, the function*

$$N \in \mathbb{N} \quad \longmapsto \quad K_{N\lambda\, N\beta}$$

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- $\mathrm{GT}_{\lambda\mu}$ is not an integral polytope in general (Clifford, King-Tollu-Toumazet, DeLoera-McAllister).

Factorization patterns

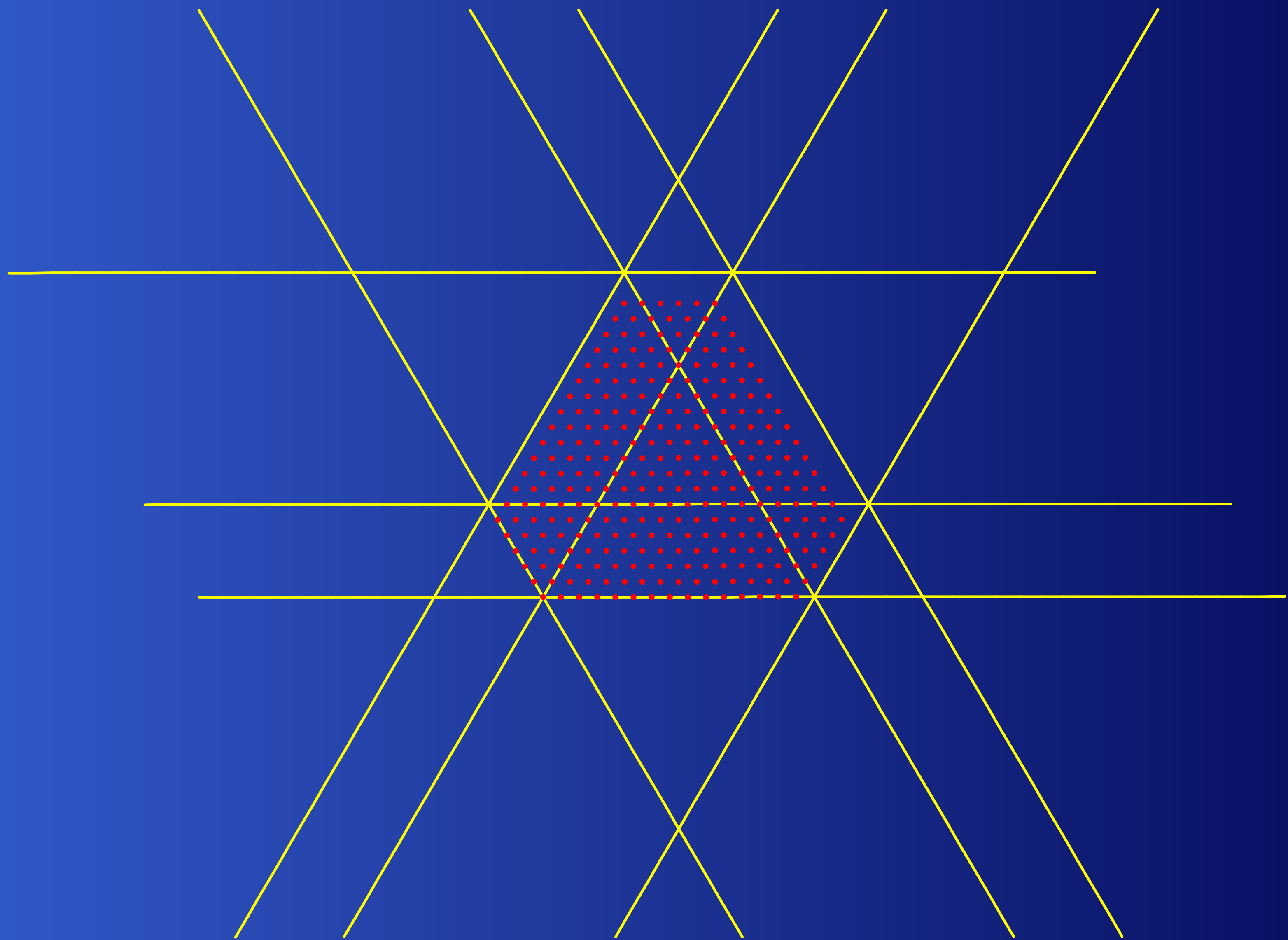
Theorem D

Suppose that H is the hyperplane supporting a facet of the permutahedron with normal $\theta(\omega_j)$.

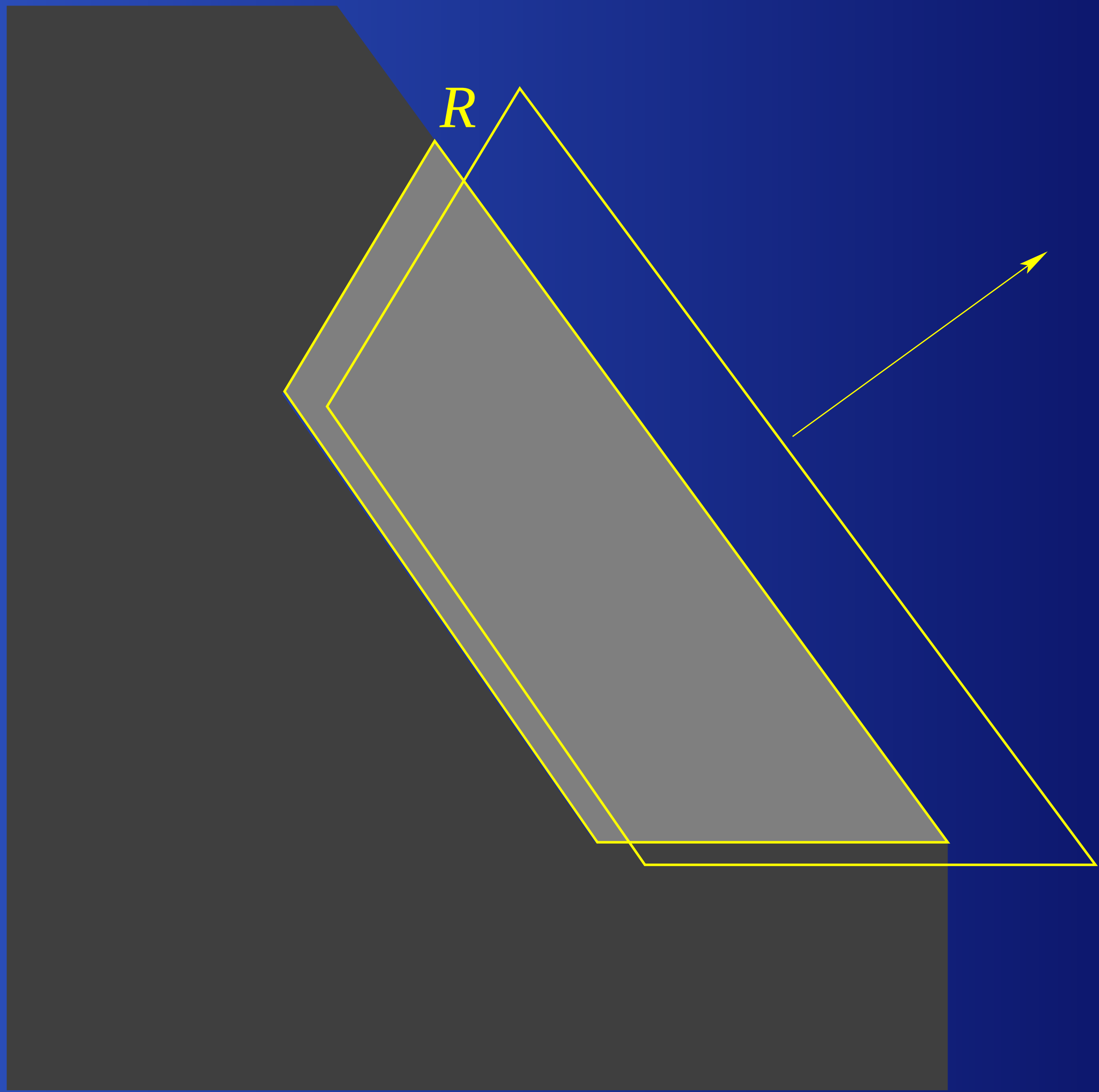
Then the polynomials giving the Kostka numbers in all the domains of the permutahedron with a facet on H are divisible by $j(k-j)-1$ linear factors.

The following diagrams will explain what those factors are.

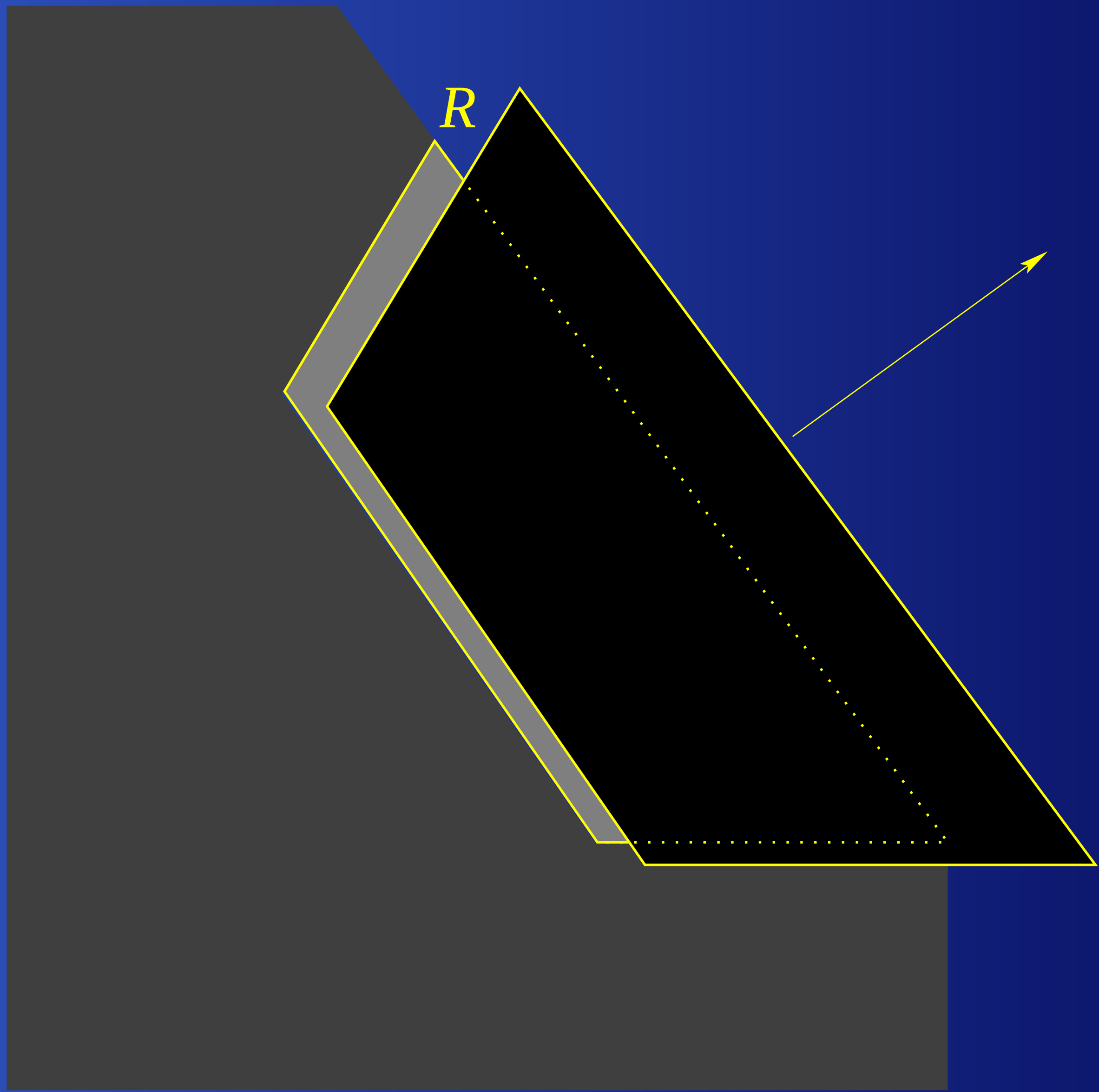
Idea of proof



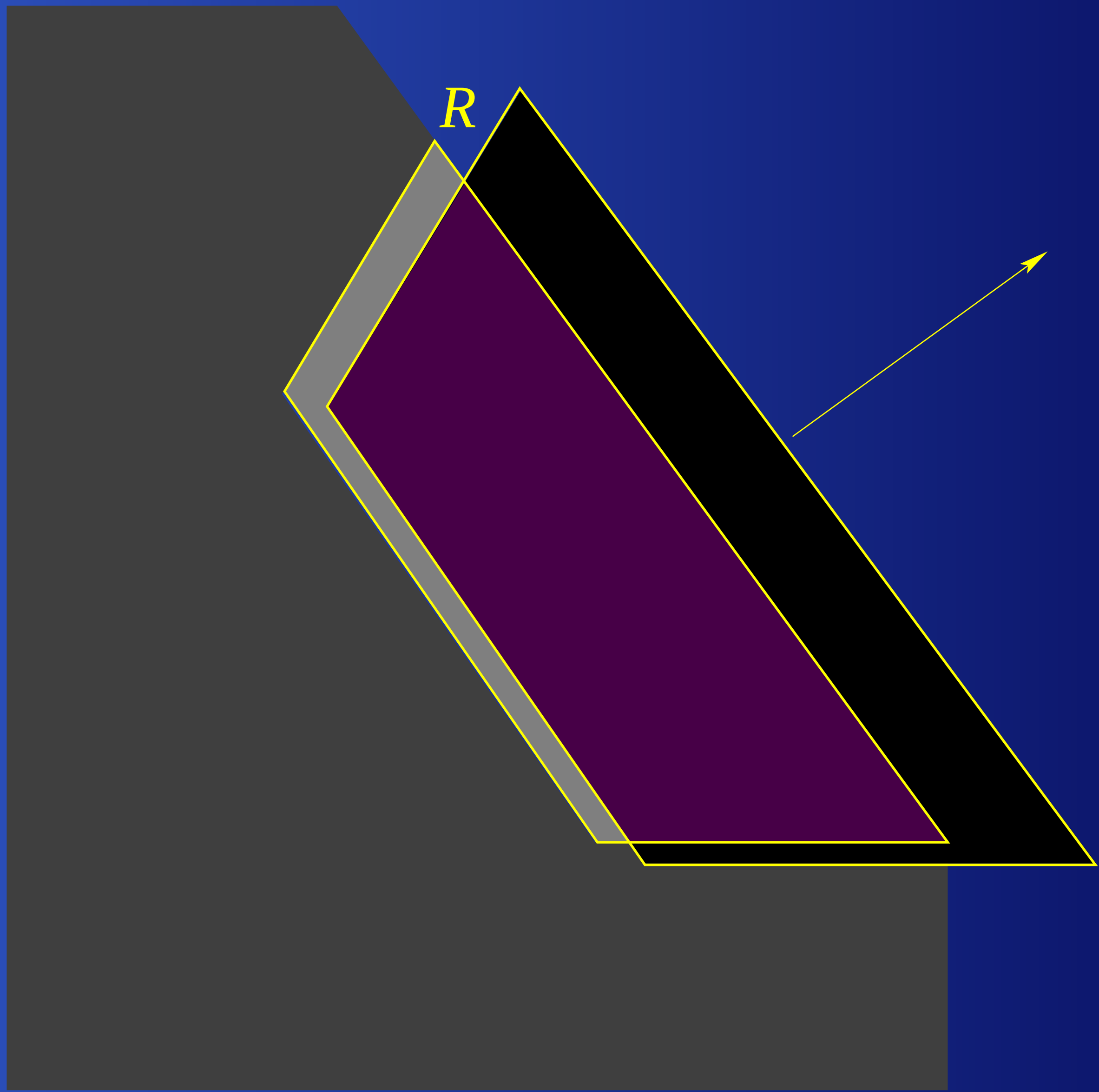
Idea of proof



Idea of proof



Idea of proof



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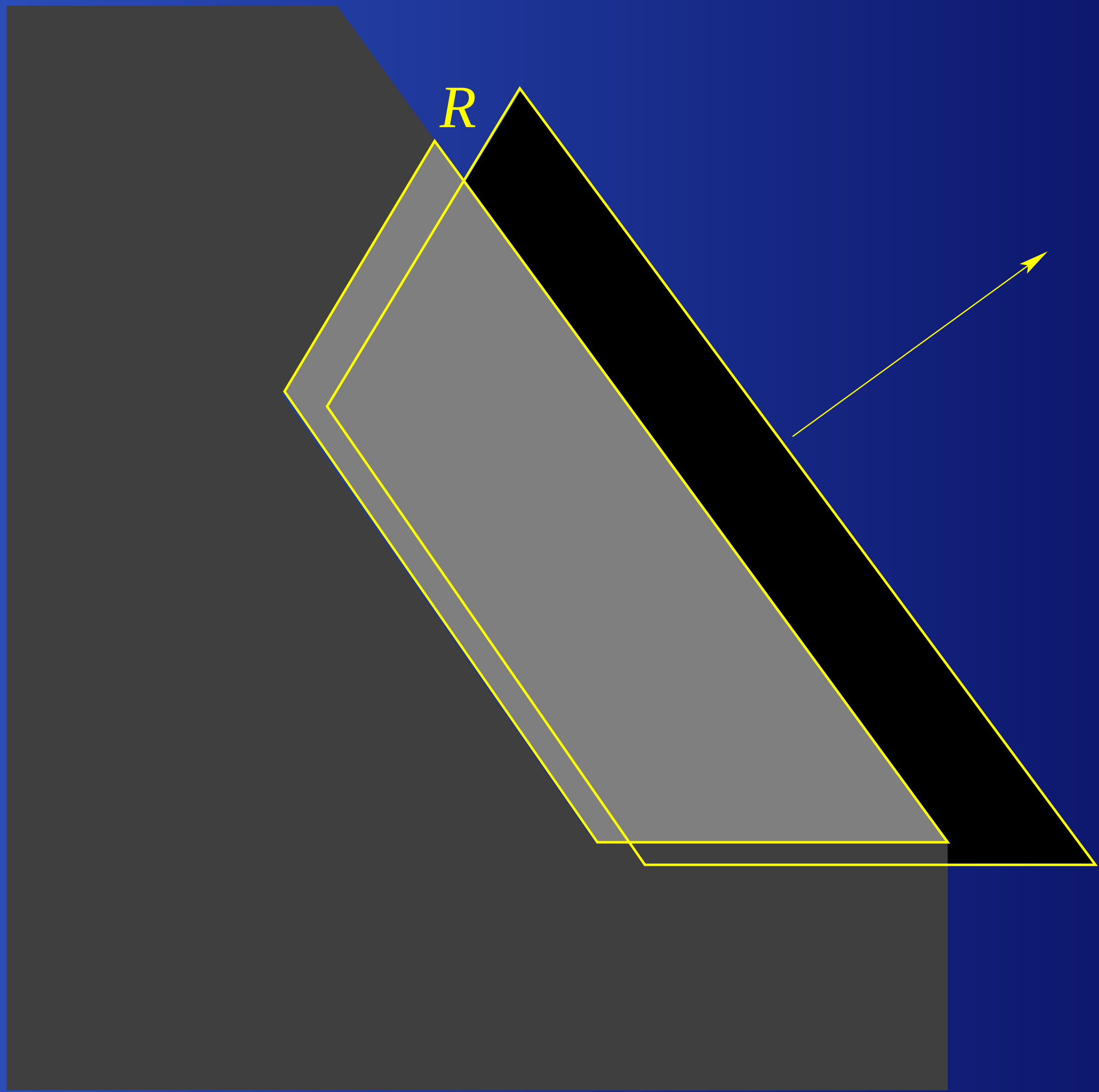
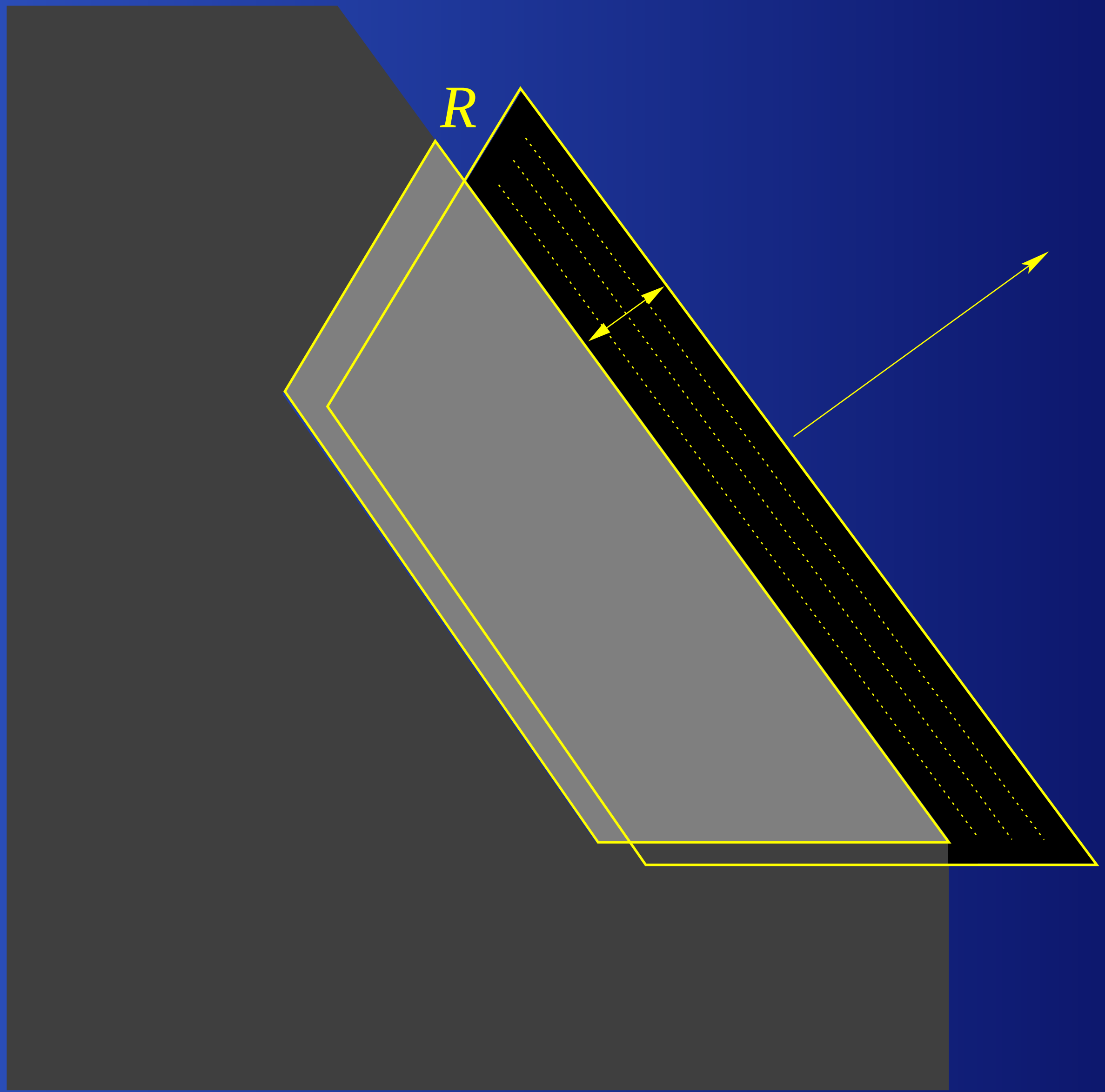


Idea of proof



Similar factorization phenomena were recently observed to hold for general vector partition functions by Szenes and Vergne.

Littlewood-Richardson coefficients

- The LR coefficients express the multiplication rule for Schur functions:

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- In the representation theory of $\mathrm{GL}_k\mathbb{C}$, the characters of the irreducible polynomial representations are Schur functions in appropriate variables.

$$V_\lambda \otimes V_\mu = \bigoplus_\nu c^\nu_{\lambda\mu} V_\nu \,.$$

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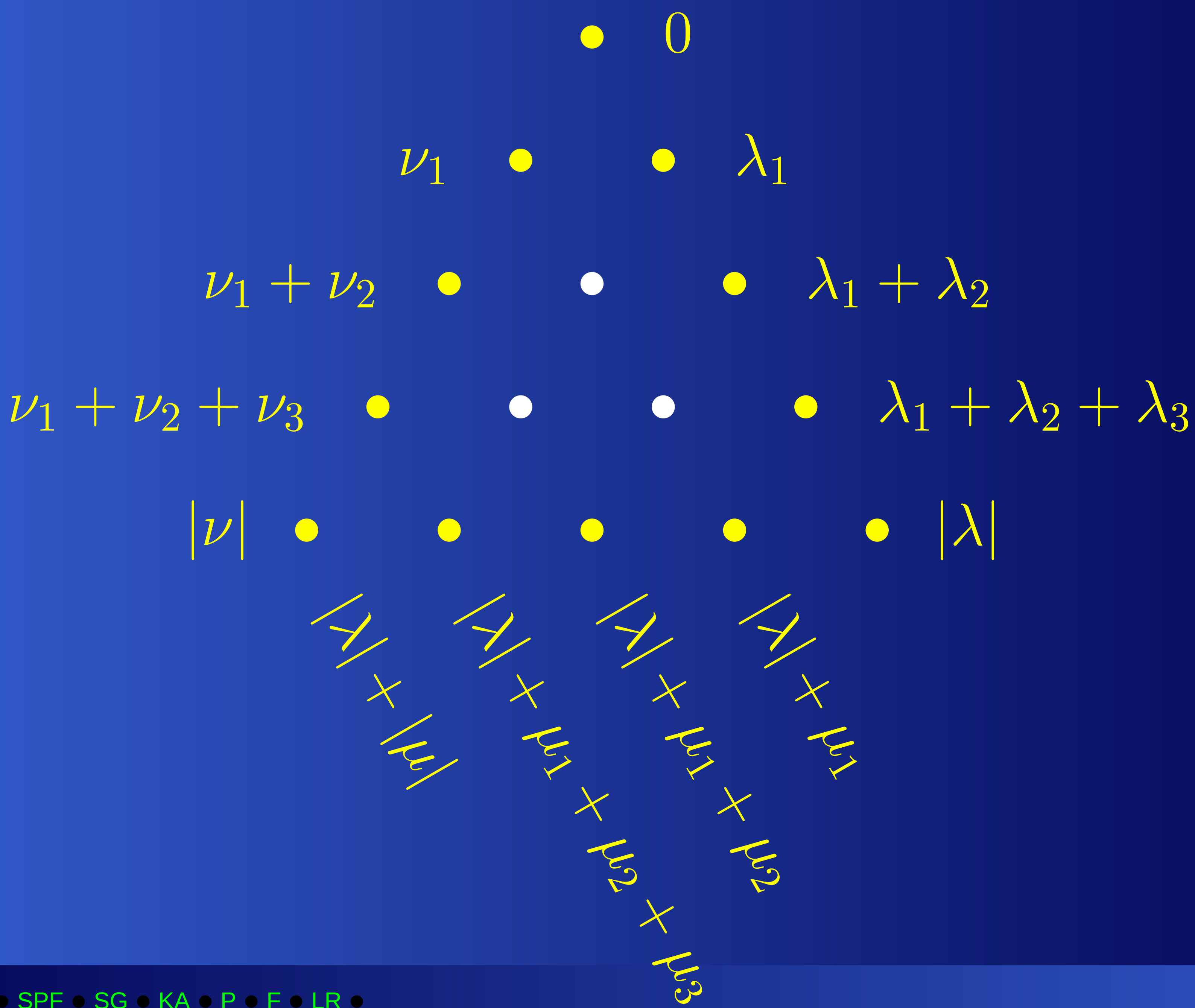
- the Kostant arrangements, over the regions of which the Littlewood-Richardson coefficients would be given by polynomial functions.

Hives


0
ν_1
λ_1
$\nu_1 + \nu_2$
$\lambda_1 + \lambda_2$
$\nu_1 + \nu_2 + \nu_3$
$\lambda_1 + \lambda_2 + \lambda_3$
$|\nu|$
$|\lambda|$
$|\lambda| + |\mu|$
$|\lambda| + \mu_1 + \mu_2 + \mu_3$
$|\lambda| + \mu_1 + \mu_2$
$|\lambda| + \mu_1$

Theorem **(Knutson-Tao, Fulton)**

Let λ, μ and ν be partitions with at most k parts such that $|\lambda| + |\mu| = |\nu|$.

The Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is the number of integral k-hives satisfying the boundary conditions and the hive conditions.

Steinberg's formula

$$c^{\nu}_{\lambda\mu} = \sum_{\sigma\in\mathfrak{S}_k} \sum_{\tau\in\mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma\tau)} K(\sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta))\,.$$

Partition functions and polynomiality

- Using hives, we can find a vector partition function for the LR coefficients, so they are given by quasipolynomial functions in λ, μ and ν over the cones of a chamber complex.

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- Using hives, we can find a vector partition function for the LR coefficients, so they are given by quasipolynomial functions in λ, μ and ν over the cones of a chamber complex.
- We can construct a hyperplane arrangement from Steinberg's formula over whose regions the LR coefficients are given by a polynomial in λ, μ and ν.
- We can relate the chamber complex to the Steinberg arrangement and show that the quasipolynomials are really polynomials.

Stretching for LR coefficients

- This shows in particular that the function

$$N \in \mathbb{N} \quad \longmapsto \quad c_{N\lambda\ N\mu}^{N\nu}$$

is polynomial in N.

This was known previously (Derksen-Weyman, Knutson).

- This function is the Ehrhart polynomial of the hive polytope for λ, μ and ν.

Conjectures

Conjecture **(Kirillov, King-Tollu-Toumazet)**

For all partitions λ, μ such that $K_{\lambda\mu} > 0$ there exists a polynomial $P_{\lambda\mu}(N)$ in N with nonnegative rational coefficients such that $P_{\lambda\mu}(0) = 1$ and $P_{\lambda\mu}(N) = K_{N\lambda\, N\mu}$ for all positive integers N.

Open problem

k	#(facets)	deg	$j=1$	$j=2$	$j=3$	$j=4$
3	6	1	1 (6)			
4	14	3	2 (8)	3 (6)		
5	30	6	3 (10)	5 (20)		
6	62	10	4 (12)	7 (30)	8 (20)	
7	126	15	5 (14)	9 (42)	11 (70)	
8	254	21	6 (16)	11 (56)	14 (112)	15 (70)
9	510	28	7 (18)	13 (72)	17 (168)	19 (252)

Open problem Determine what the other factors are on the boundary of the permutahedron.

Conclusion

- We have found vector partition functions expressing the Kostka numbers and LR coefficients as quasipolynomials over the cells of a complex of cones.
- We have found a combinatorial description for the domains of quasipolynomiality of the Kostka numbers.
- We have proved that the quasipolynomials are actually polynomials.
- Many of these polynomials exhibit interesting factorization patterns.