Geometric approaches to computing Kostka numbers and Littlewood-Richardson coefficients

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Outline

- Introduction
- Vector partition functions
- Weight multiplicities (Kostka numbers)
- The A_3 picture
- Littlewood-Richardson coefficients
- A q-analogue of the Kostant partition function

Introduction

Kostka numbers

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- The **Kostka number** $K_{\lambda\beta}$ is the number of semistandard Young tableaux of shape λ and content β.
- $K_{\lambda\beta}$ is also the multiplicity with which the weight β appears in the irreducible representation of $\mathrm{GL}_k\mathbb{C}$ (or $\mathrm{SL}_k(\mathbb{C})$) with highest weight λ.

Littlewood-Richardson coefficients

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Littlewood-Richardson coefficients

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- The LR coefficients express the multiplication rule:

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu \,.$$

- They also appear when writing skew Schur functions in terms of the Schur function basis:

$$s_{\nu/\lambda} = \sum_\mu c^\nu_{\lambda\mu} s_\mu \,.$$

- In the representation theory of $\mathrm{GL}_k\mathbb{C}$, the LR coefficient $c^{\nu}_{\lambda\mu}$ gives the multiplicity with which the irreducible representation V_ν of $\mathrm{GL}_k\mathbb{C}$ appears in the tensor product of the irreducible representations V_λ and V_μ .

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- LR coefficients also appear in algebraic geometry: Schubert classes form a linear basis of the cohomology ring of the Grassmannian, and the LR coefficients again express the multiplication rule.

Roots and weights for A_{k-1}

- Roots

$$\Delta = \{e_i - e_j \ : \ 1 \leq i \neq j \leq k\}\,.$$

- Positive roots

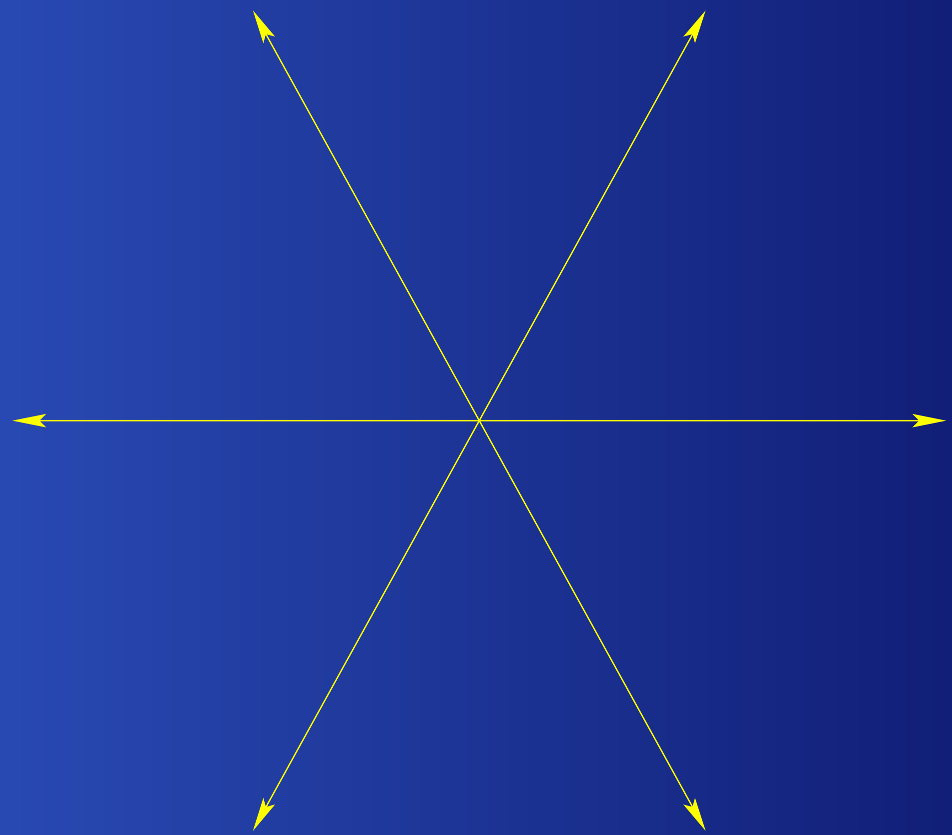
$$\Delta_+ = \{e_i - e_j \ : \ 1 \leq i < j \leq k\}\,.$$

- Simple roots

$$\Pi = \{\underbrace{e_i - e_{i+1}}_{\alpha_i} \ : \ 1 \leq i \leq k-1\}\,.$$

- Fundamental weights : $\omega_1, \ldots, \omega_{k-1}$ defined by $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$.

Example: A_2 ($\mathfrak{sl}_3\mathbb{C}$)



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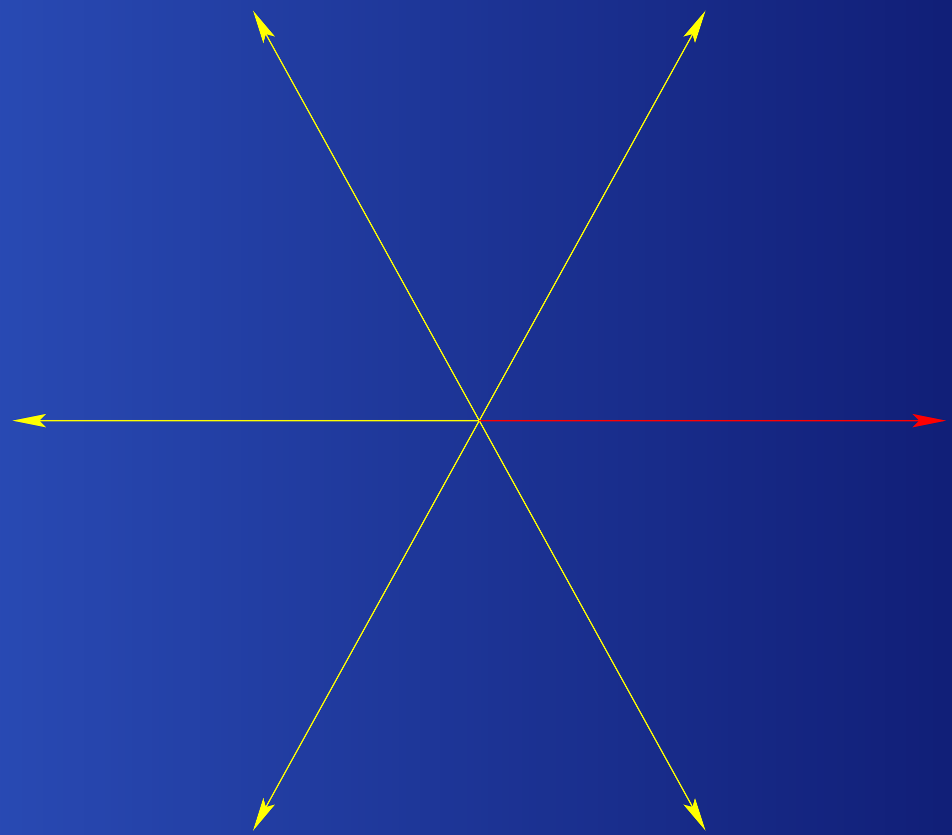

$$\alpha_1 = e_1 - e_2$$

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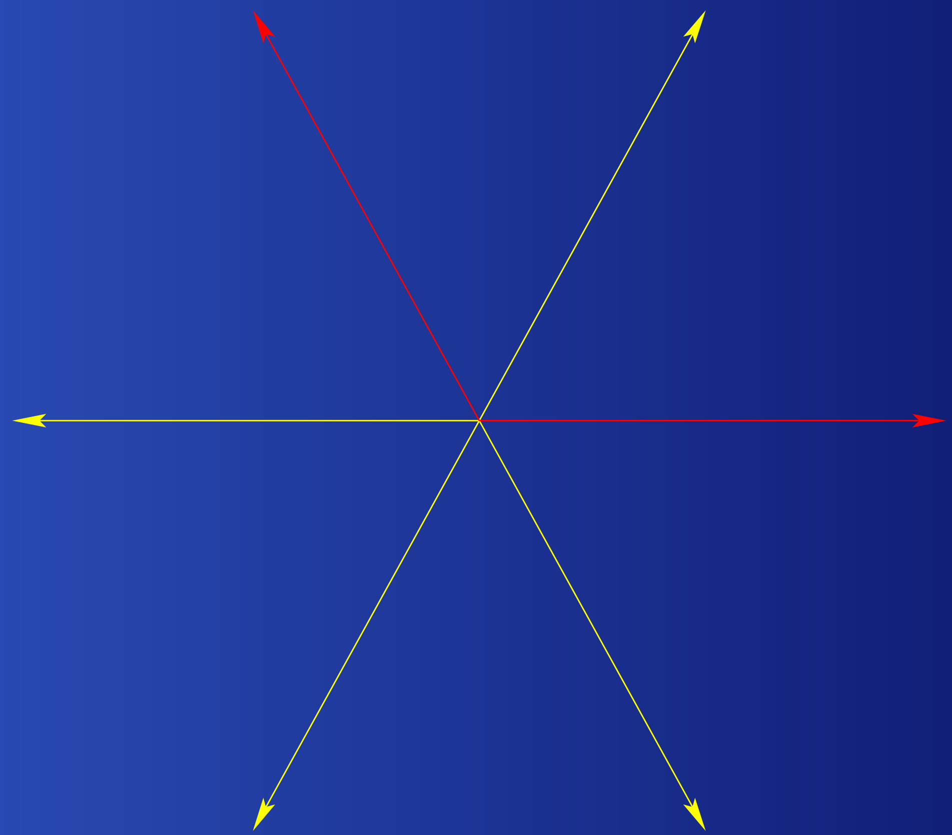
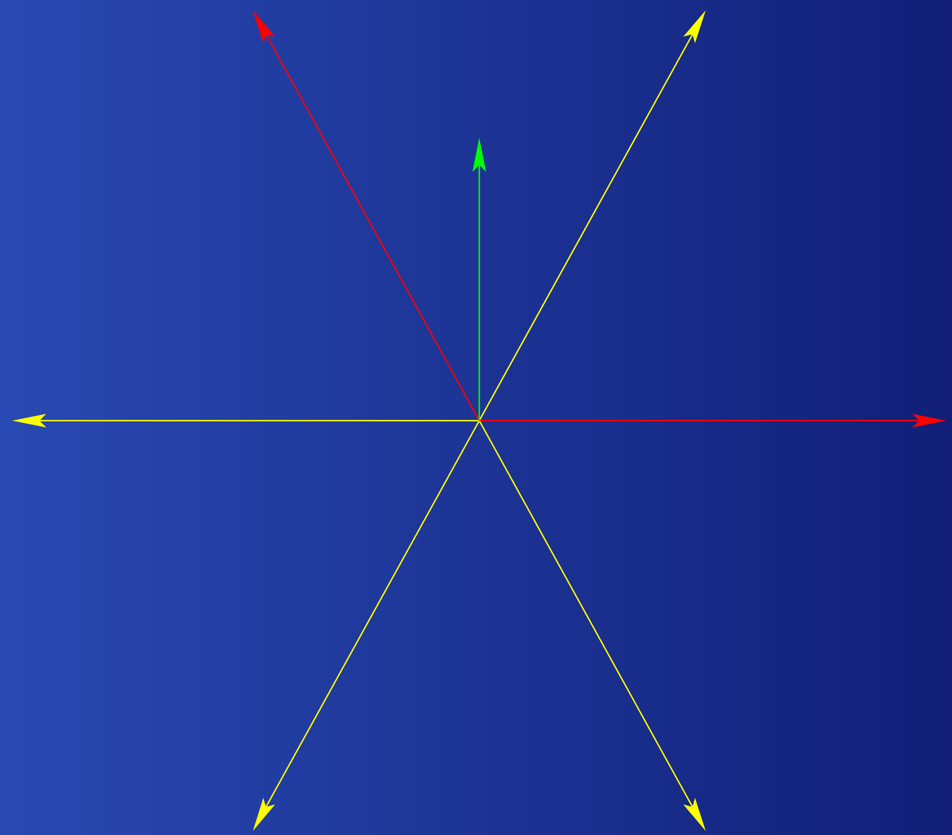


$$\alpha_2 = e_2 - e_3$$

Example: A_2 ($\mathfrak{sl}_3\mathbb{C}$)



$$\omega_1 = e_1 - \frac{1}{3}(1\ \ 1\ \ 1)$$

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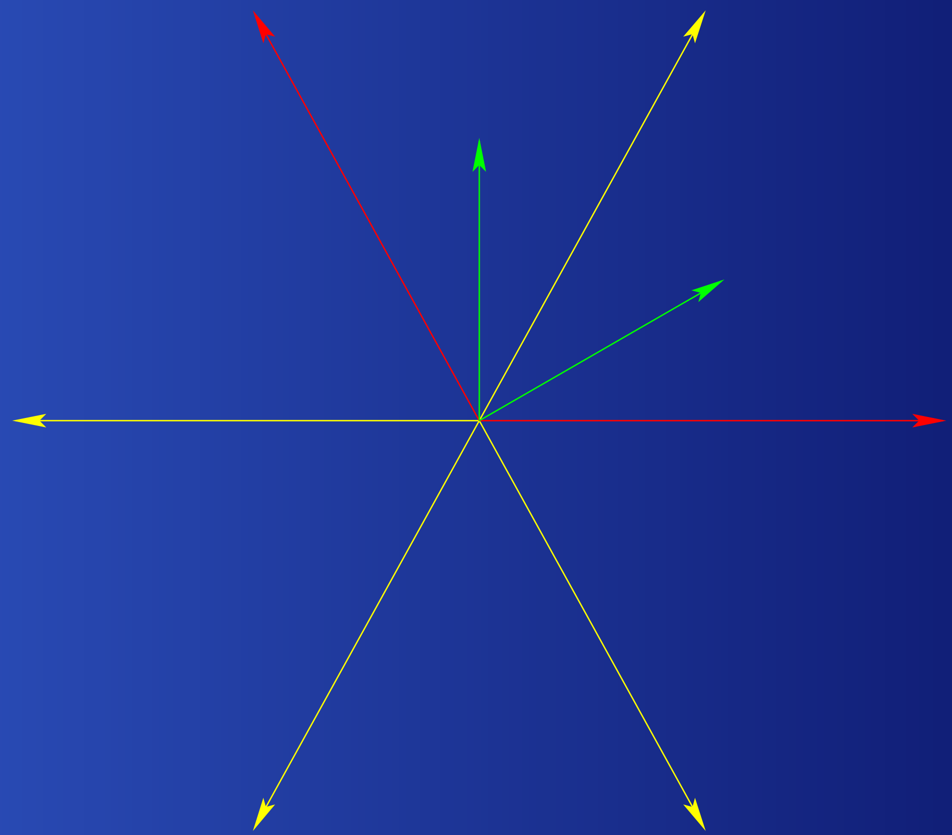

$$\omega_2 = e_1 + e_2 - \frac{2}{3}(1 \;\; 1 \;\; 1)$$

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- For $\mathfrak{sl}_k\mathbb{C}$, the reflection with respect to the hyperplane with normal vector $e_i - e_j$ exchanges coordinates i and j .
- The Weyl group of $\mathfrak{sl}_k\mathbb{C}$ is therefore the symmetric group $\mathfrak{S}_k$ acting on $\{e_1, \ldots, e_k\}$.

Irreducible representations of $\mathfrak{sl}_k\mathbb{C}$

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- The irreducible polynomial representations of $\mathfrak{gl}_k\mathbb{C}$ are indexed by partitions with at most k parts, i.e. $\lambda \in \mathbb{N}^k$ with $\lambda_1 \geq \cdots \geq \lambda_k$.

Schur functions

$$s_\lambda(x_1, \ldots, x_k) = \sum_{T \in \mathrm{SSYT}(\lambda; k)} \mathbf{x}^T .$$

$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$
$x_1^2 x_2$	$x_1^2 x_3$	$x_1 x_2^2$	$x_1 x_2 x_3$	$x_1 x_2 x_3$	$x_1 x_3^2$	$x_2^2 x_3$	$x_2 x_3^2$

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$$\begin{aligned} s_{\ydiagram{2,1}}(x_1,x_2,x_3) &= x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 \\ &\quad + 2\,x_1x_2x_3\,. \end{aligned}$$

Characters

- From the definition of the Schur functions, we have that

$$s_\lambda = \sum_\beta K_{\lambda\beta}\,\mathbf{x}^\beta\,,$$

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where $K_{\lambda\beta}$ is the number of ways of filling a SSYT of shape λ with integers distributed according to composition β.

- The character of the irreducible representation of $\mathfrak{gl}_k\mathbb{C}$ with highest weight λ is the Schur function $s_\lambda(x_1, \ldots, x_k)$.

Weight space decomposition

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- This translates combinatorially as the symmetric function identity

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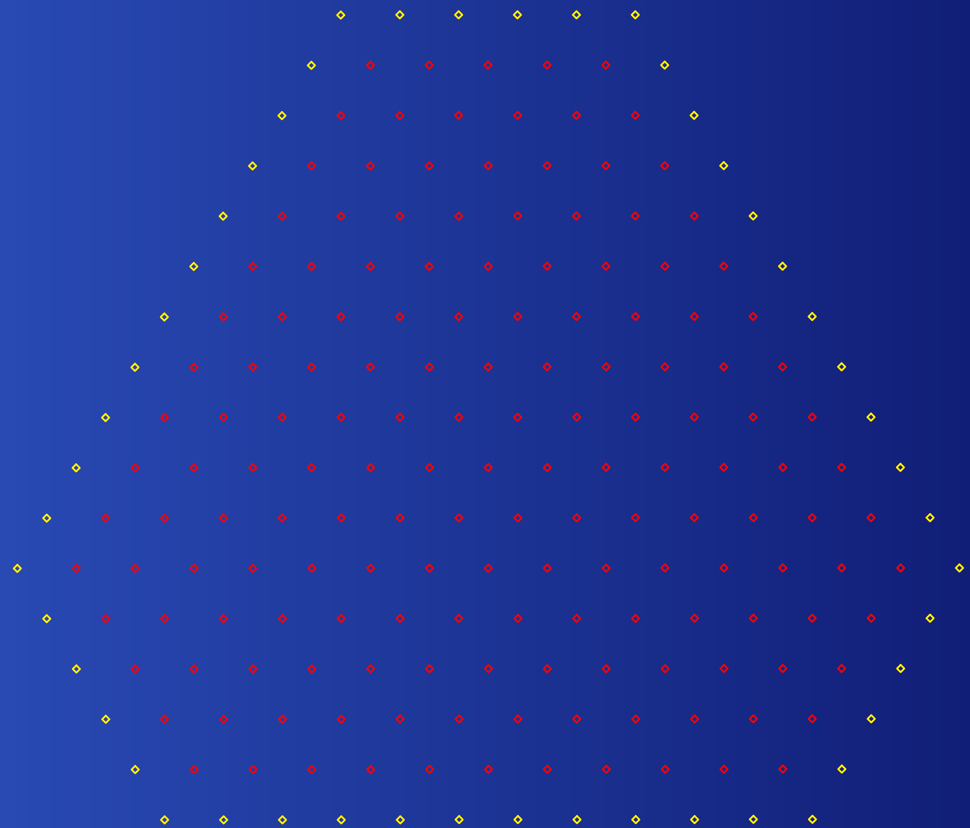
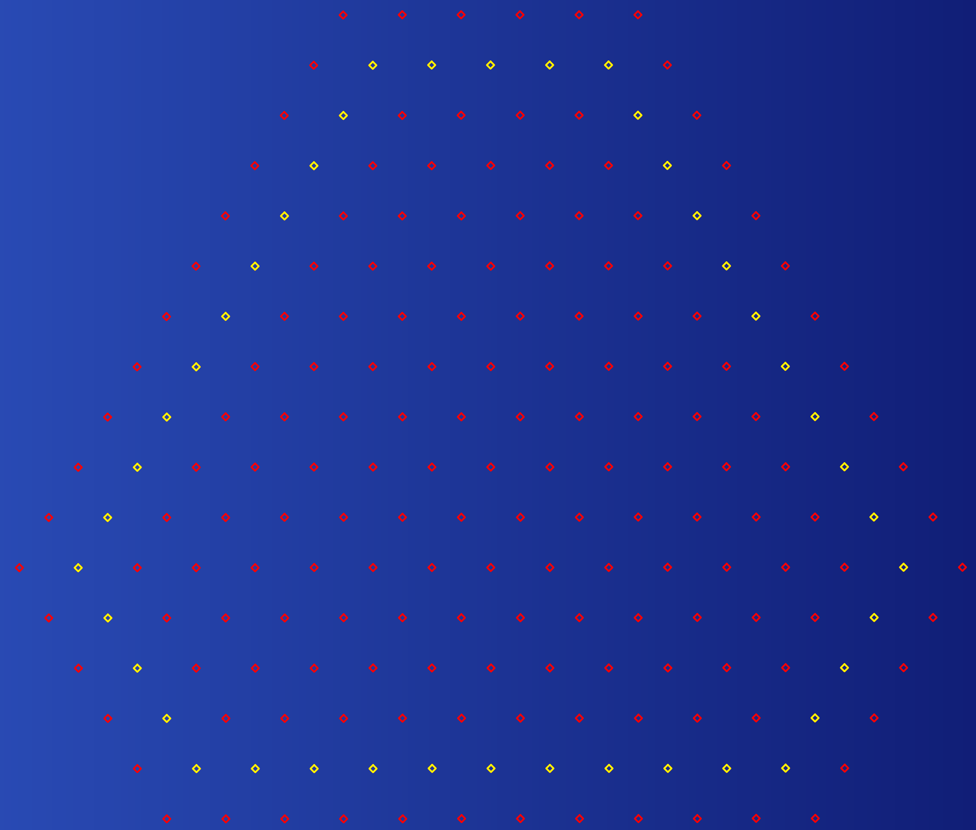
- The β's for which $(V_\lambda)_\beta \neq 0$ are exactly those for which
  - $\beta \in \Lambda_W$,
  - $\lambda - \beta \in \Lambda_R$,
  - $\beta \in \mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$.
- We call the convex hull $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$ the permutahedron.

$\lambda = (9, -2, -7)$



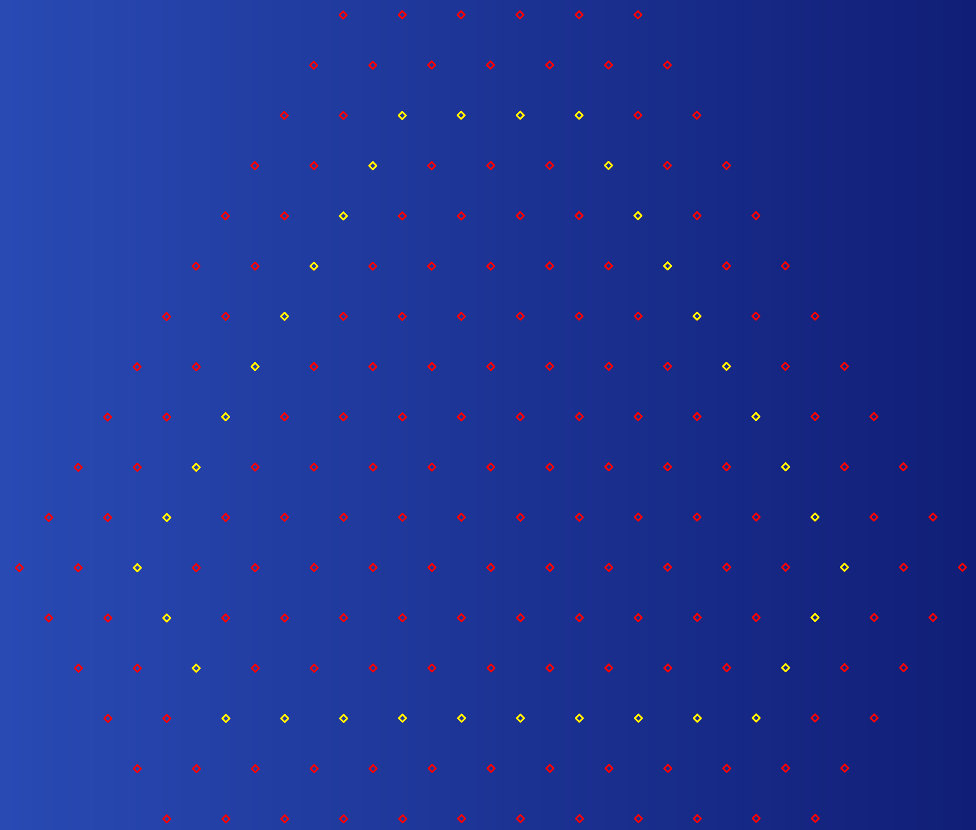
$$m_\lambda(\beta) = 1$$

$\lambda = (9,\ -2,\ -7)$



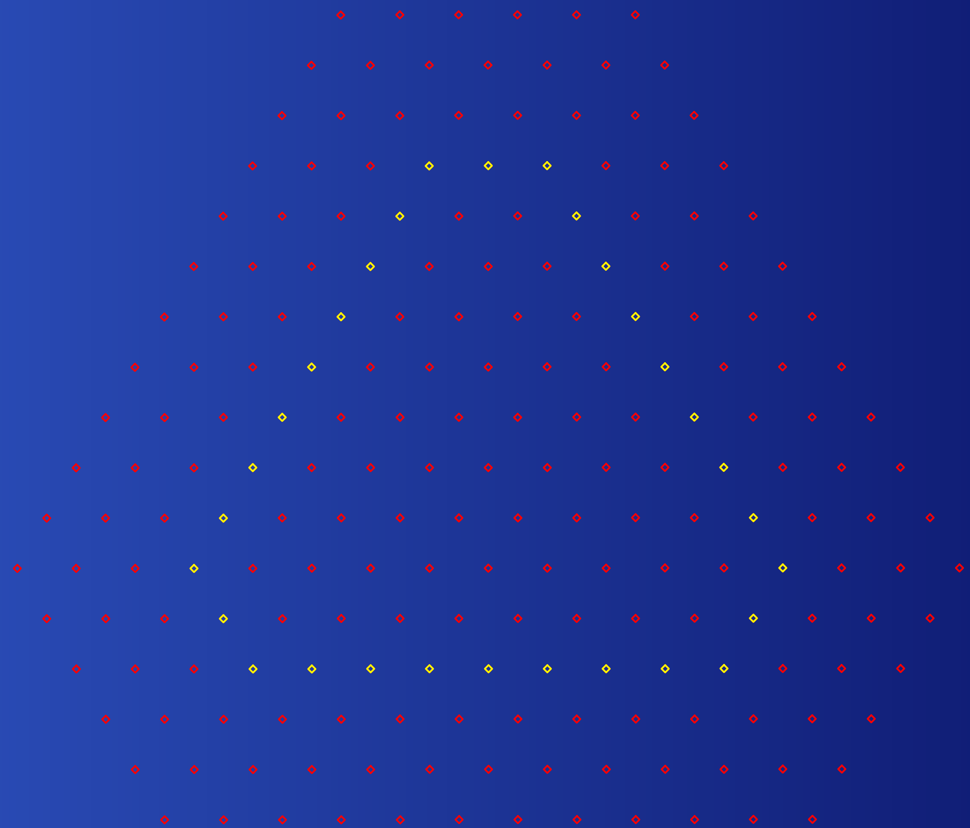
$$m_\lambda(\beta) = 2$$

$\lambda = (9, -2, -7)$



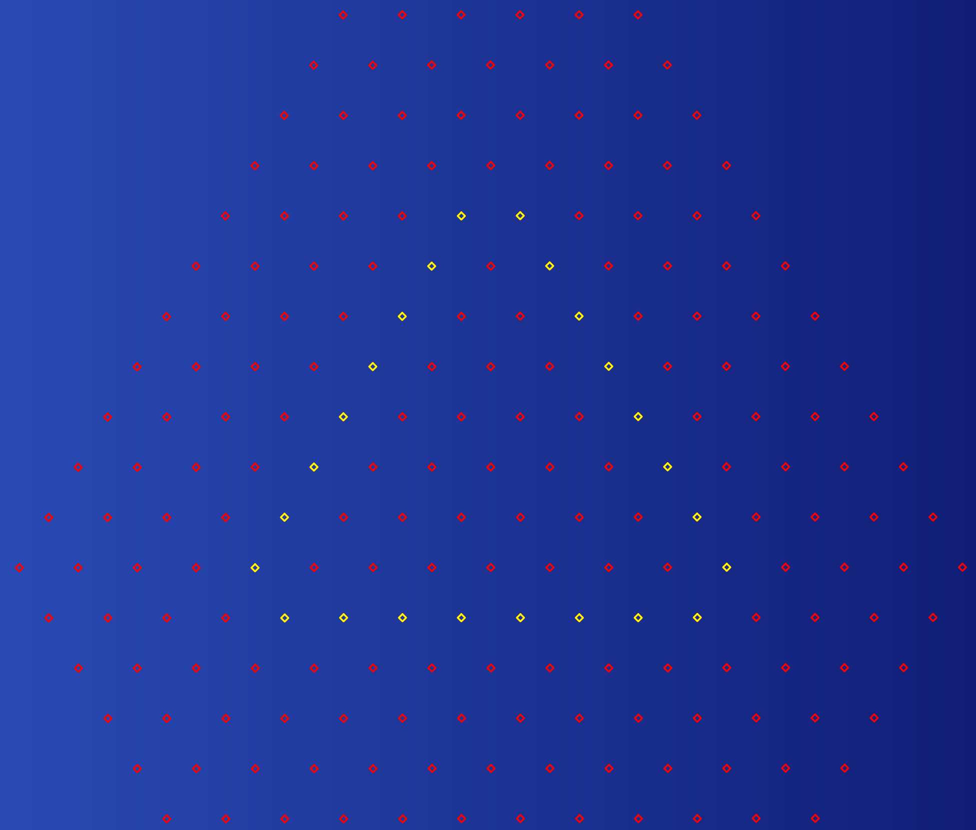
$$m_\lambda(\beta) = 3$$

$\lambda = (9, -2, -7)$



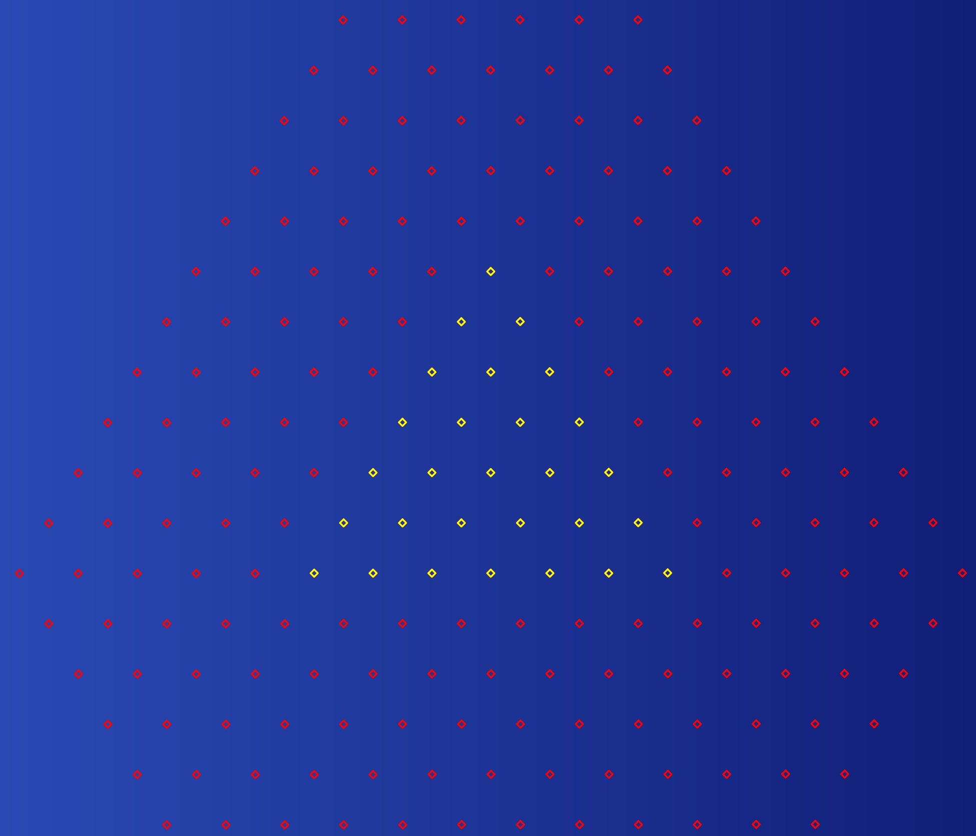
$$m_\lambda(\beta) = 4$$

$\lambda = (9, -2, -7)$



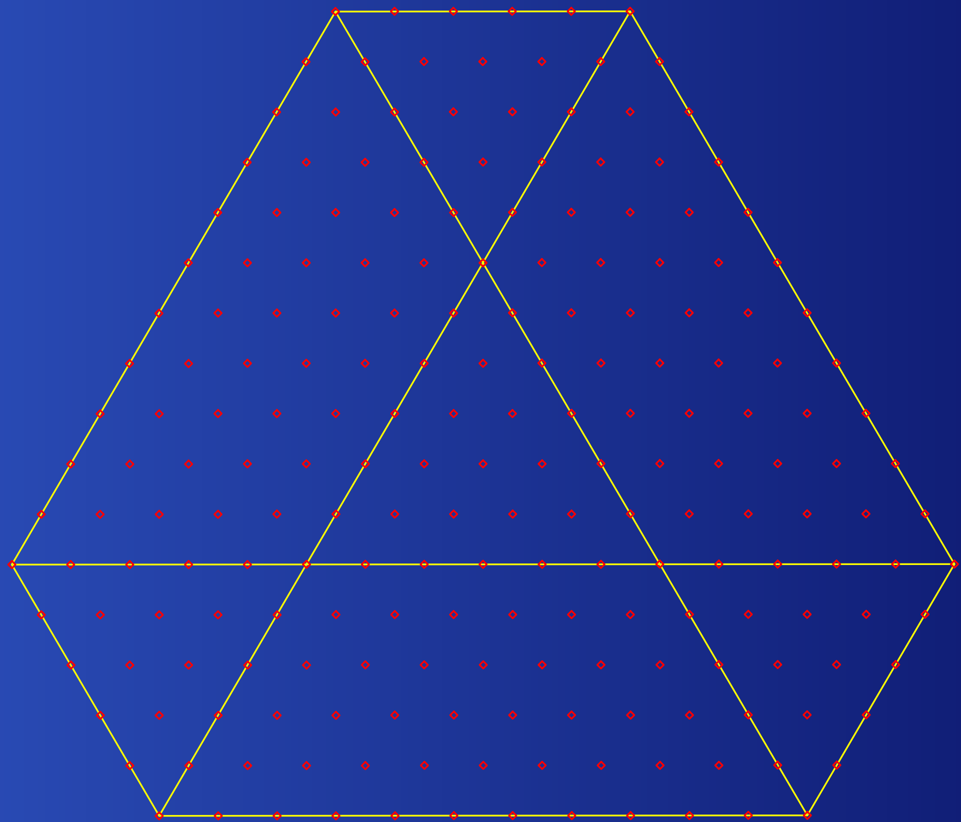
$$m_\lambda(\beta) = 5$$

$\lambda = (9, -2, -7)$



$$m_\lambda(\beta) = 6$$

$\lambda = (9, \ -2, \ -7)$



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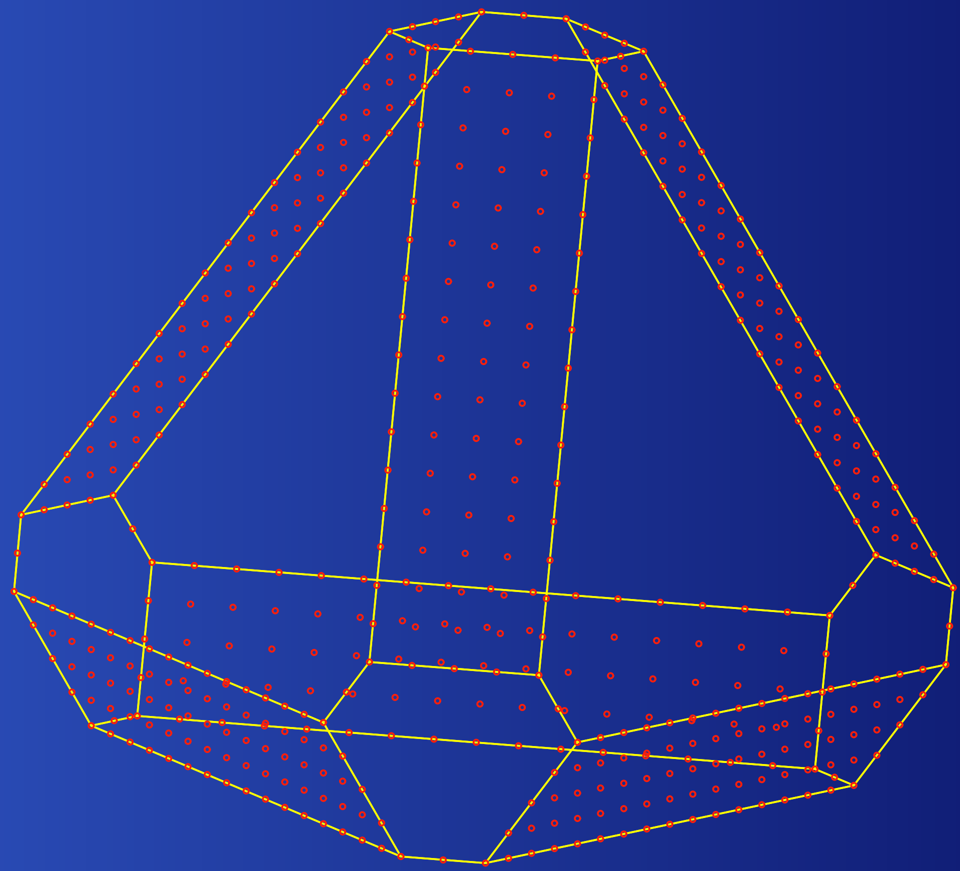
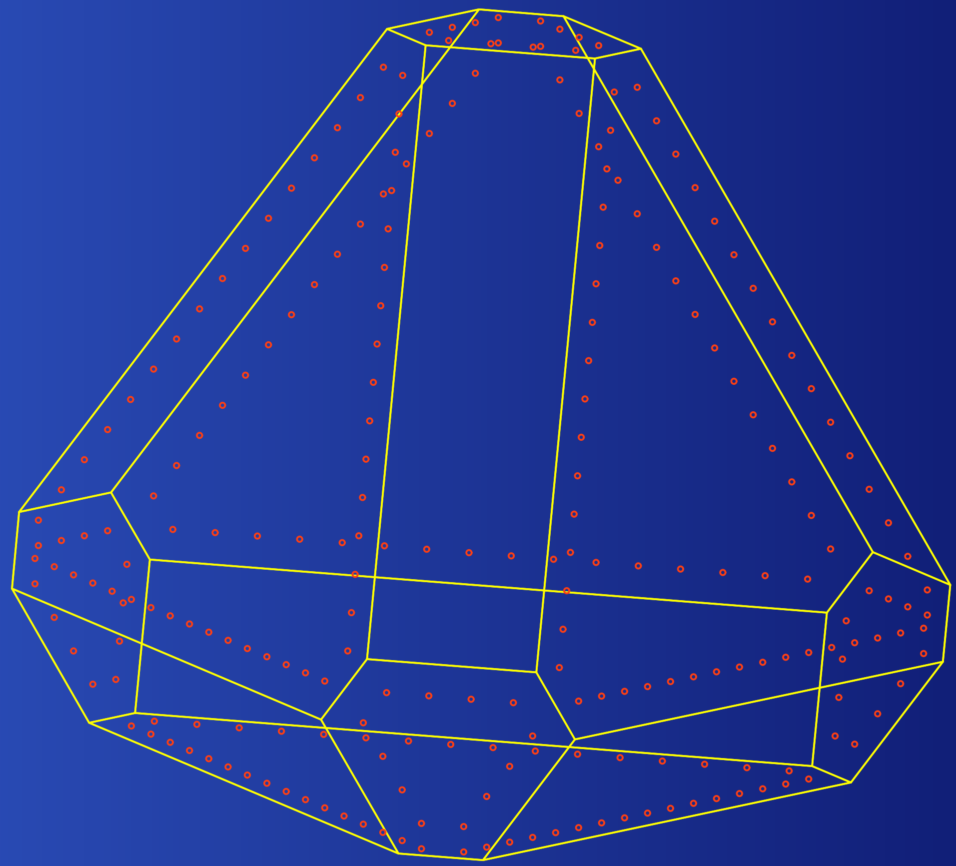
- There is a finite family of polynomials in general for A_n

$\lambda = (14, -2, -4, -8)$



$m_\lambda(\beta) = 1$

$\lambda = (14, -2, -4, -8)$



$$m_\lambda(\beta) = 2$$

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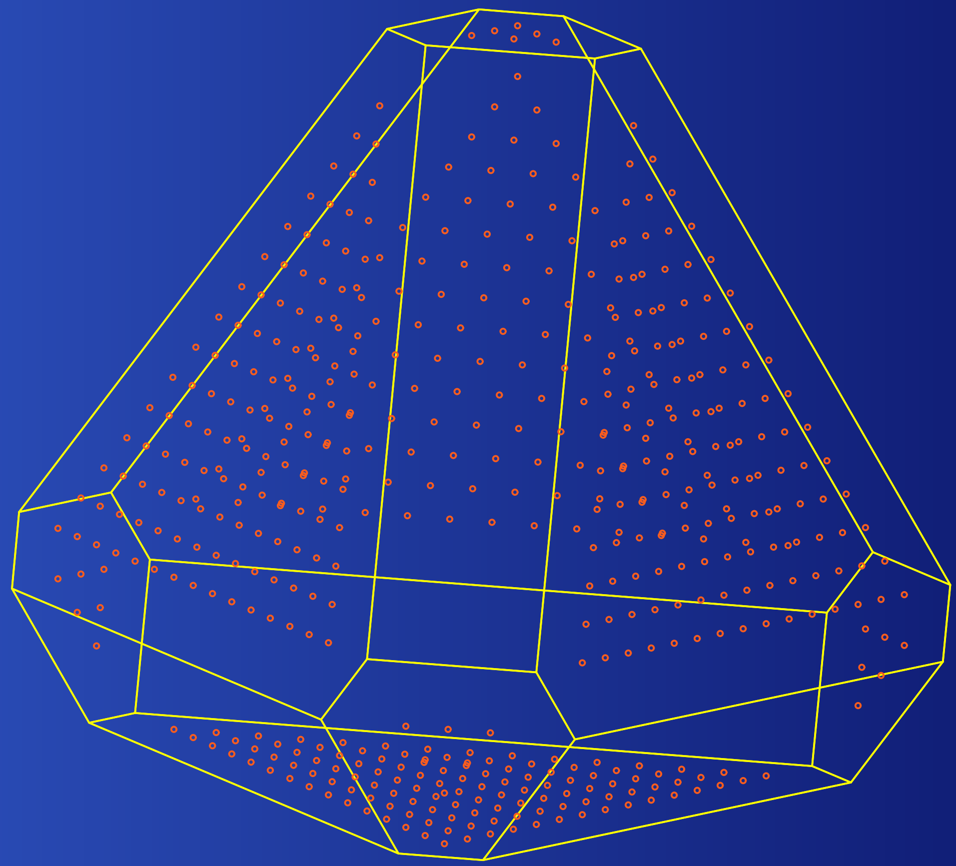

$$m_\lambda(\beta) = 3$$

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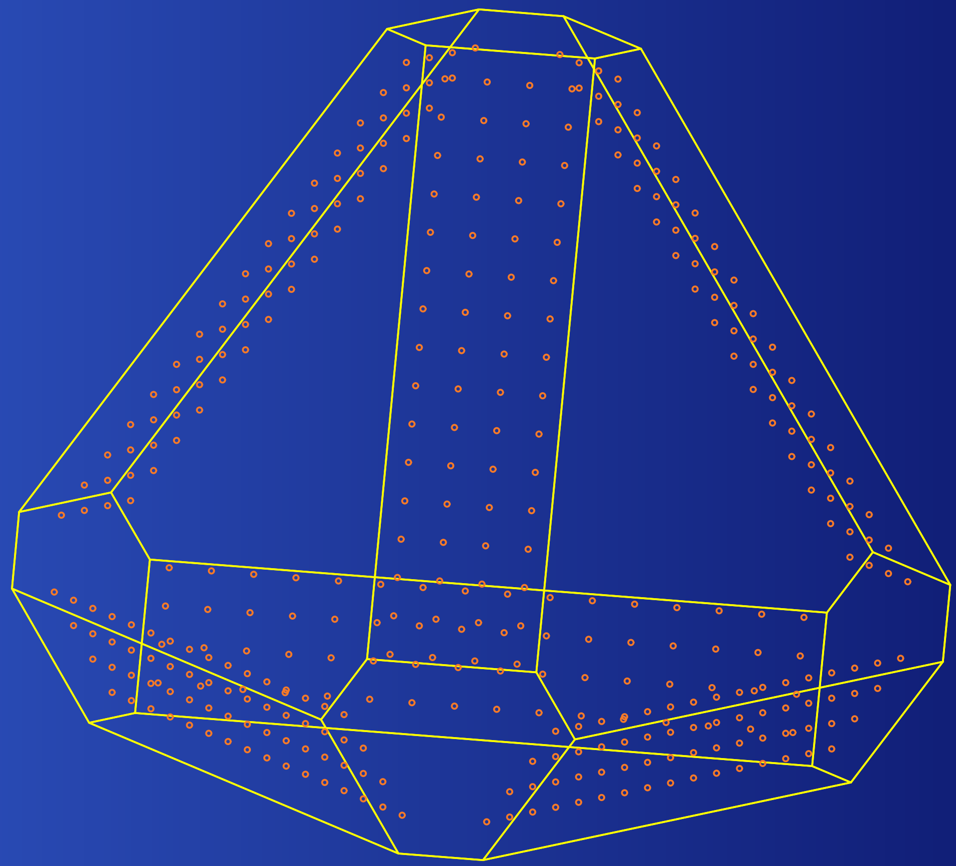

$$m_\lambda(\beta) = 4$$

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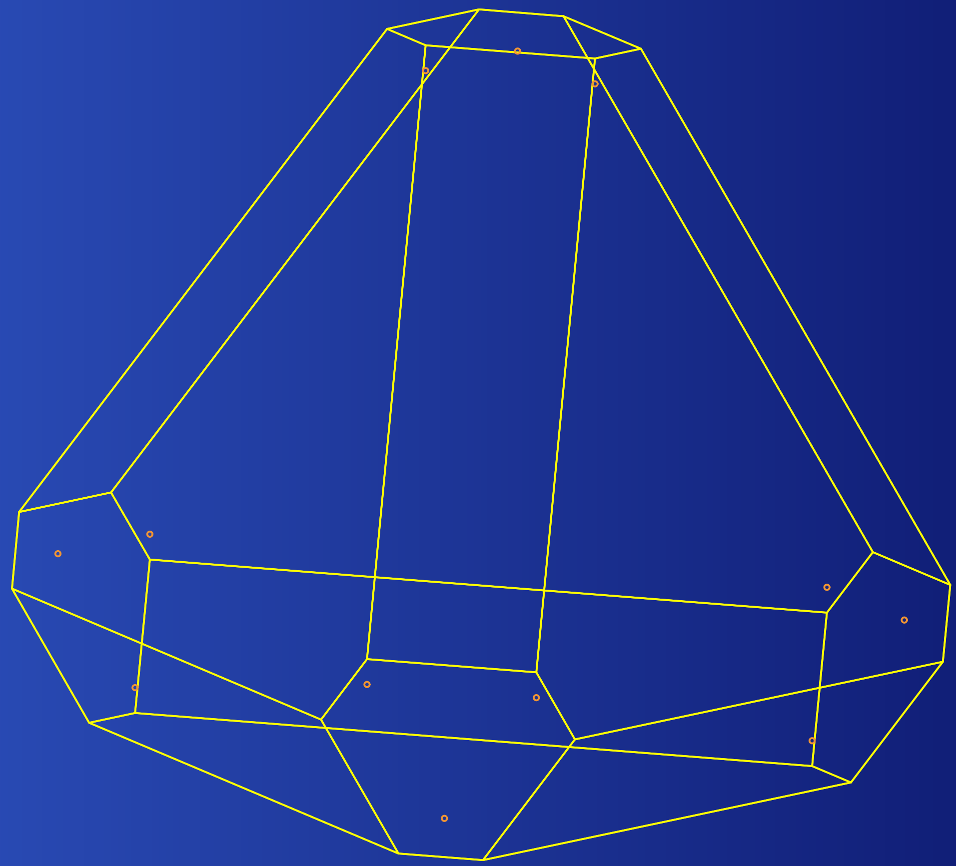

$$m_\lambda(\beta) = 5$$

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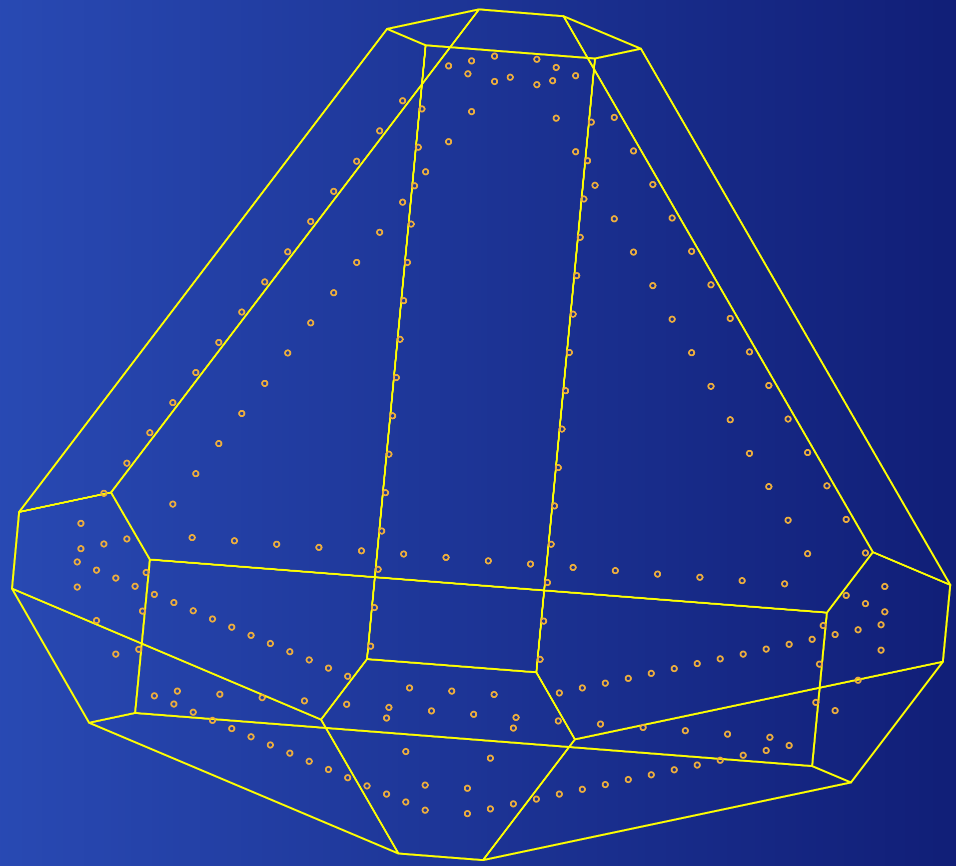
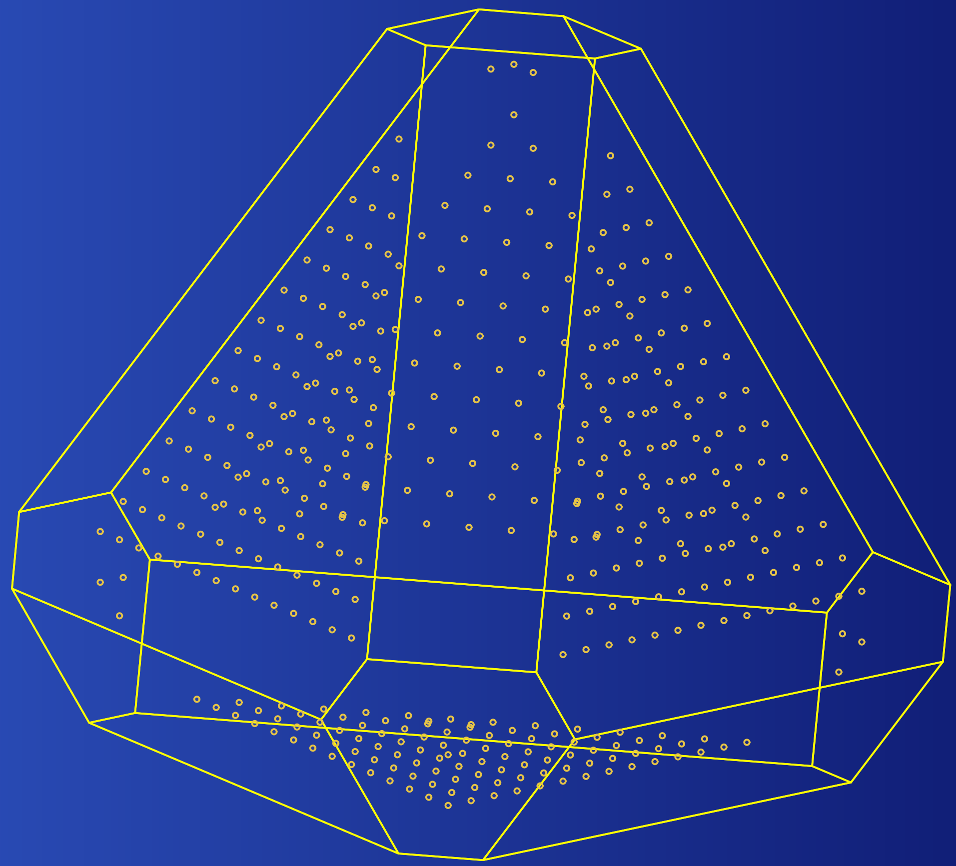


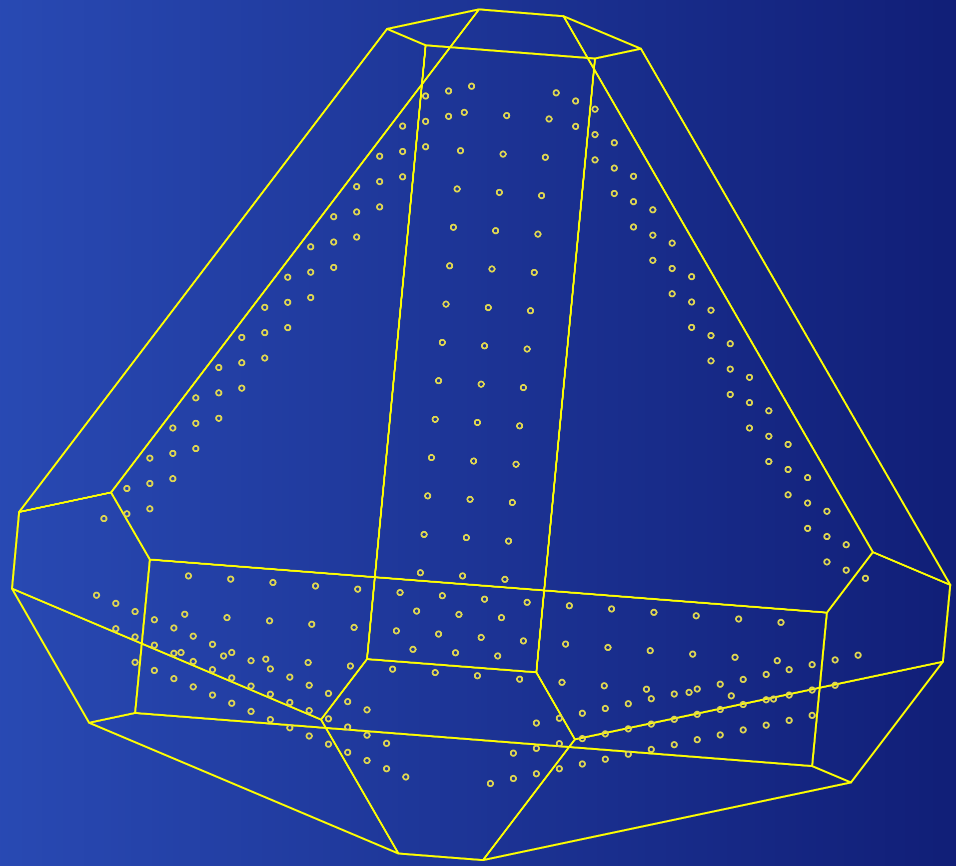
$$m_\lambda(\beta) = 7$$

$\lambda = (14, -2, -4, -8)$



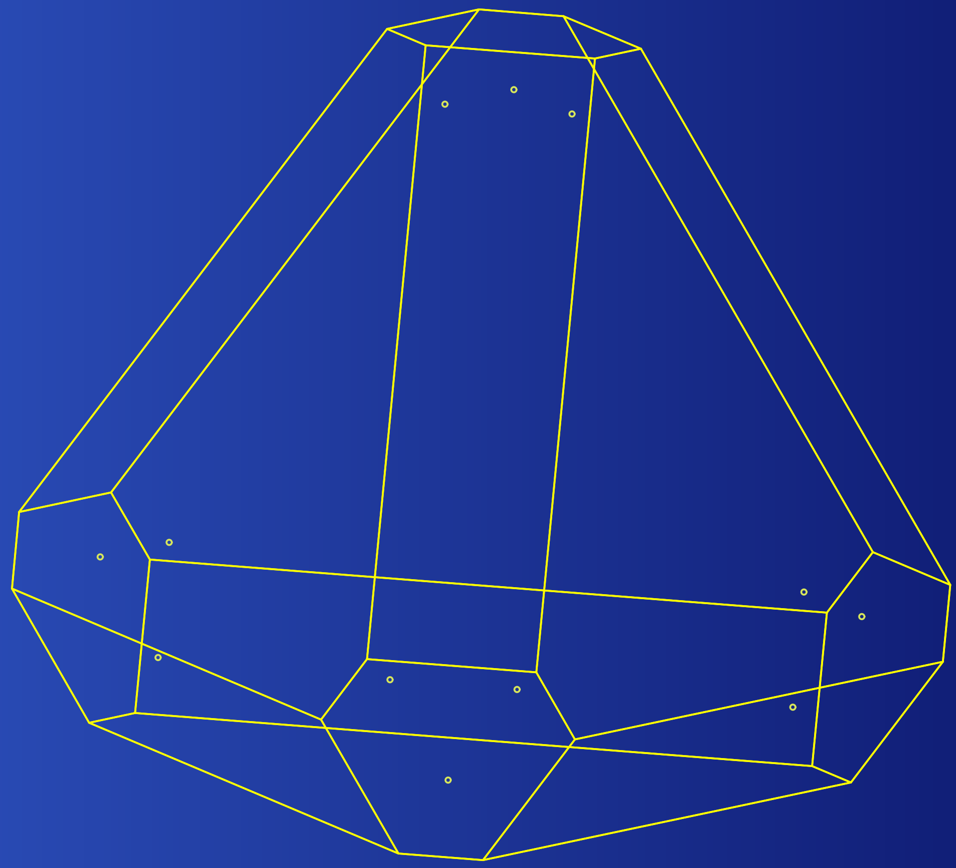
$$m_\lambda(\beta) = 9$$

$\lambda = (14, -2, -4, -8)$



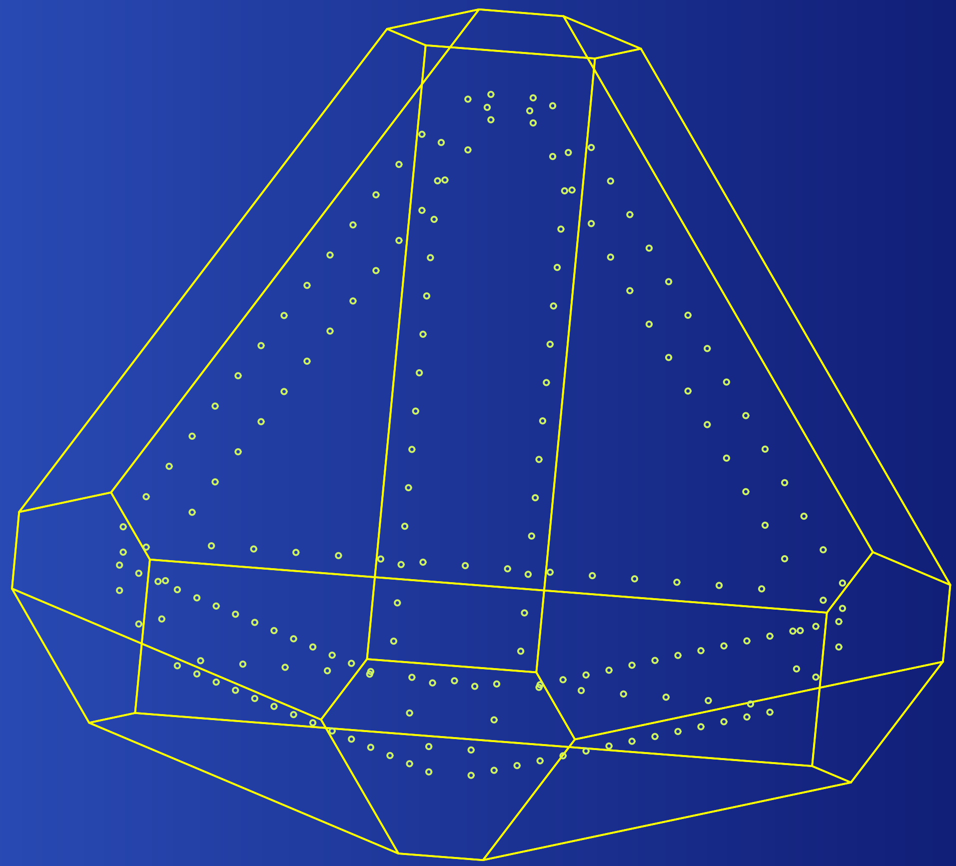
$$m_\lambda(\beta) = 10$$

$\lambda = (14, -2, -4, -8)$



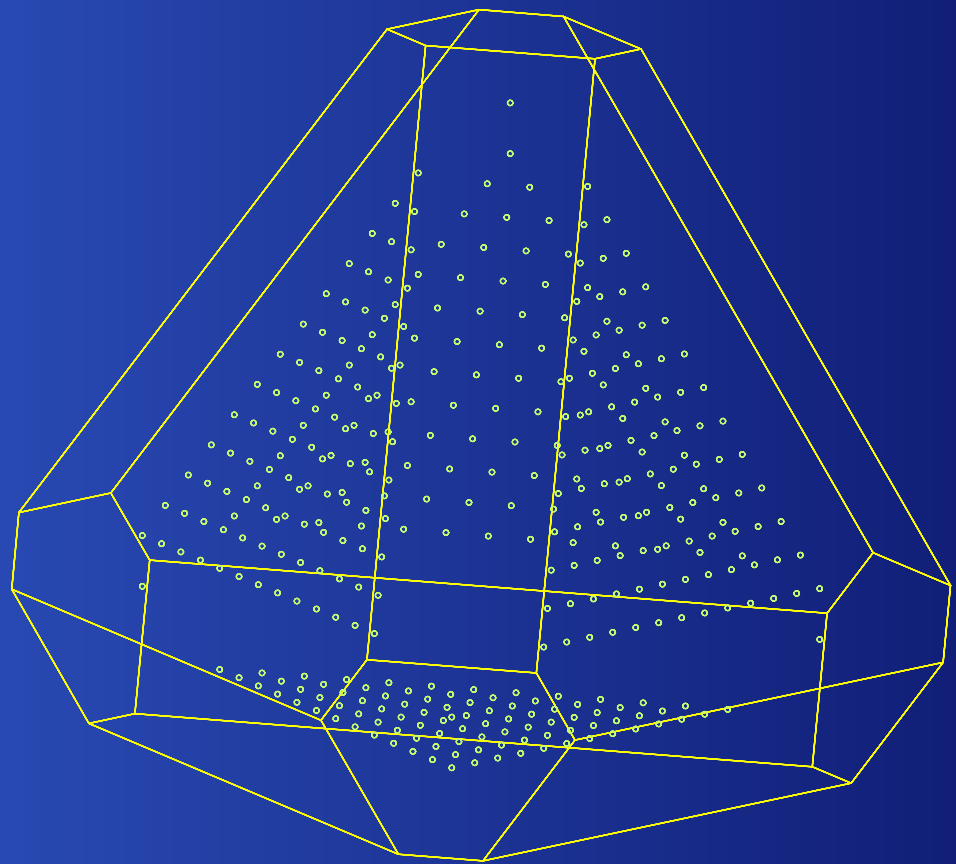
$$m_\lambda(\beta) = 12$$

$\lambda = (14, -2, -4, -8)$



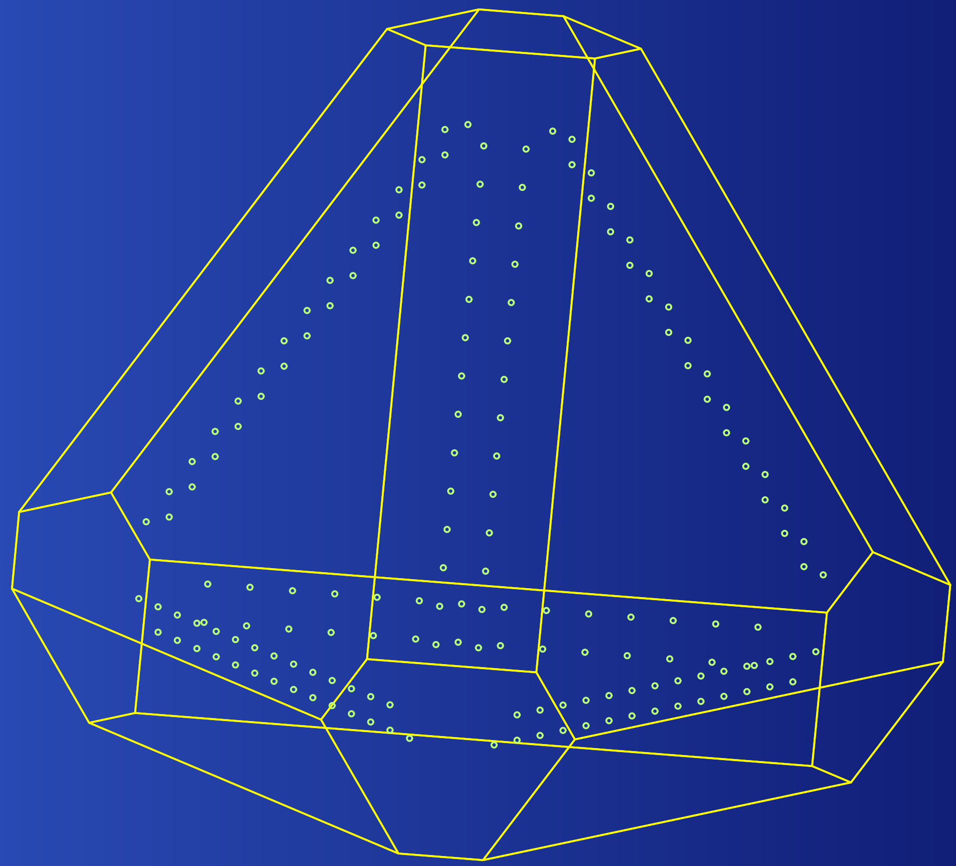
$$m_\lambda(\beta) = 15$$

$\lambda = (14, -2, -4, -8)$



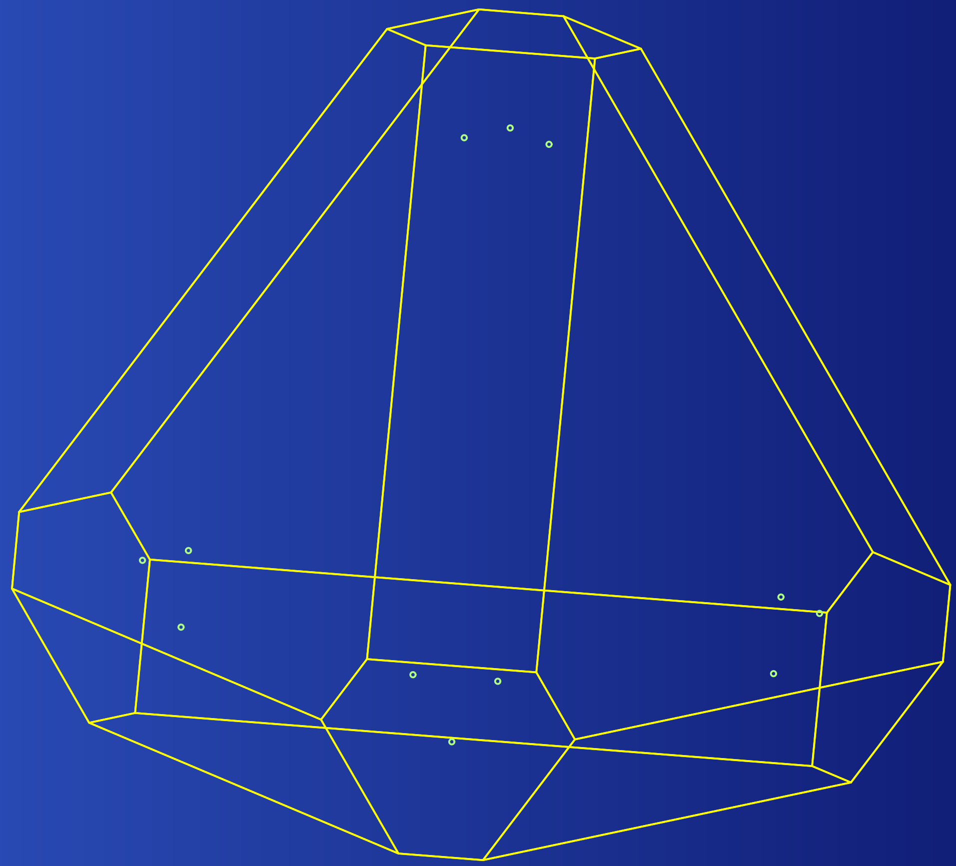
$$m_\lambda(\beta) = 18$$

$\lambda = (14, -2, -4, -8)$



$$m_\lambda(\beta) = 19$$

$\lambda = (14, -2, -4, -8)$



$$m_\lambda(\beta) = 22$$

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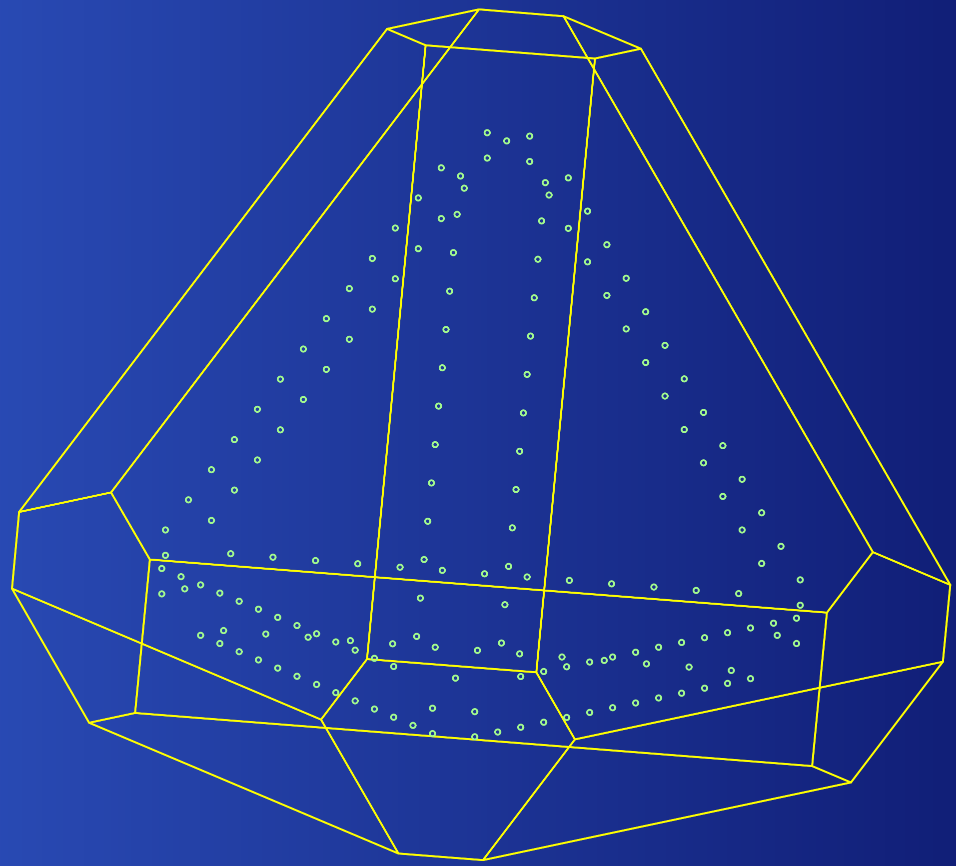
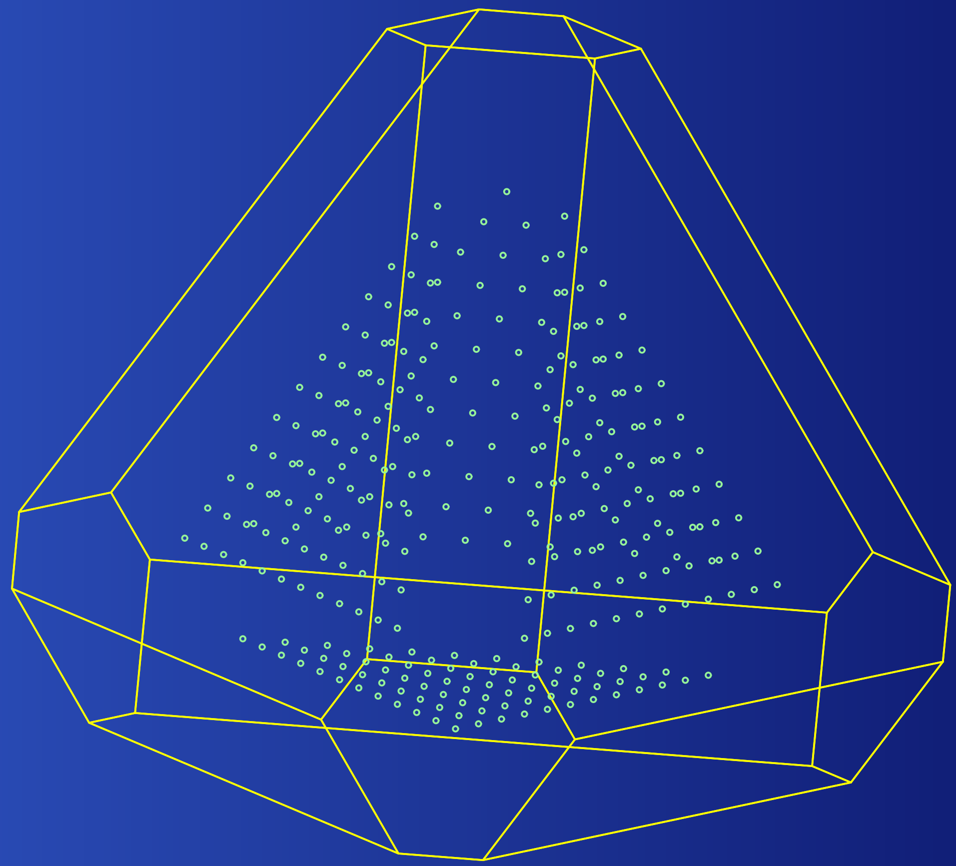


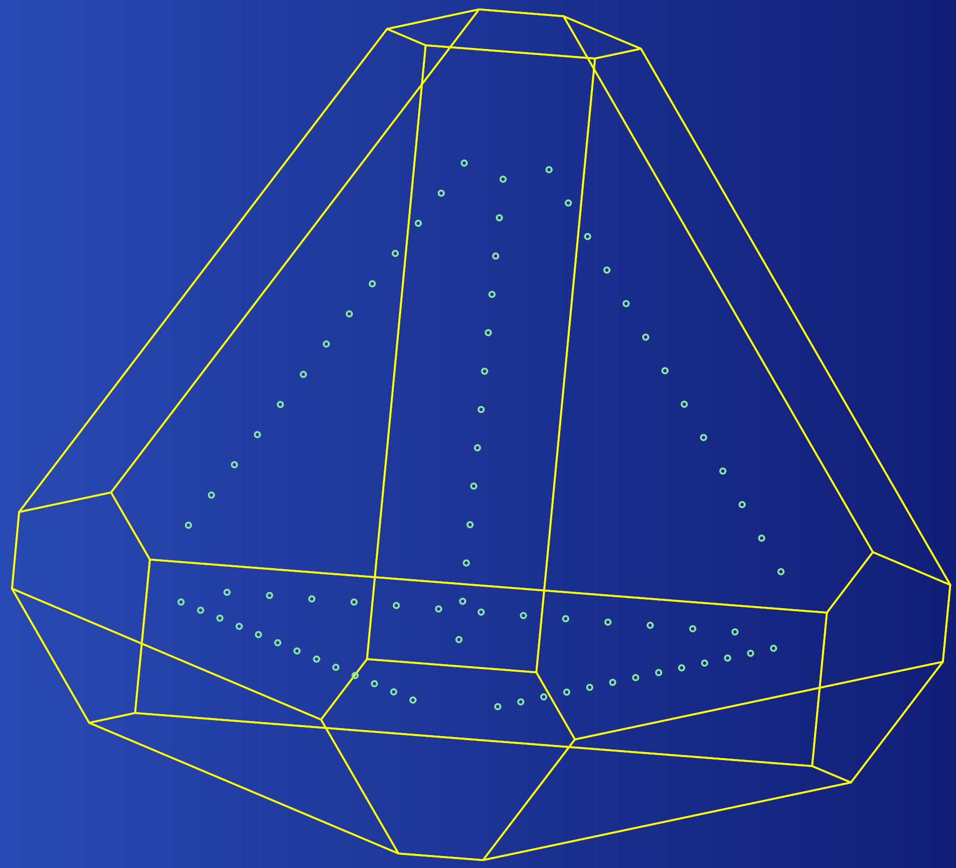
$$m_\lambda(\beta) = 26$$

$\lambda = (14, -2, -4, -8)$



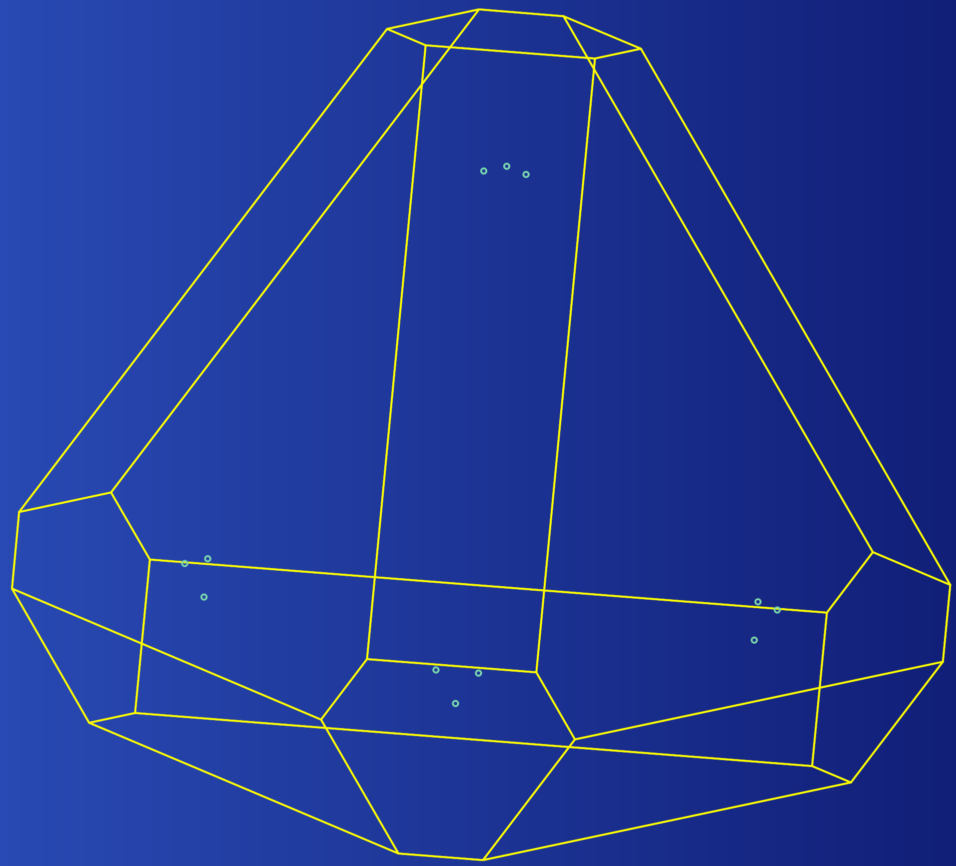
$$m_\lambda(\beta) = 30$$

$\lambda = (14, -2, -4, -8)$



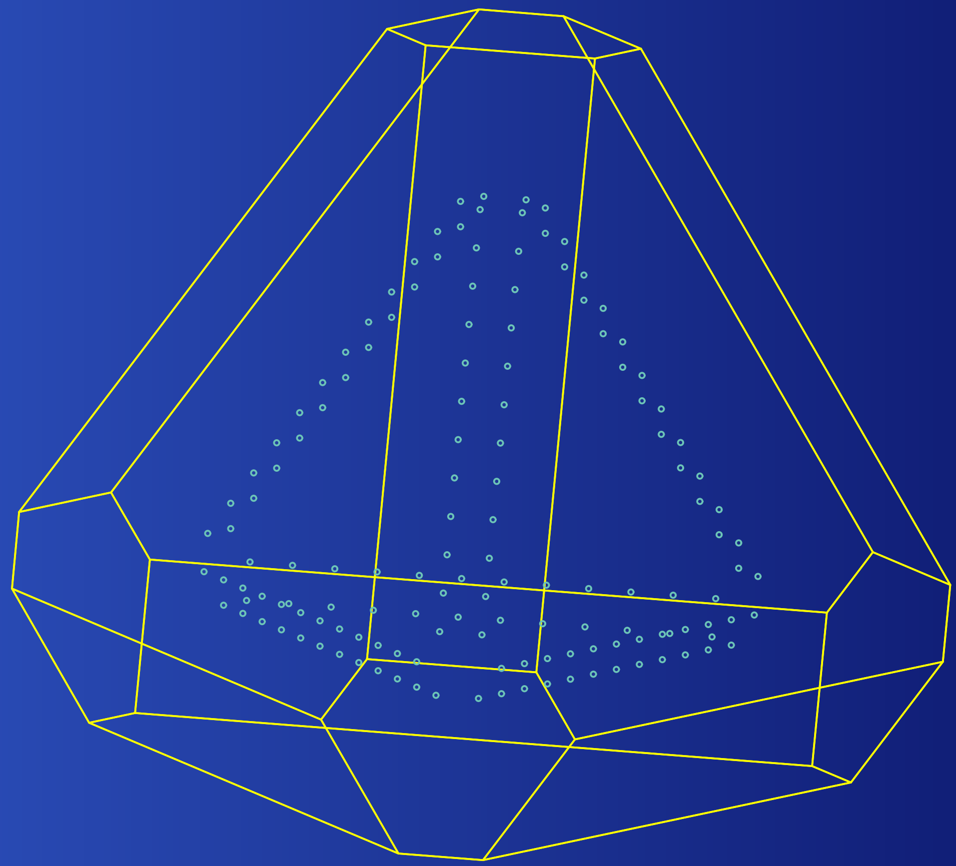
$$m_\lambda(\beta) = 31$$

$\lambda = (14, -2, -4, -8)$



$$m_\lambda(\beta) = 35$$

$\lambda = (14, -2, -4, -8)$



$$m_\lambda(\beta) = 40$$

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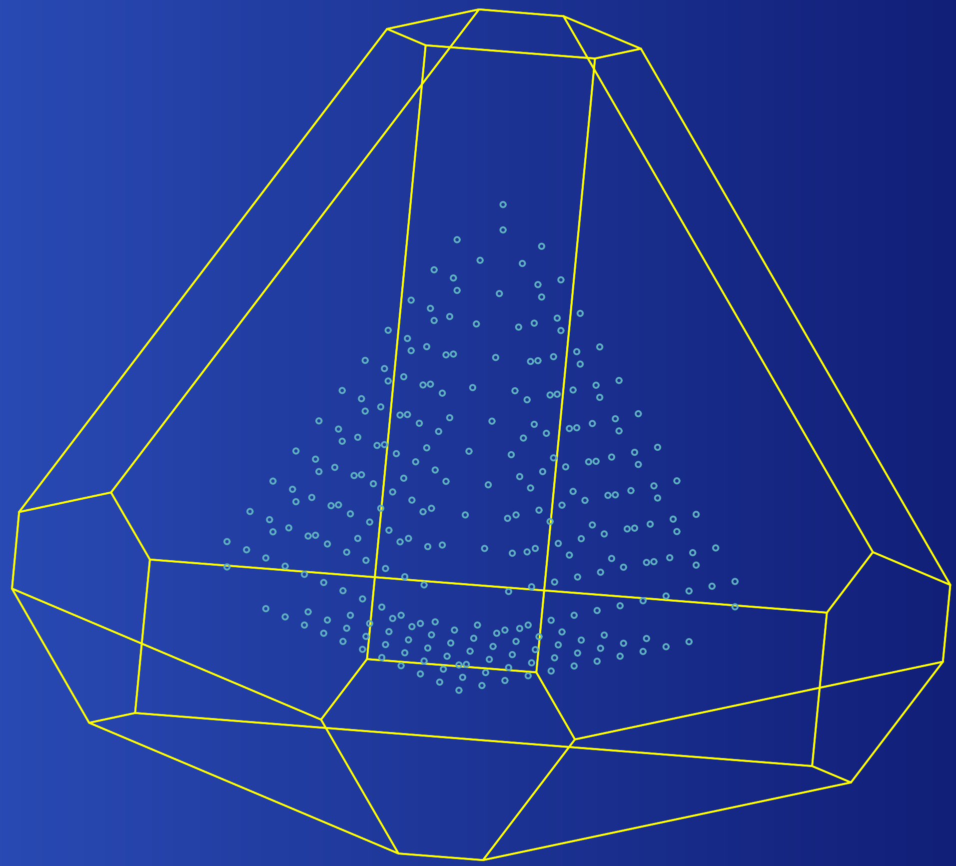

$$m_\lambda(\beta) = 45$$

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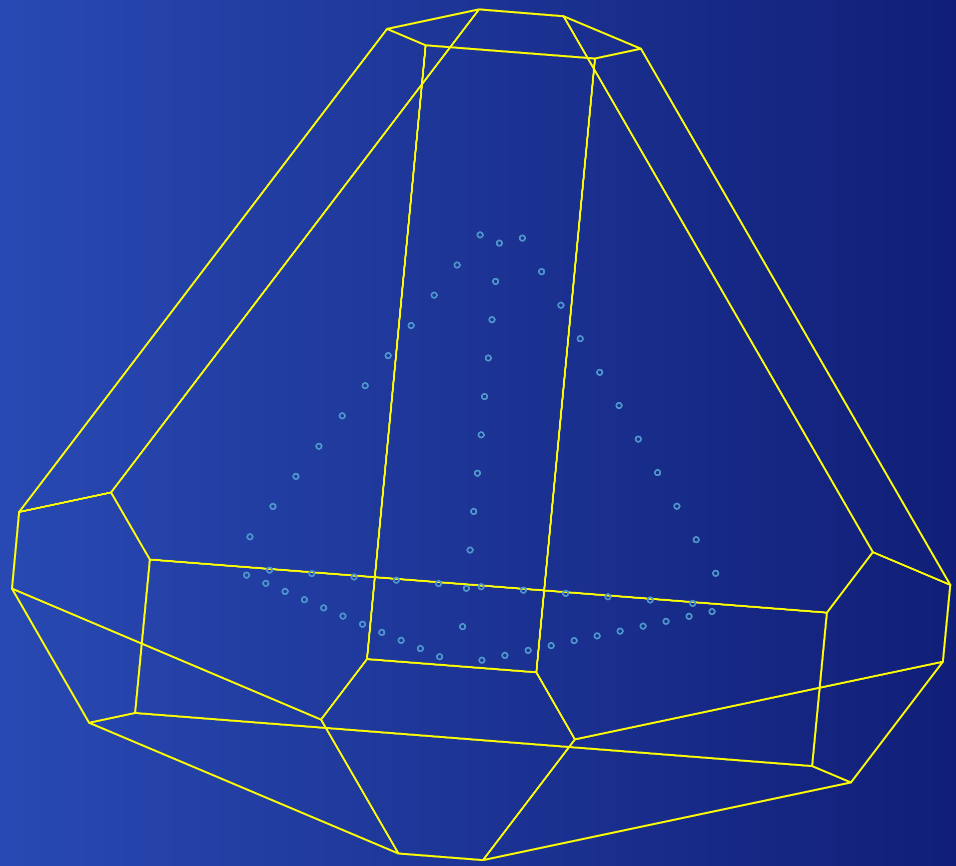

$$m_\lambda(\beta) = 50$$

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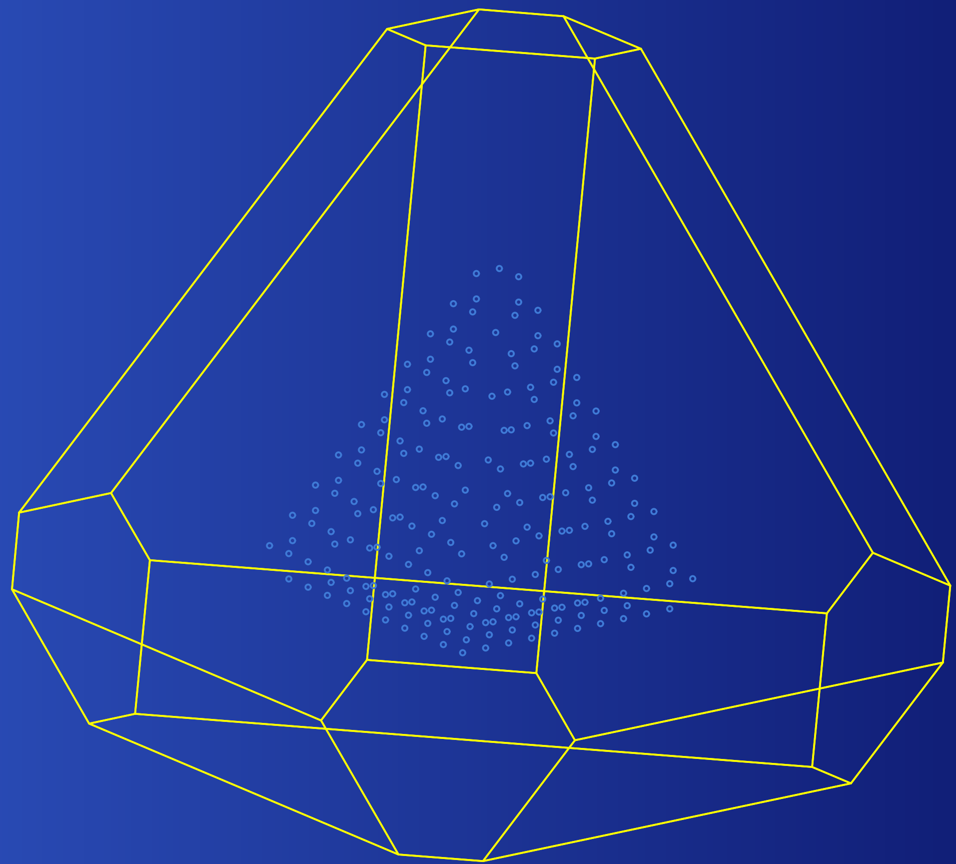

$$m_\lambda(\beta) = 55$$

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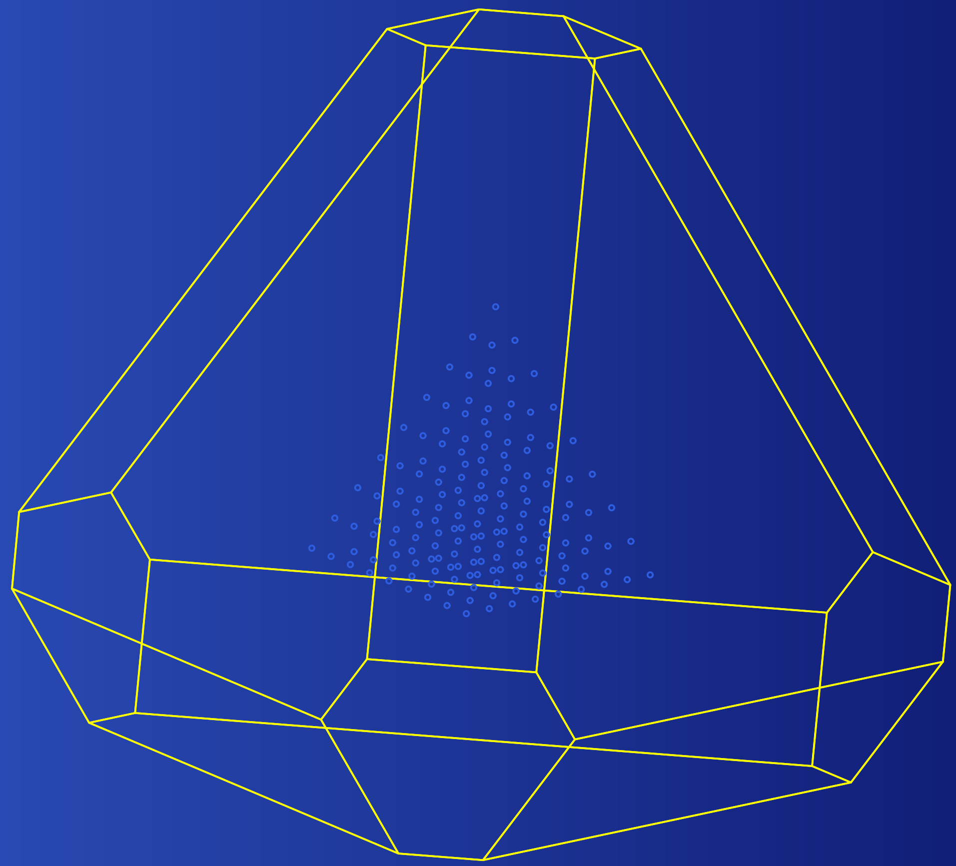
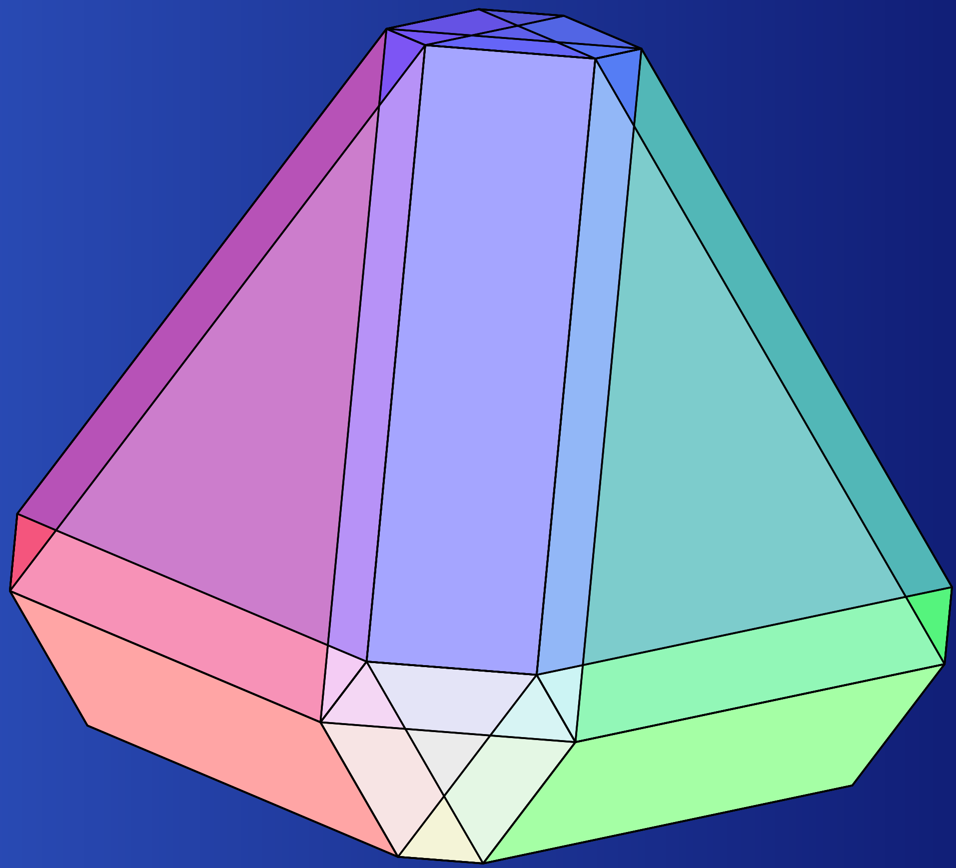


$$m_\lambda(\beta) = 60$$

$\lambda = (14, -2, -4, -8)$



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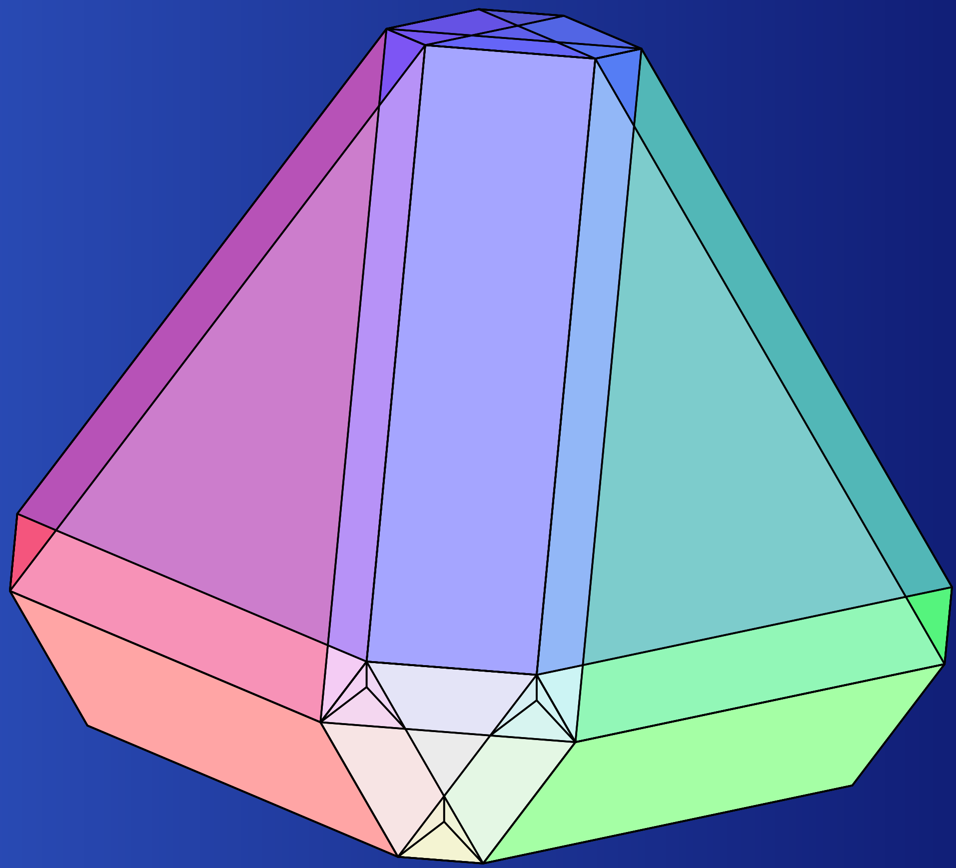

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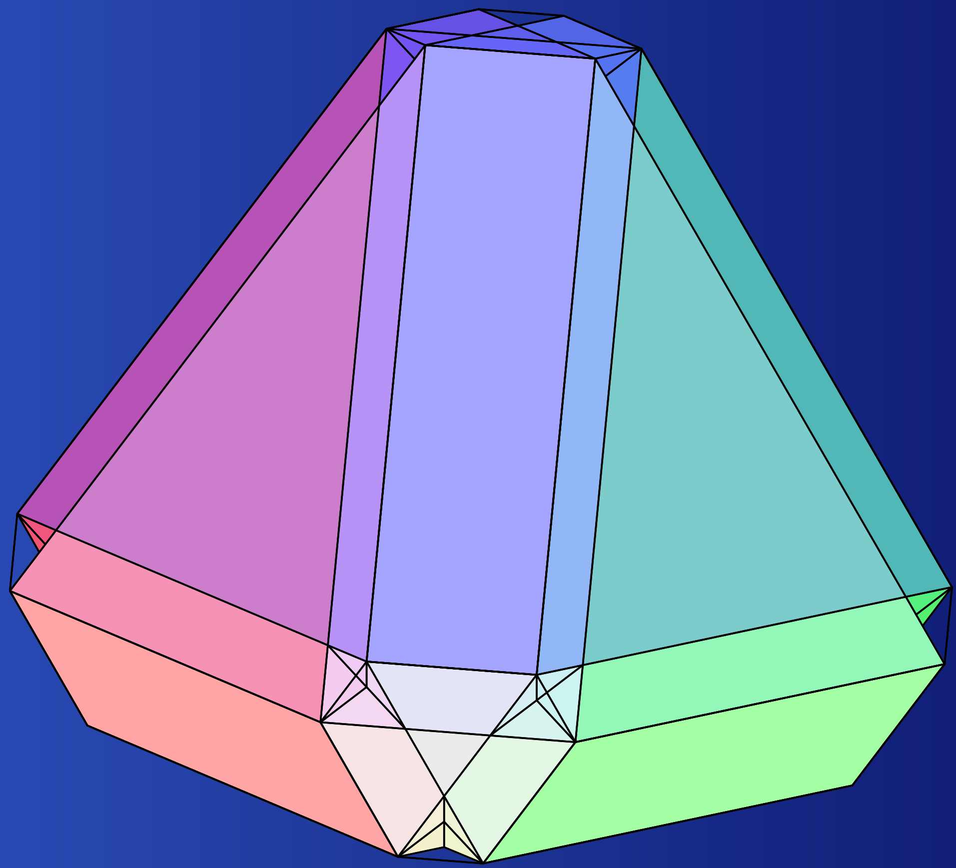

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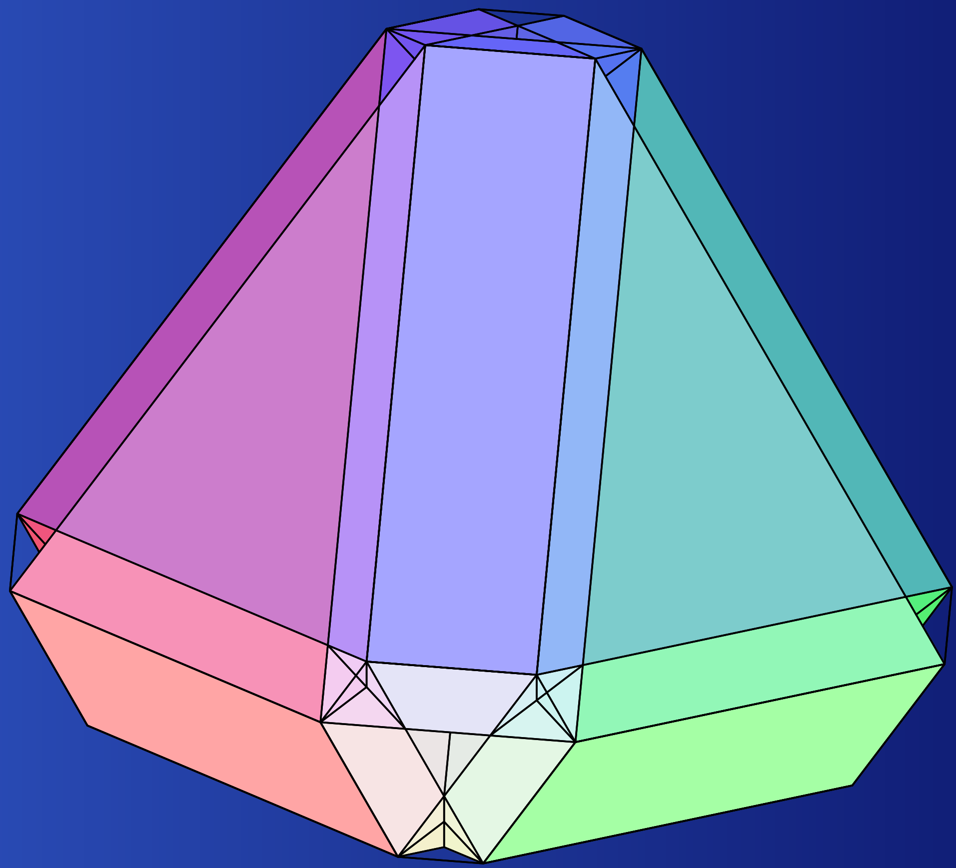

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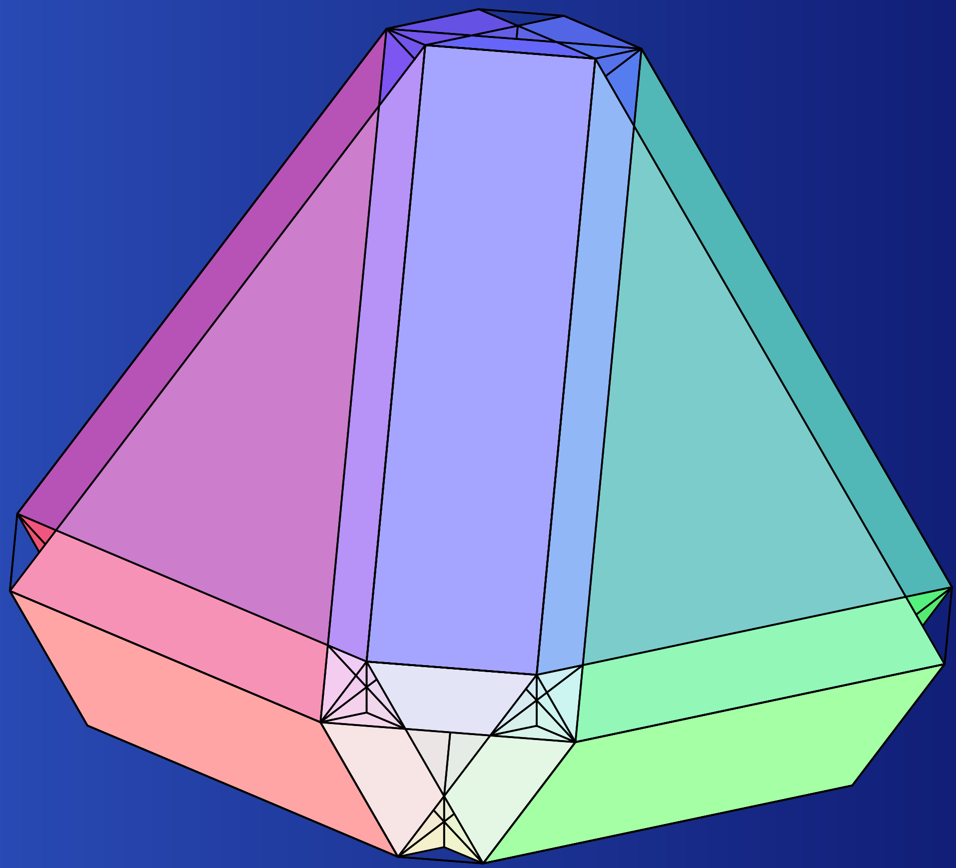

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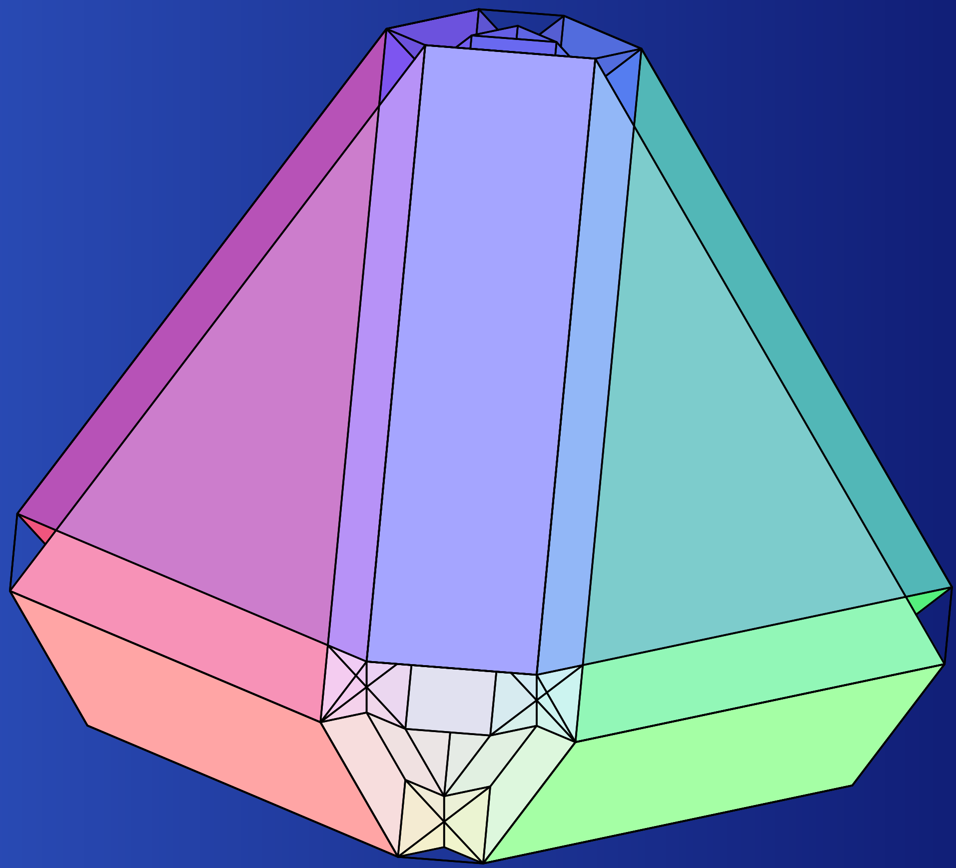

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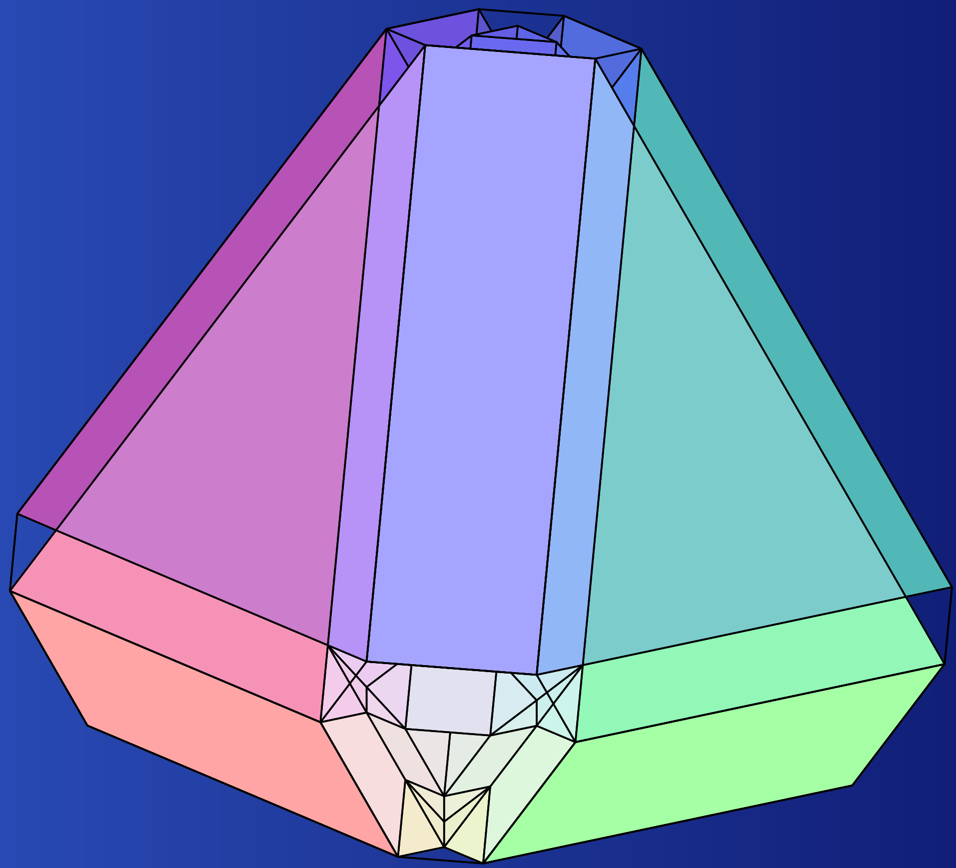

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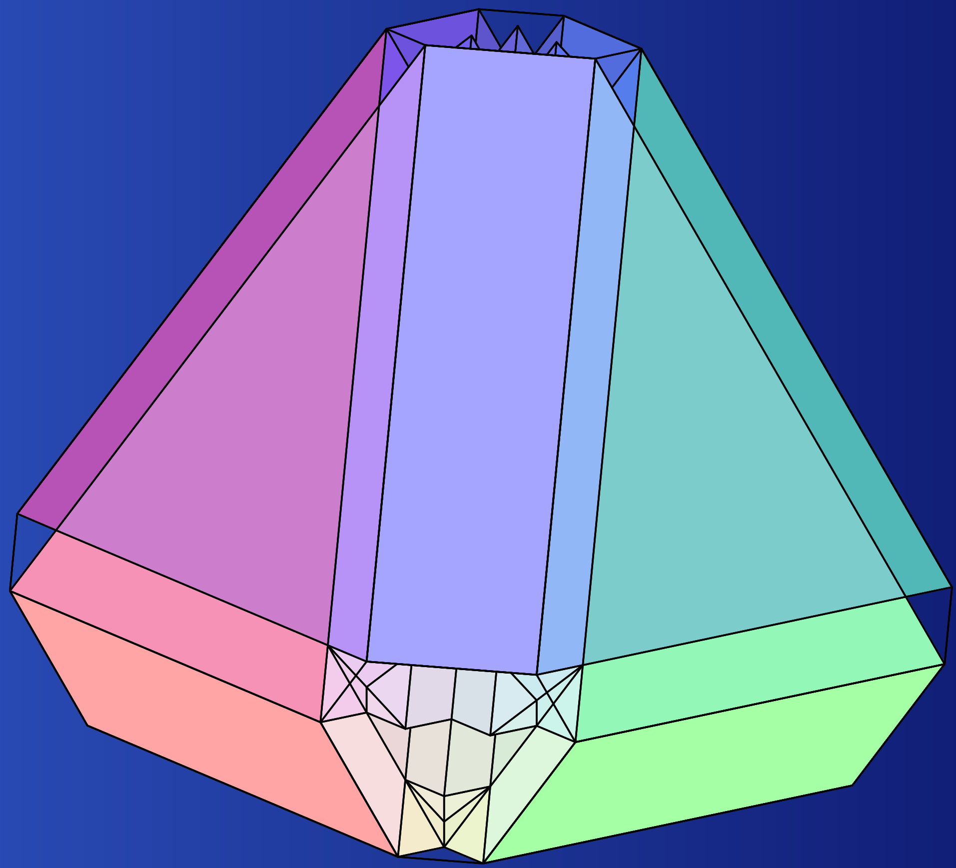

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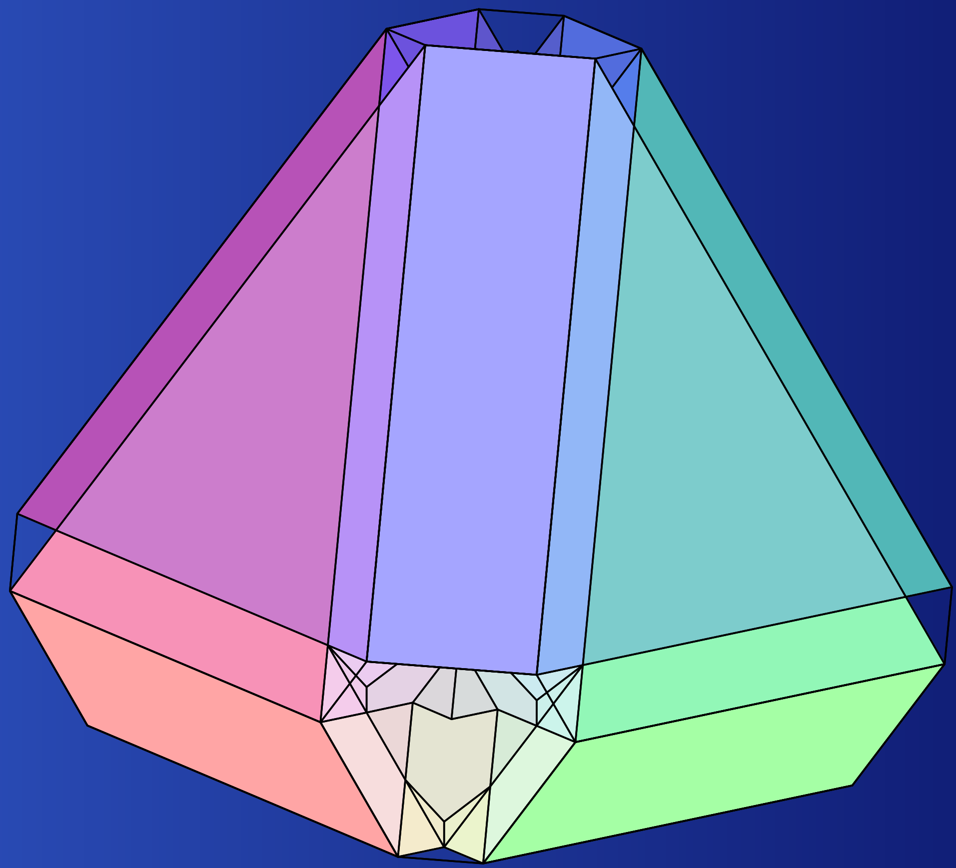

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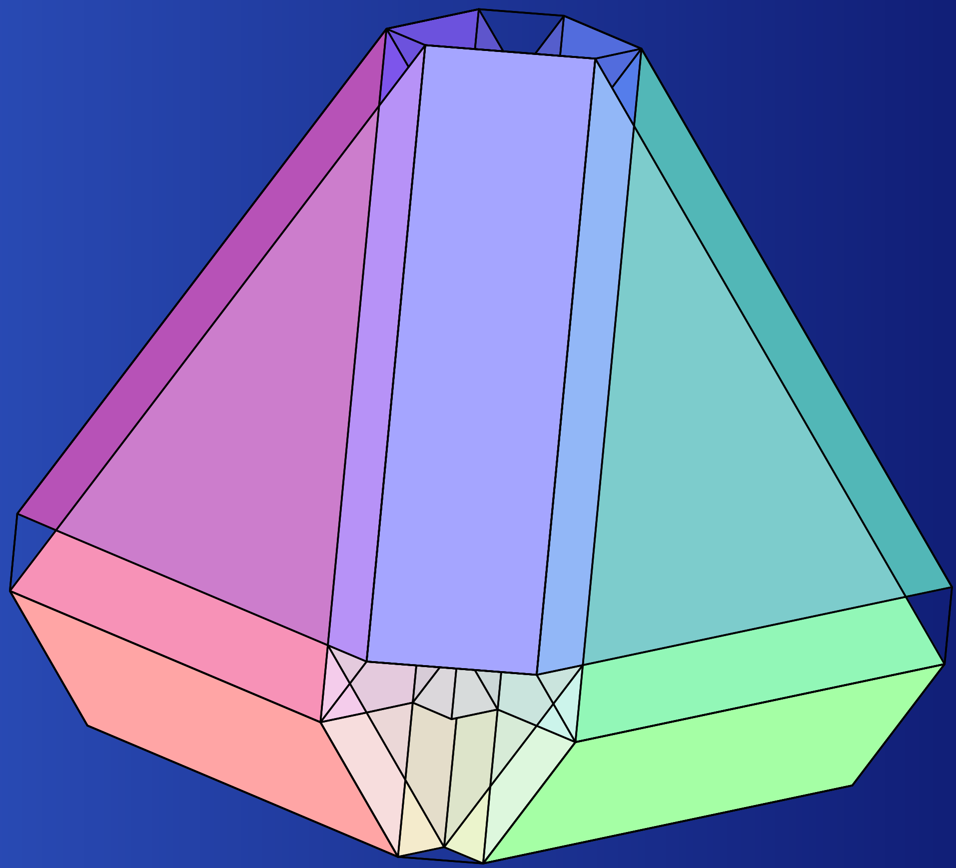

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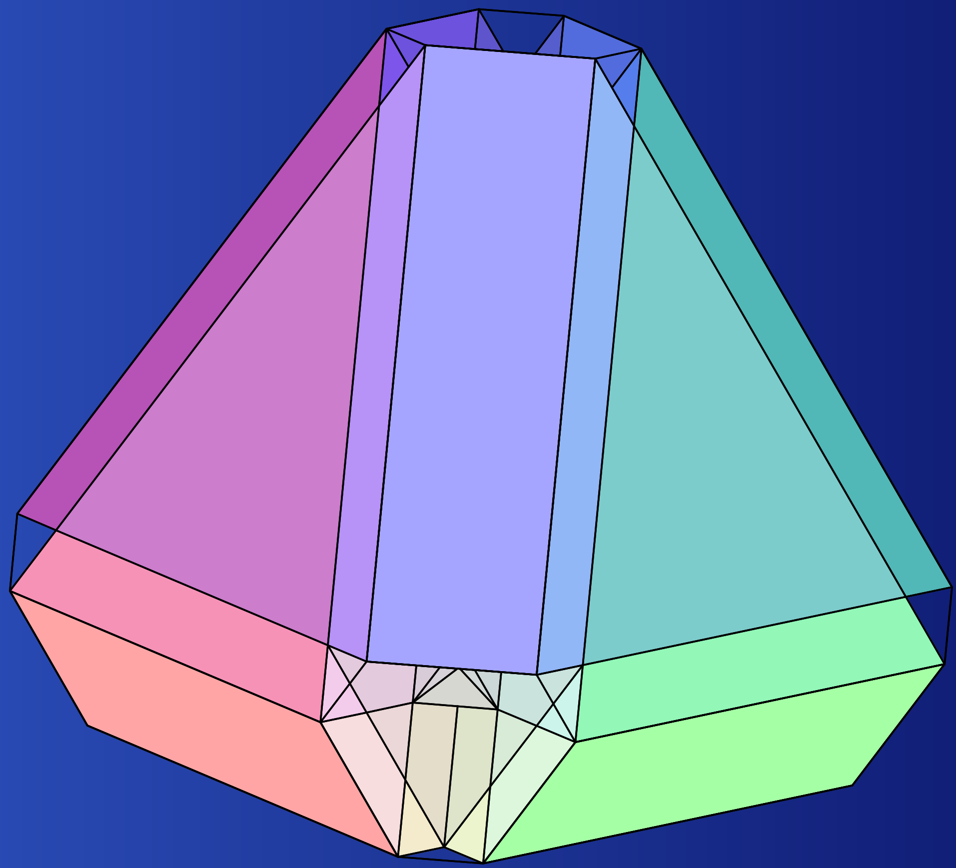

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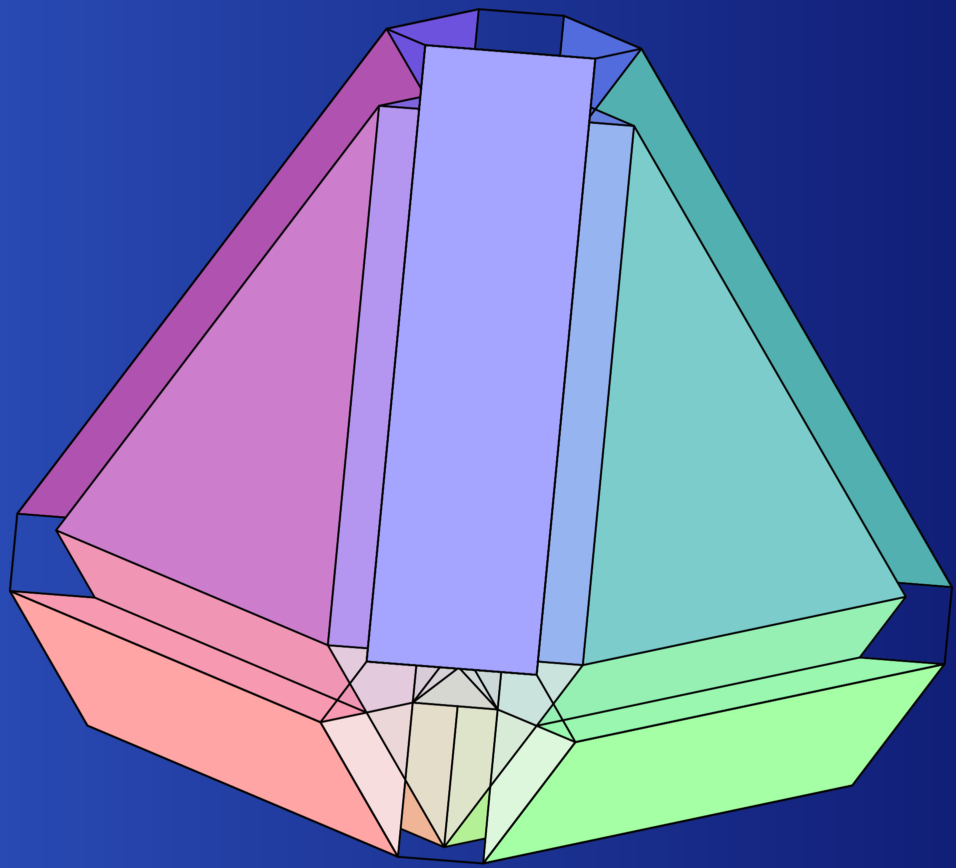
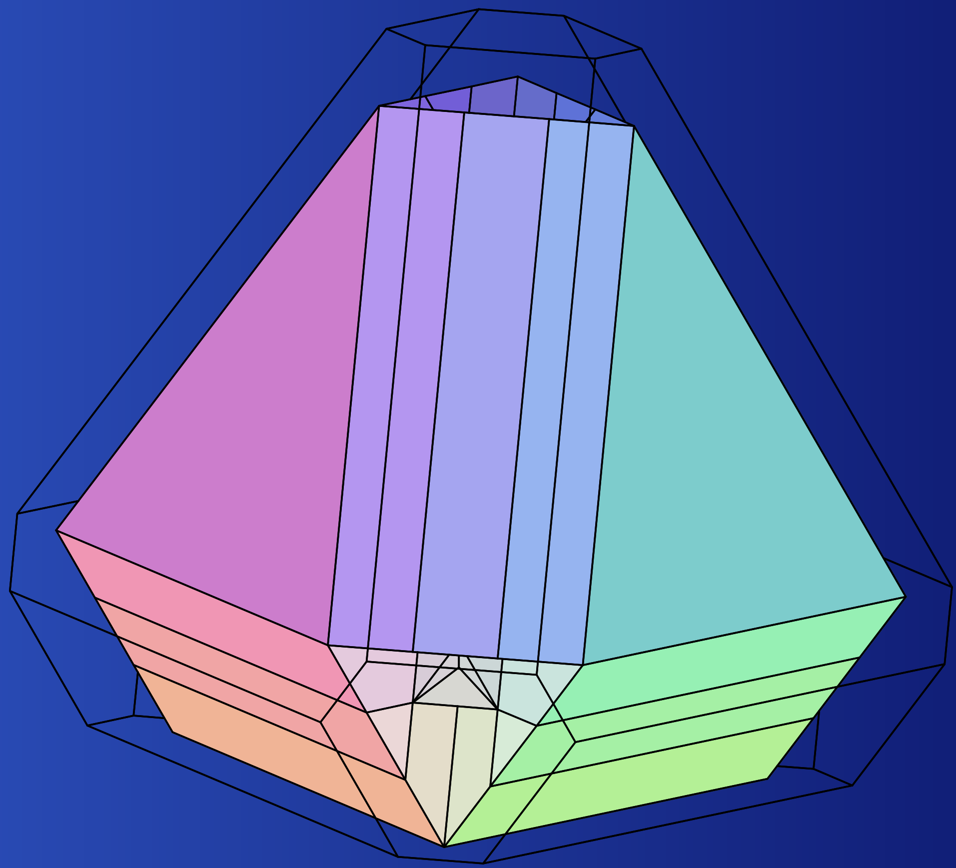


$\lambda = (14,\ -2,\ -4,\ -8)$



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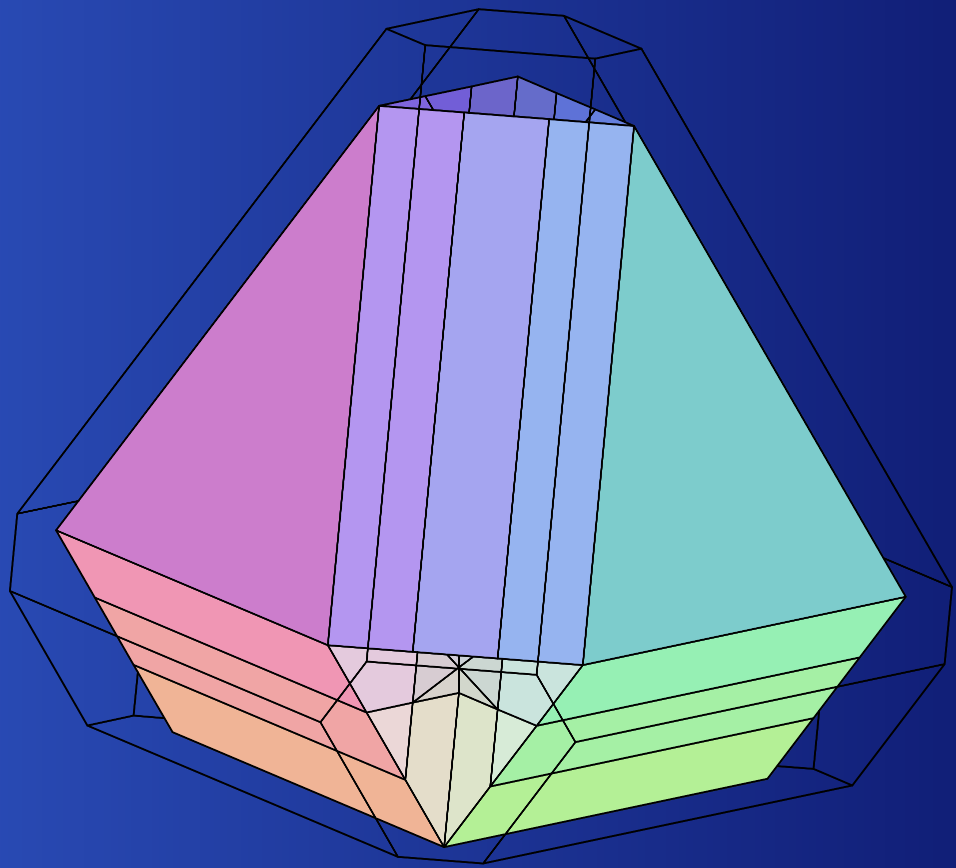

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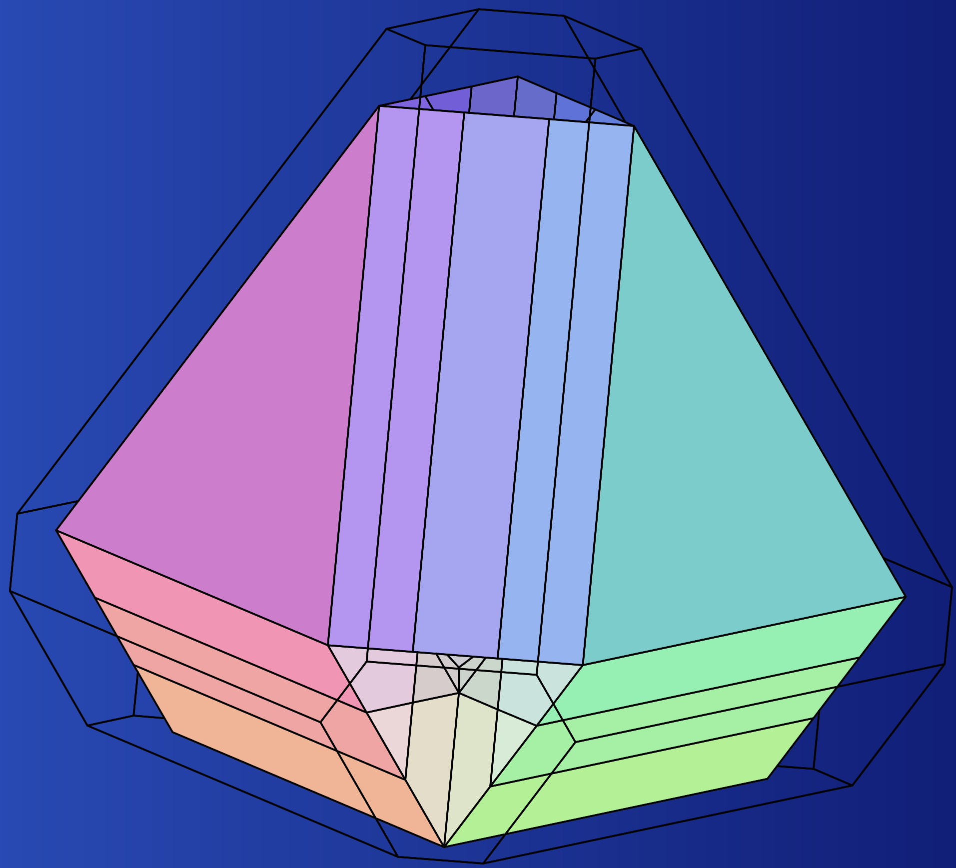

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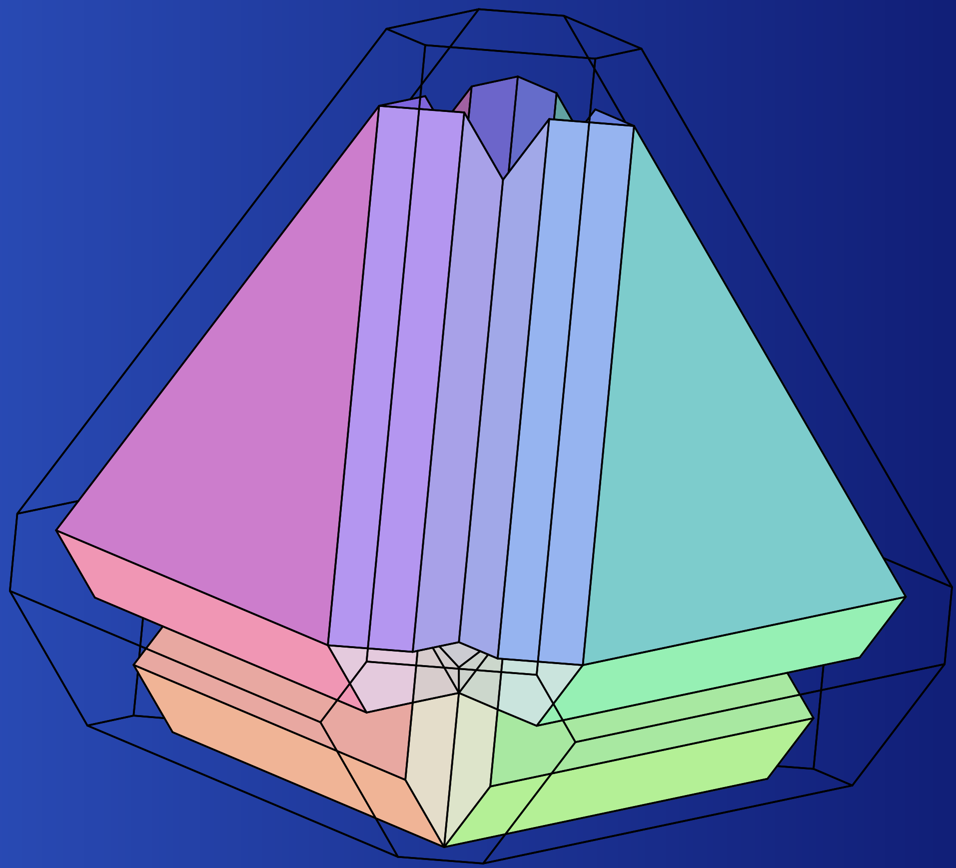

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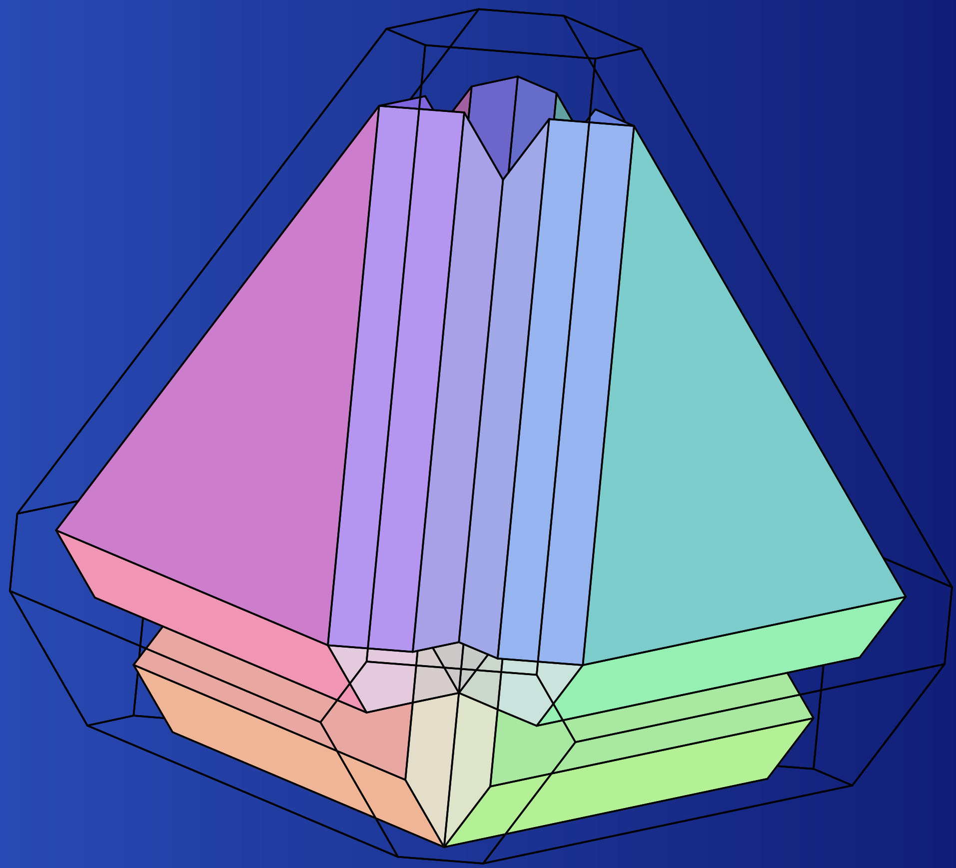

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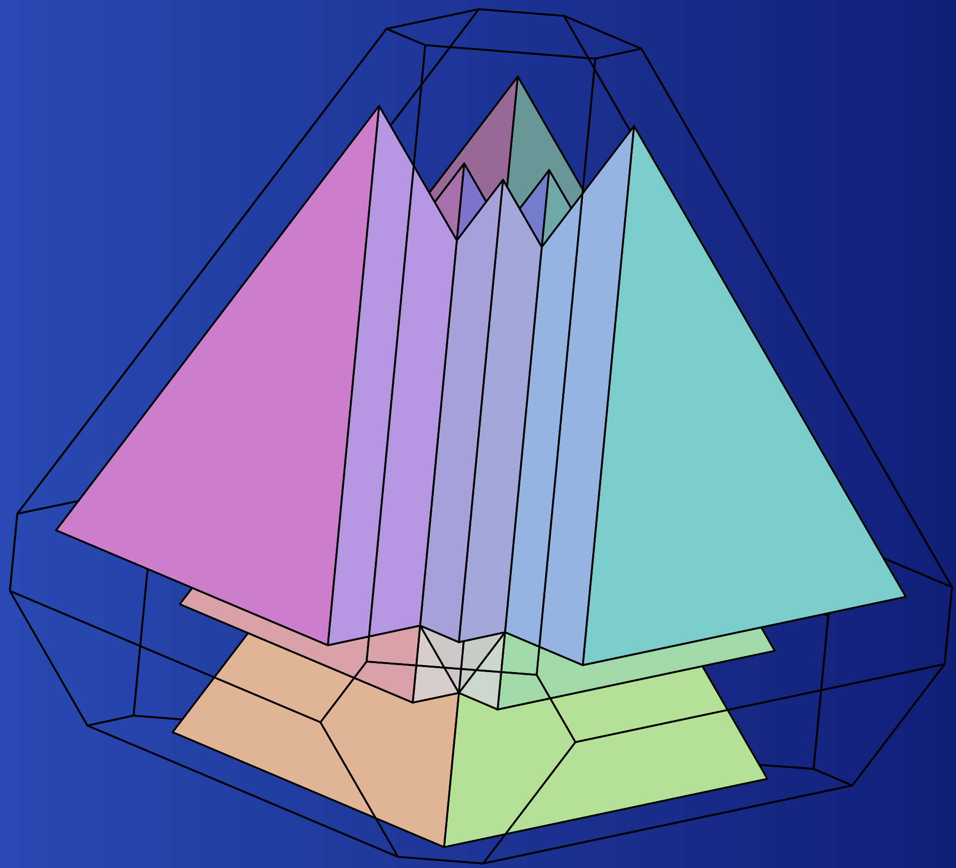

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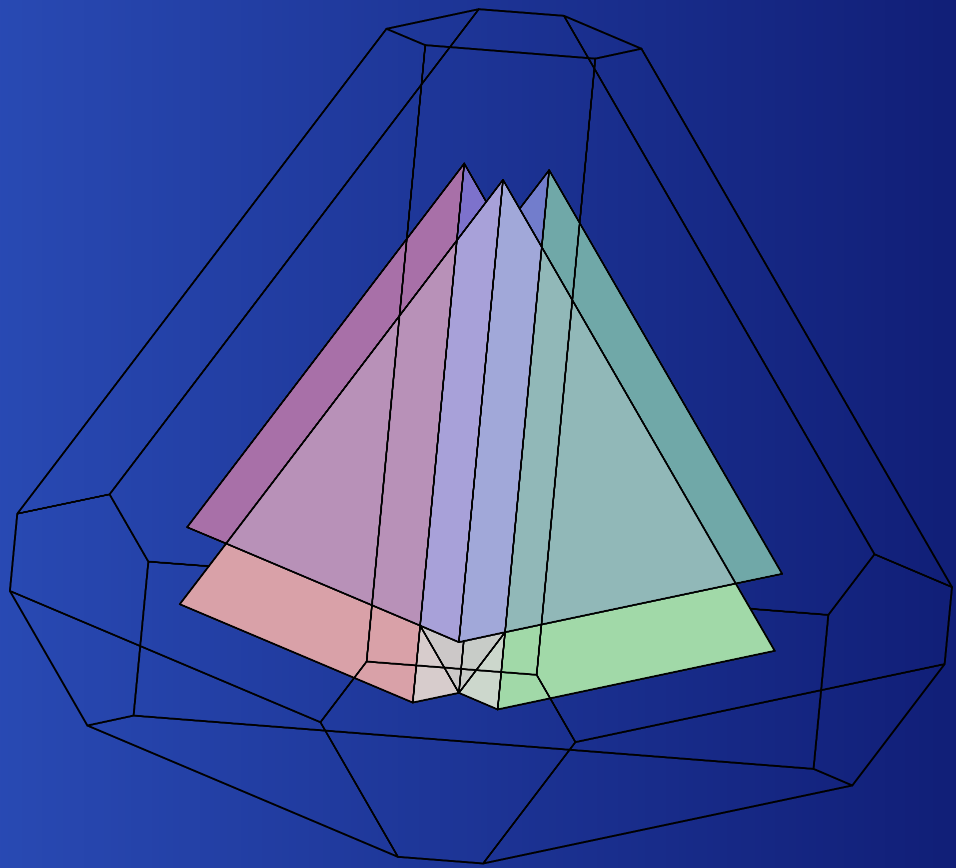

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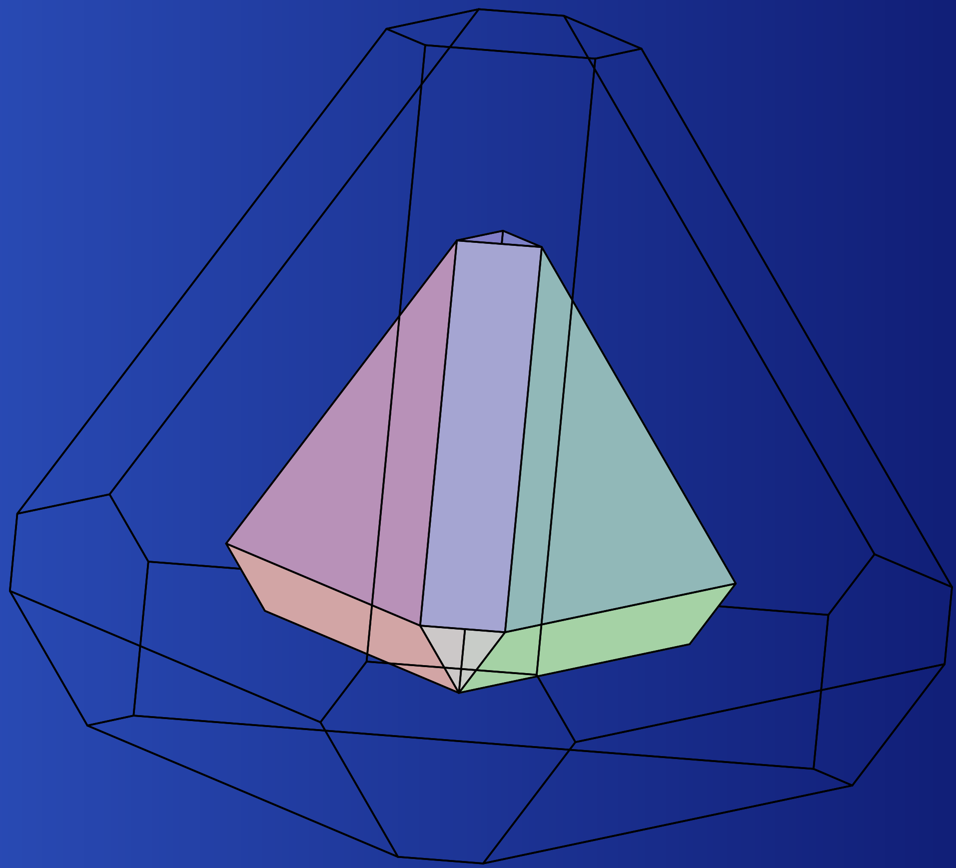

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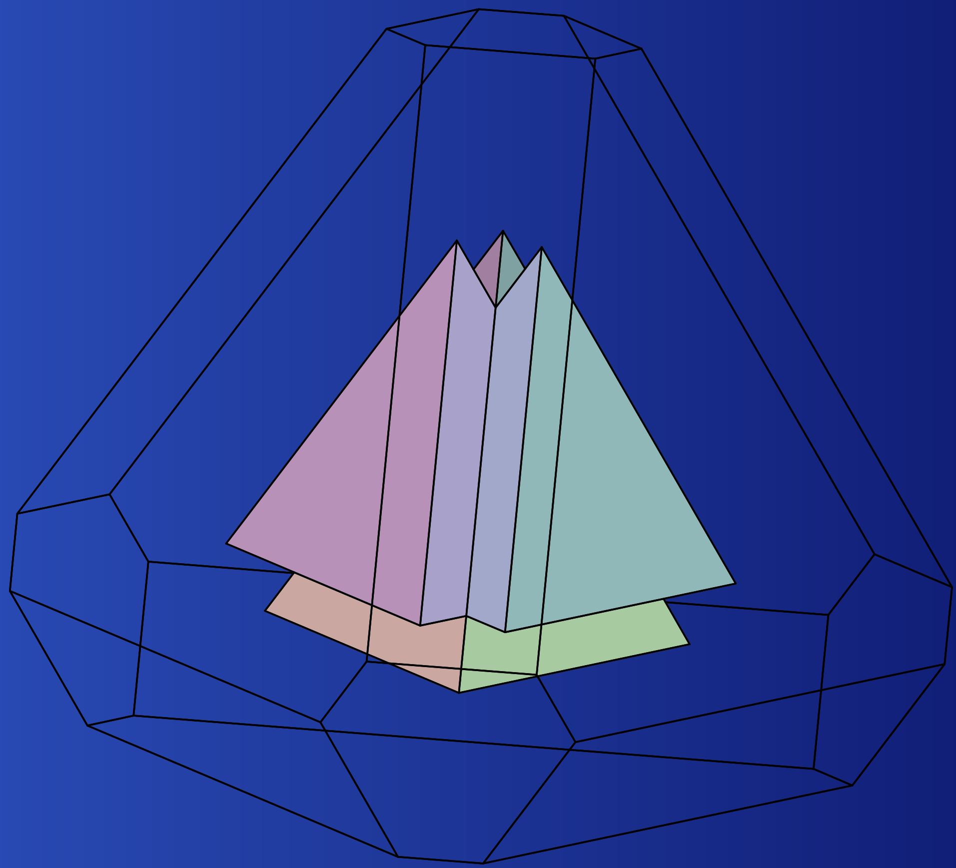

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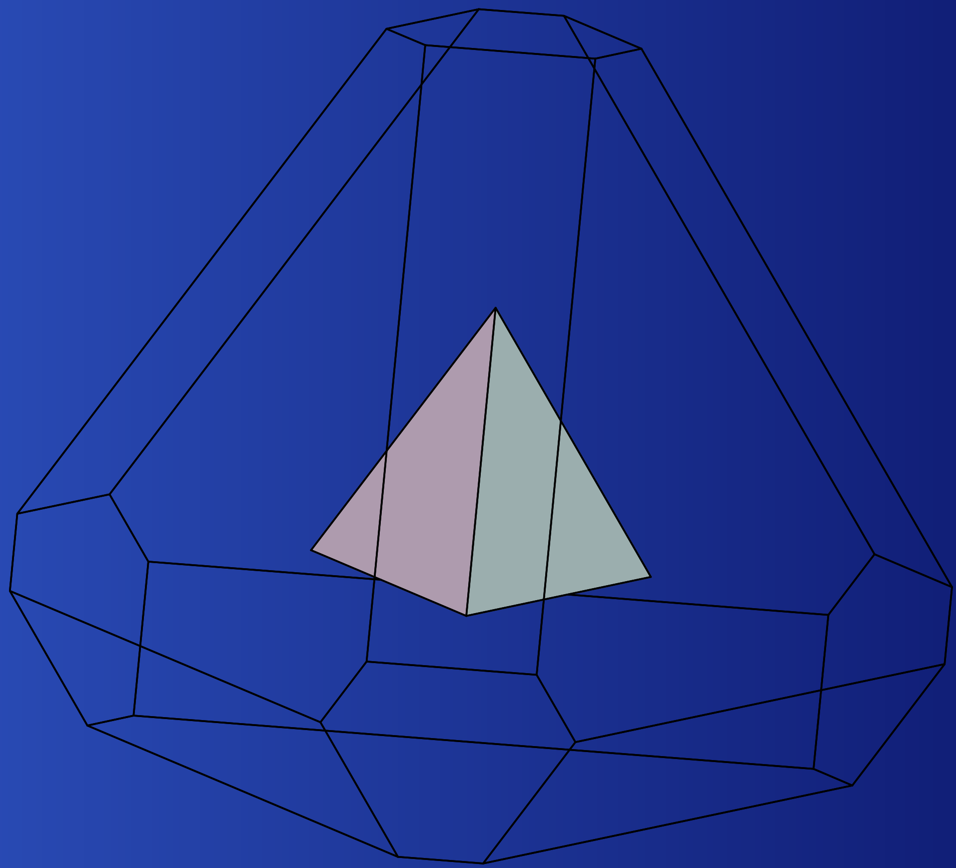

- The normals to the facets of the permutahedron $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$ are the conjugates $\theta(\omega_i)$ of the fundamental weights.

- An important vector:

$$\delta \quad = \quad \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha \quad = \quad \sum_{i=1}^{k-1}\omega_i$$

- δ corresponds to the staircase partition.

Vector partition functions

Vector partition functions

Let M be a $d \times n$ matrix over the integers. The vector partition function associated to M is the function

$$\begin{aligned} \phi_M : \quad \mathbb{Z}^d &\longrightarrow \mathbb{N} \\ b &\mapsto |\{x \in \mathbb{N}^n \; : \; Mx = b\}| \end{aligned}$$

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Example

If $M = \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ and $b = \binom{3}{1}$ then $\phi_M(b) = 3$

since $b = \binom{1}{0} + \binom{0}{1} + \binom{2}{0} = \binom{1}{0} + \binom{2}{1} = 3\binom{1}{0} + \binom{0}{1}$.

Polytopes and partition functions

- If M is such that $\ker M \cap \mathbb{R}^n_{\geq 0} = 0$, then

$$P_b = \{x \in \mathbb{R}^n_{\geq 0} \ : \ Mx = b\}$$

is a polytope.

$\phi_M(b)$ is the number of integral points in P_b .

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  $\phi_M(b)$ is the number of integral points in P_b .

- ϕ_M vanishes outside of $\mathrm{pos}(M)$.

The structure of partition functions

- ϕ_M is piecewise quasipolynomial of degree $n - \mathrm{rank}(M)$. (Sturmfels)

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The structure of partition functions

- ϕ_M is piecewise quasipolynomial of degree $n - \mathrm{rank}(M)$. (Sturmfels)
- The domains of quasipolynomiality form a complex of convex polyhedral cones, the chamber complex of ϕ_M .
- Alekseevskaya, Gelfand and Zelevinsky described how to determine the chamber complex of a partition function from its matrix.

Determining the chamber complex

We can assume without loss of generality that M has full rank d.

- Find all the $d \times d$ nonsingular submatrices M_σ of M.

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- Find all the $d \times d$ nonsingular submatrices M_σ of M .
- Determine the cone $\tau_\sigma = \mathrm{pos}(M_\sigma)$ spanned by the columns of M_σ .
- The chamber complex of ϕ_M is the common refinement of the τ_σ .

The Kostant partition function for A_3

$$\Delta_+^{(A_3)} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

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$K(v) = \phi_{M_{A_3}}(v)$ for

$$M_{A_3} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

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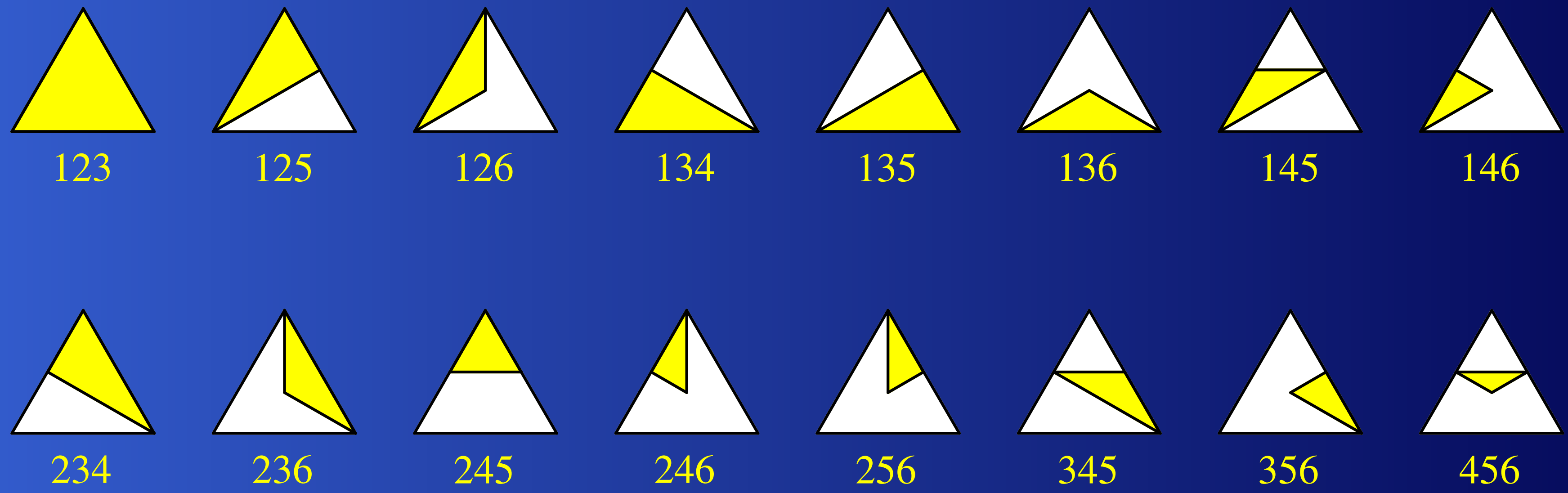
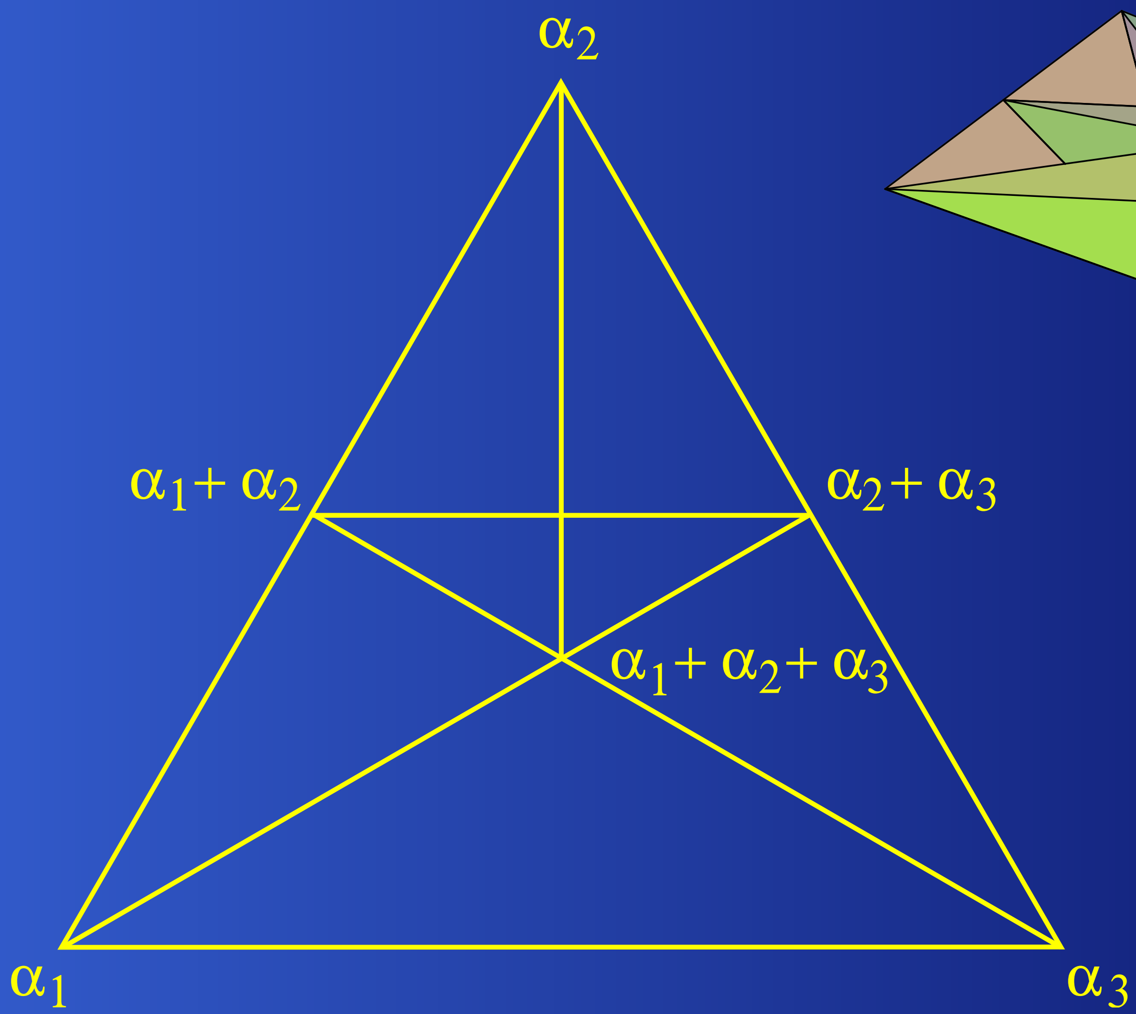
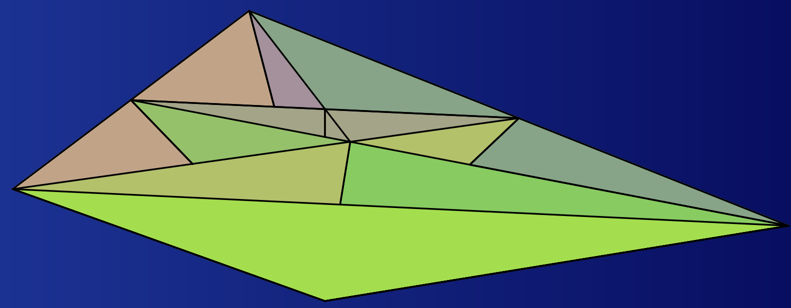
$$\begin{aligned} \mathcal{B} \;&=\; \{123, 125, 126, 134, 135, 136, 145, 146, \\ &\quad 234, 236, 245, 246, 256, 345, 356, 456\}\,. \end{aligned}$$


123
125
126
134
135
136
145
146
234
236
245
246
256
345
356
456


α_2
$\alpha_1+\alpha_2$
$\alpha_2+\alpha_3$
$\alpha_1+\alpha_2+\alpha_3$
α_1
α_3



Unimodularity

A $d \times n$ matrix of full rank d is unimodular if all its $d \times d$ submatrices have determinant 0 or ± 1.

Vector partitions functions of unimodular matrices are polynomial over the cones of their chamber complexes. (Sturmfels)

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Lemma **(well-known)** *The matrix M_{A_n} is unimodular for all n.*

Corollary *The Kostant partition function for A_{k-1} is polynomial of degree $\binom{k-1}{2}$ over the cones of its chamber complex.*

Weight multiplicities (Kostka numbers)

Gelfand-Tsetlin diagrams

A Gelfand-Tsetlin diagram is an array of integers of the form

$$\begin{array}{ccccccccc} \lambda_1^{(k)} & & \lambda_2^{(k)} & & \cdots & & \lambda_{k-1}^{(k)} & & \lambda_k^{(k)} \\ & \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\ & & \ddots & & \vdots & & \iddots & & \\ & & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\ & & & & \lambda_1^{(1)} & & & & \end{array}$$

such that

Gelfand-Tsetlin diagrams

$$
\begin{array}{ccccccccc}
\lambda_1^{(k)} & & \lambda_2^{(k)} & & \cdots & & \lambda_{k-1}^{(k)} & & \lambda_k^{(k)} \\
& \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\
& & \ddots & & \vdots & & ⋰ & & \\
& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}
$$

Gelfand-Tsetlin diagrams

$$
\begin{array}{ccccccccc}
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& & \ddots & & \vdots & & \iddots & & \\
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\end{array}
$$

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& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}
$$

and

$$
\begin{array}{ccc}
\lambda_j^{(i+1)} & & \lambda_{j+1}^{(i+1)} \\
& \lambda_j^{(i)} &
\end{array}
\qquad \lambda_j^{(i+1)} \geq \lambda_j^{(i)} \geq \lambda_{j+1}^{(i+1)}
$$

for every such triangle in the diagram.

GT-diagrams and weight multiplicities

Lemma **(Gelfand-Tsetlin)**

The weight multiplicity $m_\lambda(\beta)$ *is the number of Gelfand-Tsetlin diagrams with top row* λ *and row sums satisfying*

$$\sum_{i=1}^{m} \lambda_i^{(m)} = \beta_1 + \cdots + \beta_m \qquad \textit{for } 1 \leq m \leq k.$$

Gelfand-Tsetlin polytopes

$$\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \cdots & & \lambda_{k-1} & & \lambda_k \\
& \lambda_1^{(k-1)} & & \lambda_2^{(k-1)} & & \cdots & & \lambda_{k-1}^{(k-1)} & \\
& & \ddots & & \vdots & & \iddots & & \\
& & & \lambda_1^{(2)} & & \lambda_2^{(2)} & & & \\
& & & & \lambda_1^{(1)} & & & &
\end{array}$$

GT_λ $\mathrm{GT}_{\lambda\beta}$

A partition function for the $K_{\lambda\beta}$

Theorem A

For every k, we can find integer matrices E_k and B_k such that the Kostka numbers for partitions with at most k parts can be written as

$$K_{\lambda\beta} = \phi_{E_k}\left(B_k \begin{pmatrix} \lambda \\ \beta \end{pmatrix} \right) .$$

A chamber complex for the $K_{\lambda\beta}$

- Theorem A implies that the Kostka numbers are given by quasipolynomials over the cells of a chamber complex $\mathcal{C}^{(k)}$.

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A chamber complex for the $K_{\lambda\beta}$

- Theorem A implies that the Kostka numbers are given by quasipolynomials over the cells of a chamber complex $\mathcal{C}^{(k)}$.

- The vector partition function ϕ_{E_k} puts λ and β on an equal footing: $\mathcal{C}^{(k)}$ is a complex in (λ, β)-space.

- By intersecting $\mathcal{C}^{(k)}$ with the affine subspace corresponding to fixing λ, we get the domains of quasipolynomiality for $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$.

The Kostant arrangements

The Kostant arrangements will be the main tool to

- complete the proof that the Kostka numbers are given by polynomials on the cones of a chamber complex;

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- complete the proof that the Kostka numbers are given by polynomials on the cones of a chamber complex;
- find interesting factorization patterns in the polynomials giving the Kostka numbers.

Kostant's multiplicity formula

The Kostant partition function is the function

$$K(v) = \left| \left\{ (k_\alpha)_{\alpha \in \Delta_+} \in \mathbb{N}^{|\Delta_+|} \ : \ \sum_{\alpha \in \Delta_+} k_\alpha \alpha = v \right\} \right|,$$

i.e. $K(v)$ is the number of ways that v can be written as a sum of positive roots.

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Kostant partition function is piecewise polynomial

$\Downarrow$

Kostka numbers are locally polynomial

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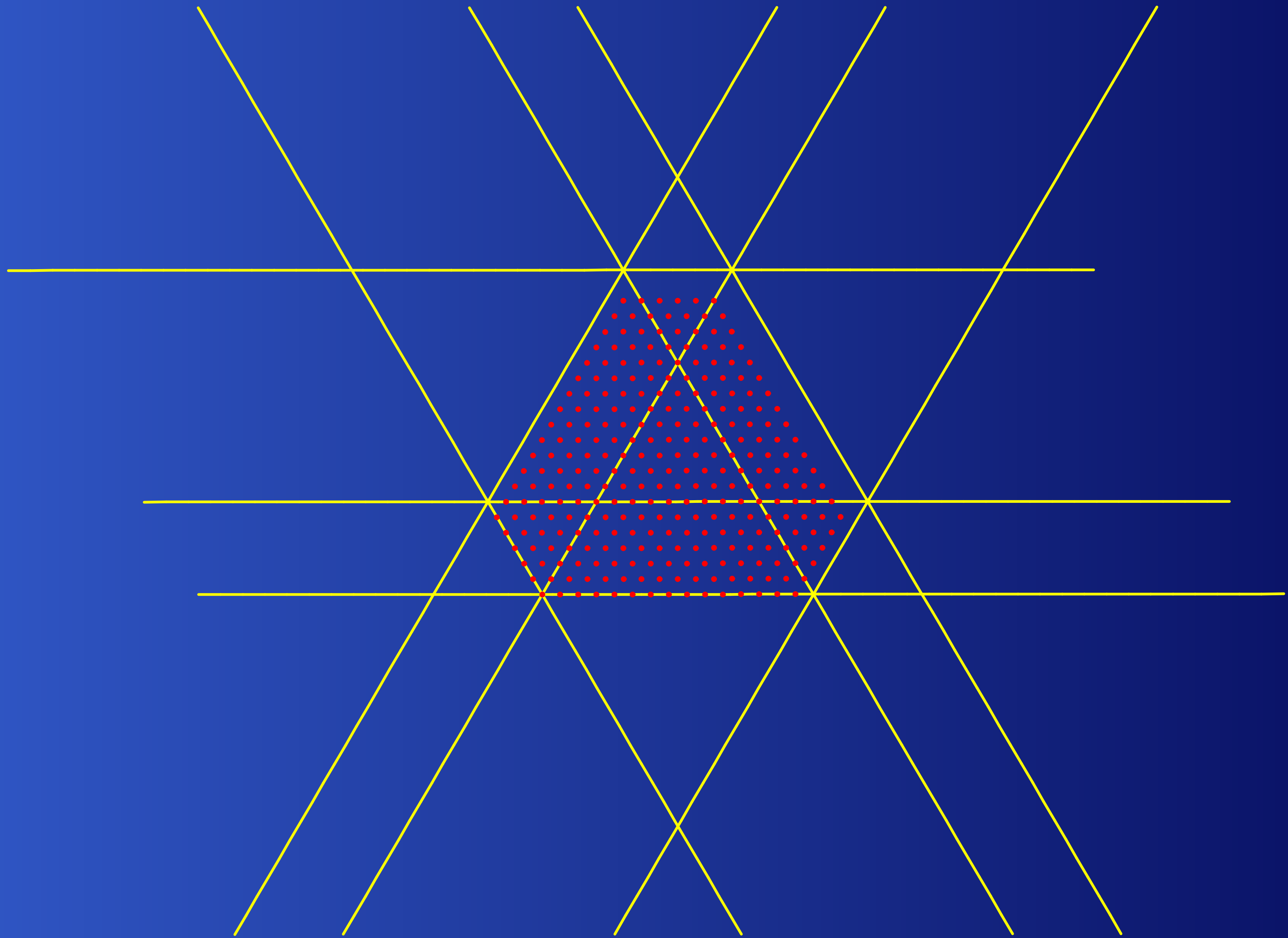
Kostant partition function is piecewise polynomial

$$\Downarrow$$

Kostka numbers are locally polynomial

- We find a family of hyperplane arrangements over whose regions the Kostka numbers are given by polynomials.

Example: $\lambda = (21, 7, 2)$



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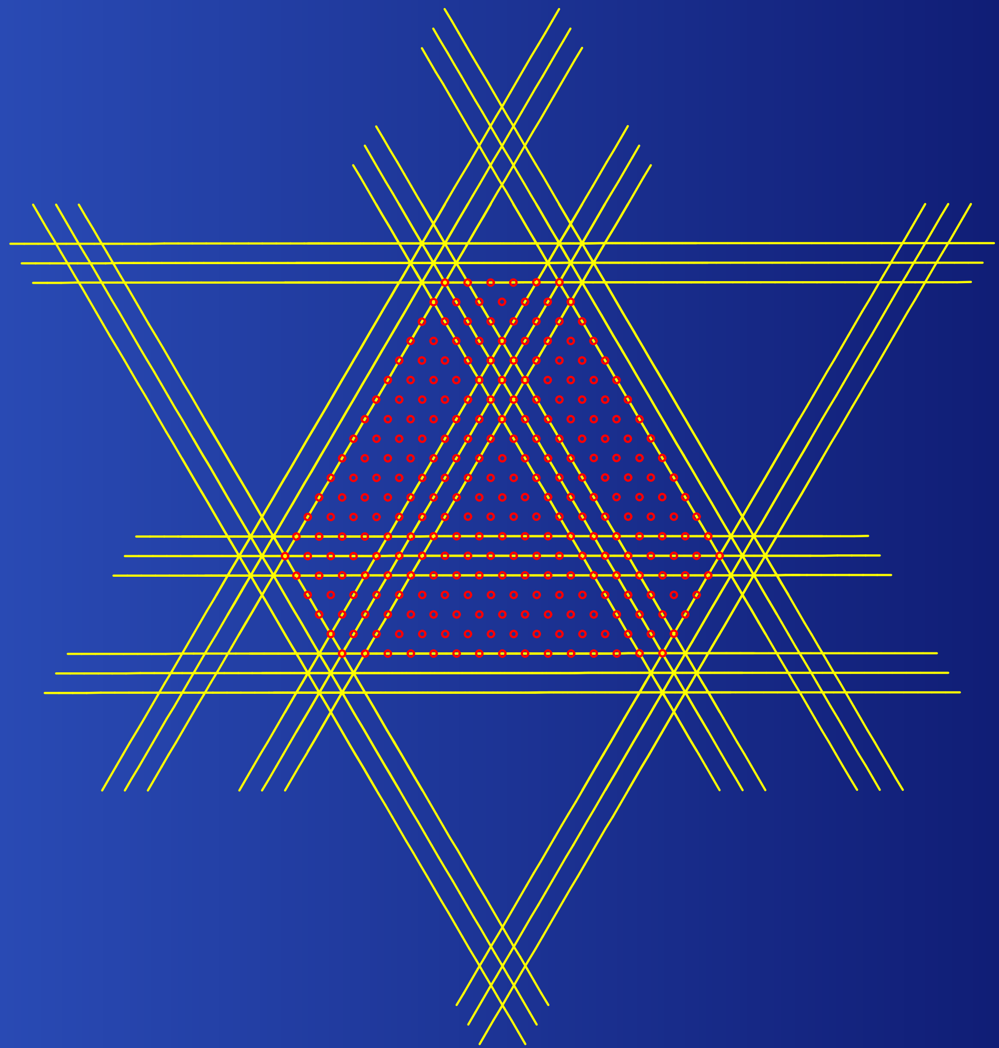

Polynomiality in the chamber complex

Theorem B

The quasipolynomials giving the Kostka numbers in the cones of $\mathcal{C}^{(k)}$ *are polynomials of total degree* $\binom{k-1}{2}$ *in the two sets of variables* (λ_i) *and* (β_j) *.*

Lemma

For each cone C of the chamber complex for the Kostka numbers, we can find a region R of any of the Kostant arrangements such that $C \cap R$ contains an arbitrarily large ball.

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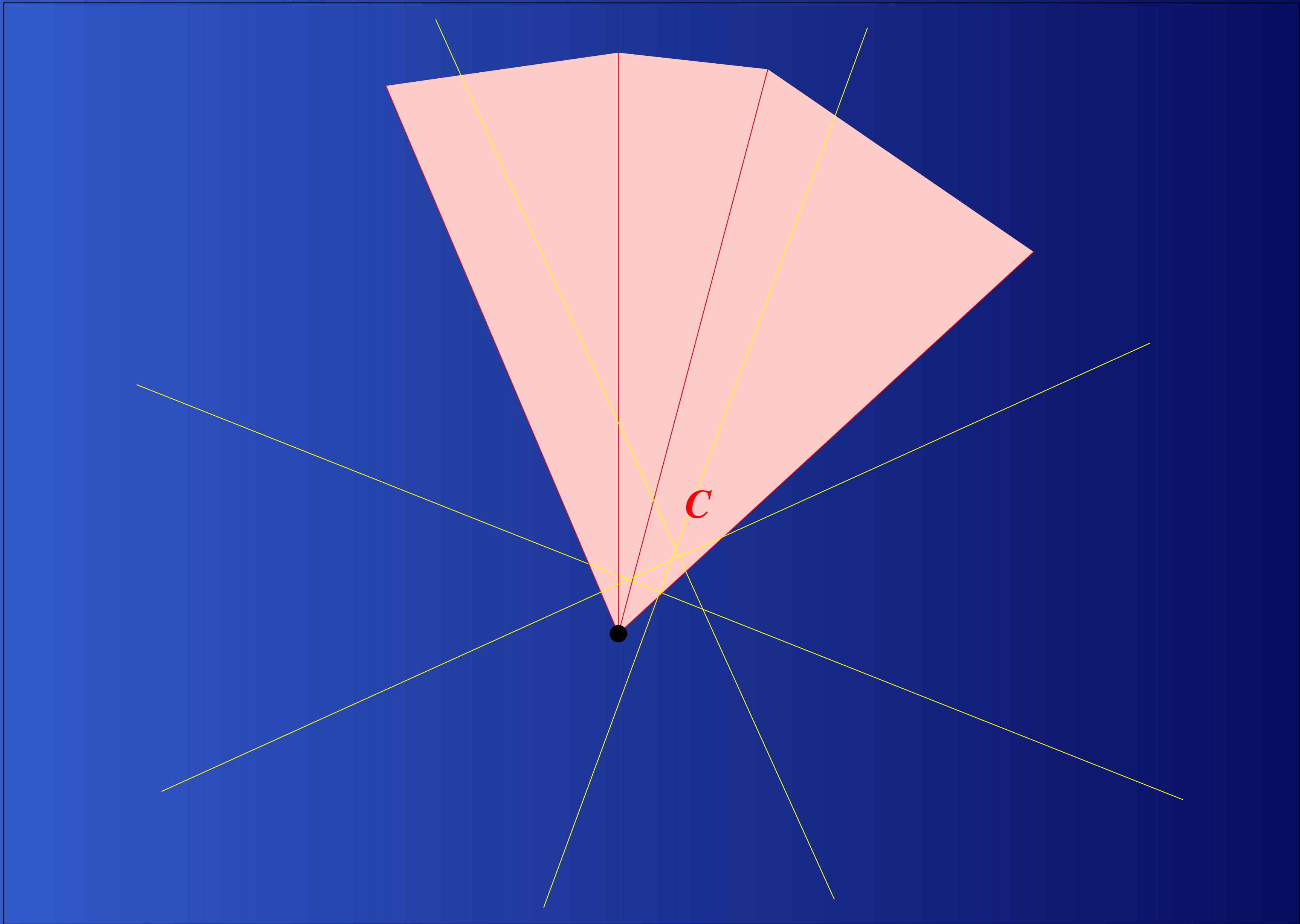
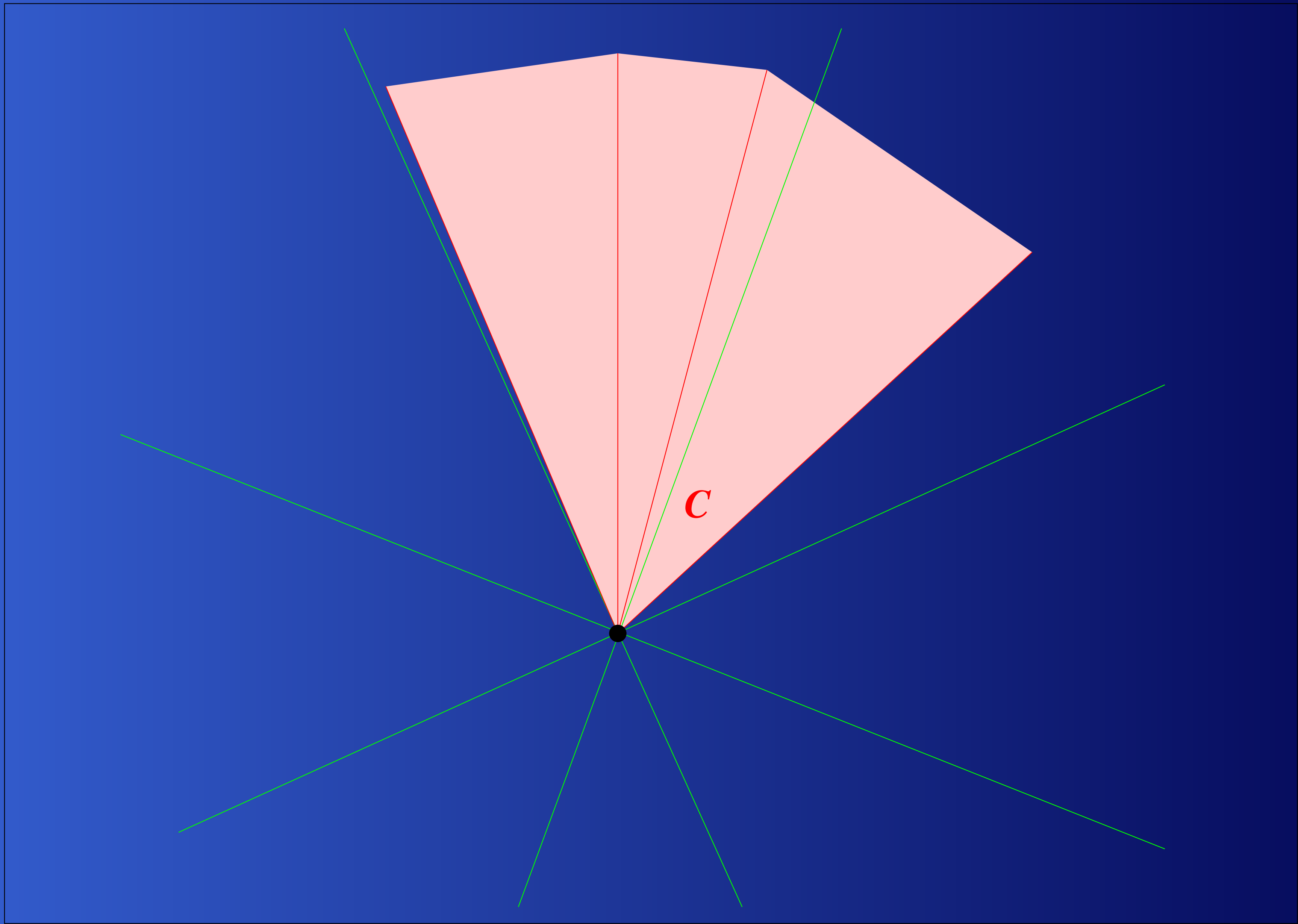
- Then the polynomial on R and the quasipolynomial on C agree on all the lattice points (λ, β) in that ball.
- The degree bounds follow from the degree bounds on the Kostant partition function.

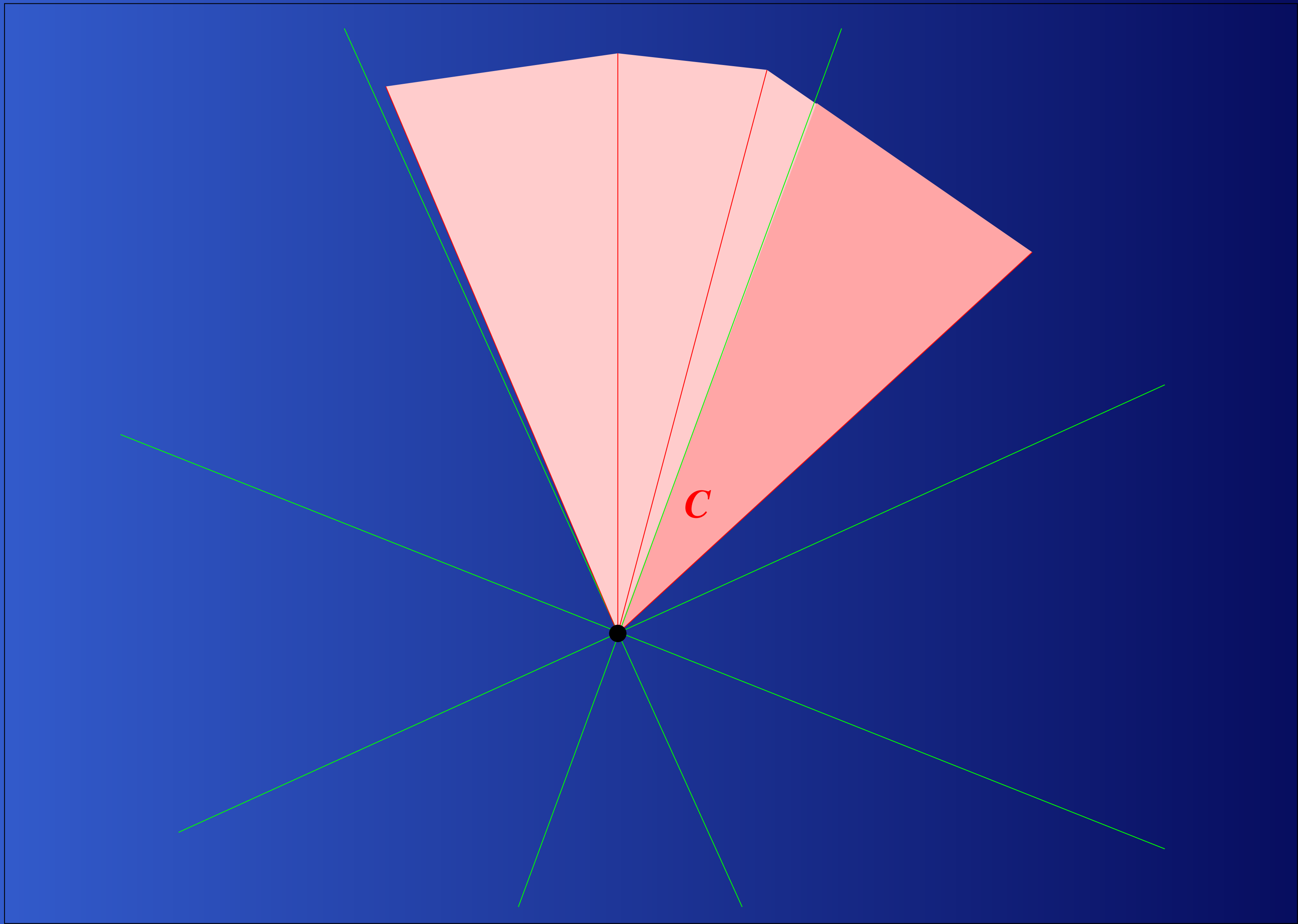
Idea of proof



Idea of proof



Idea of proof



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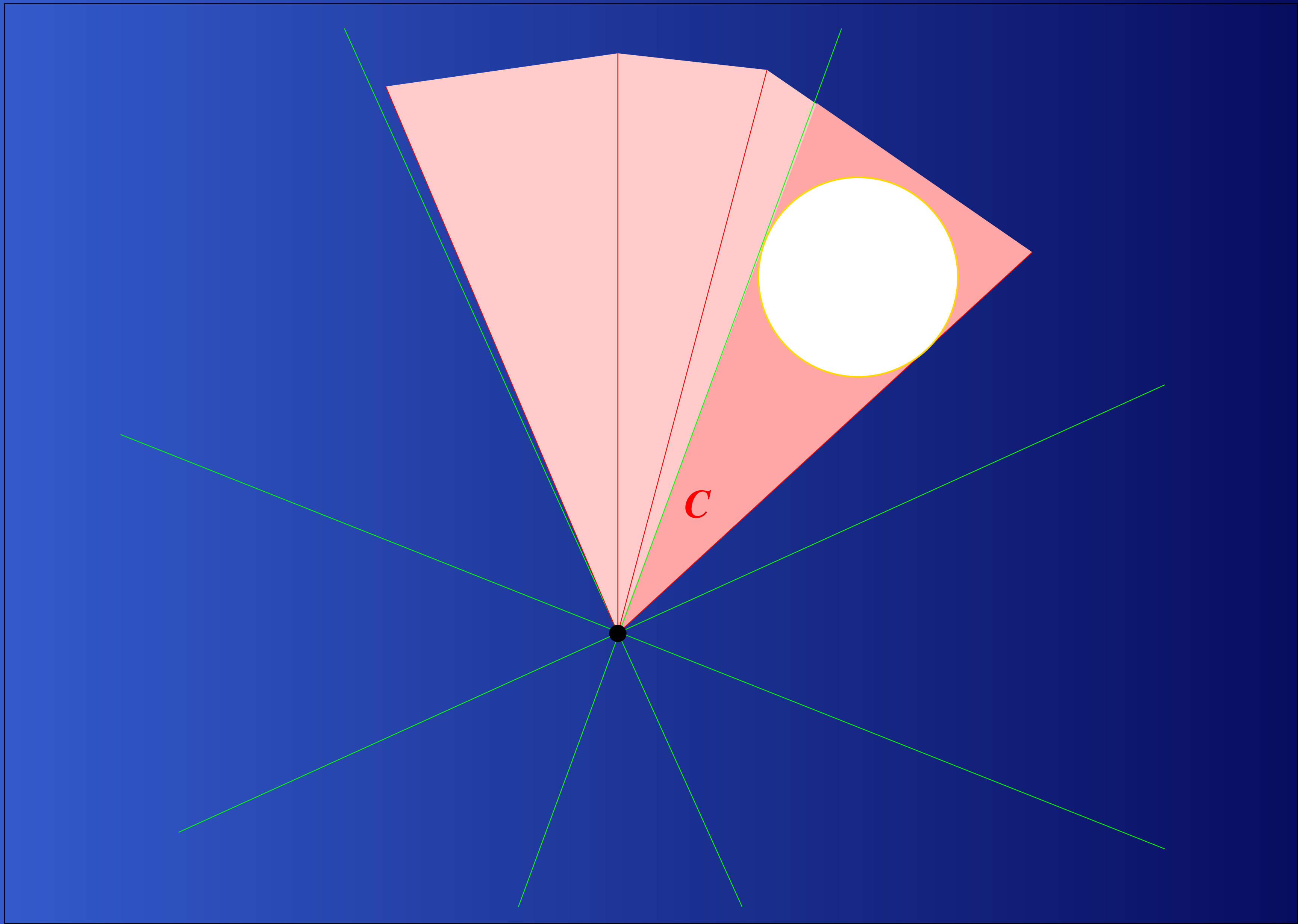

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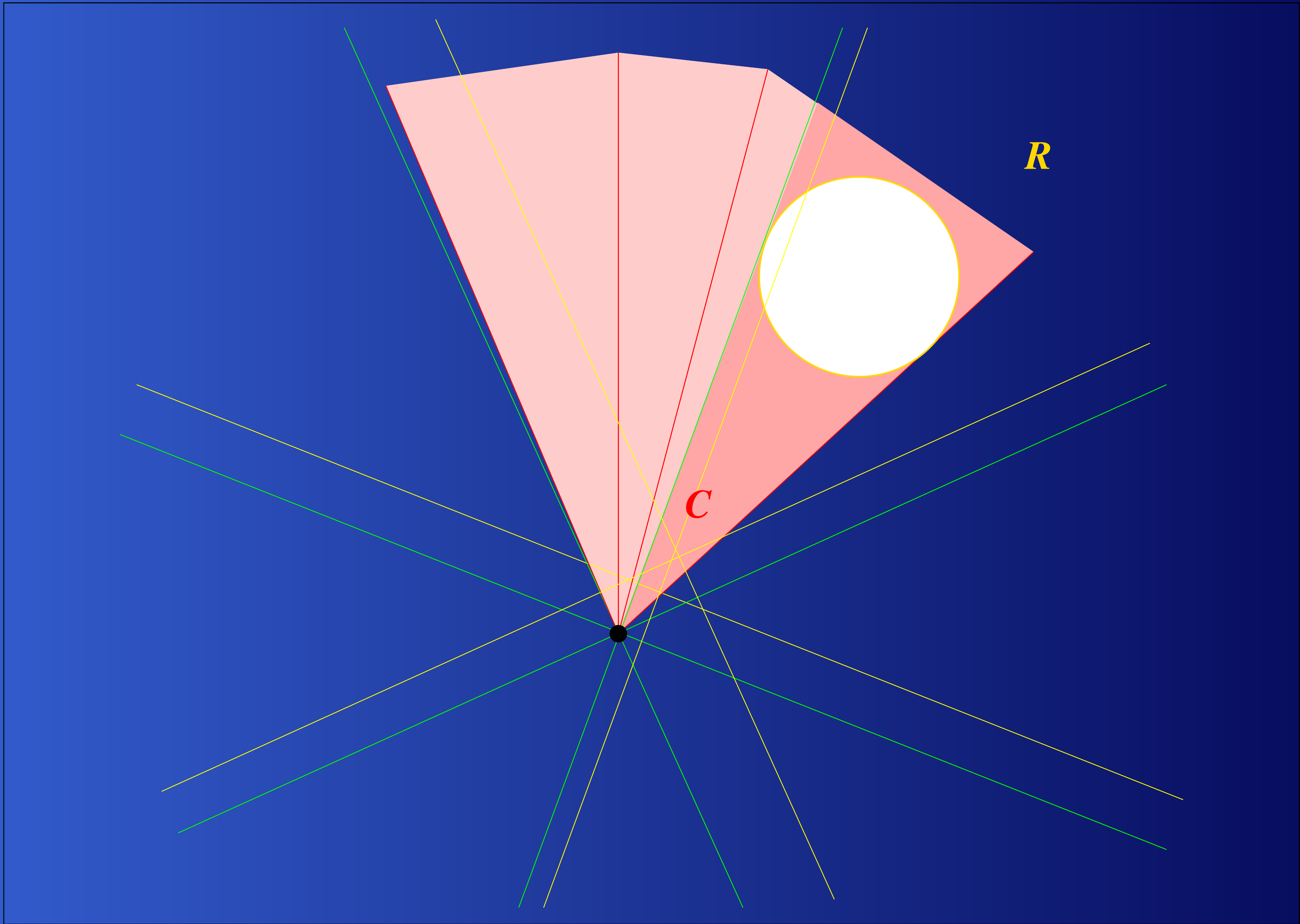

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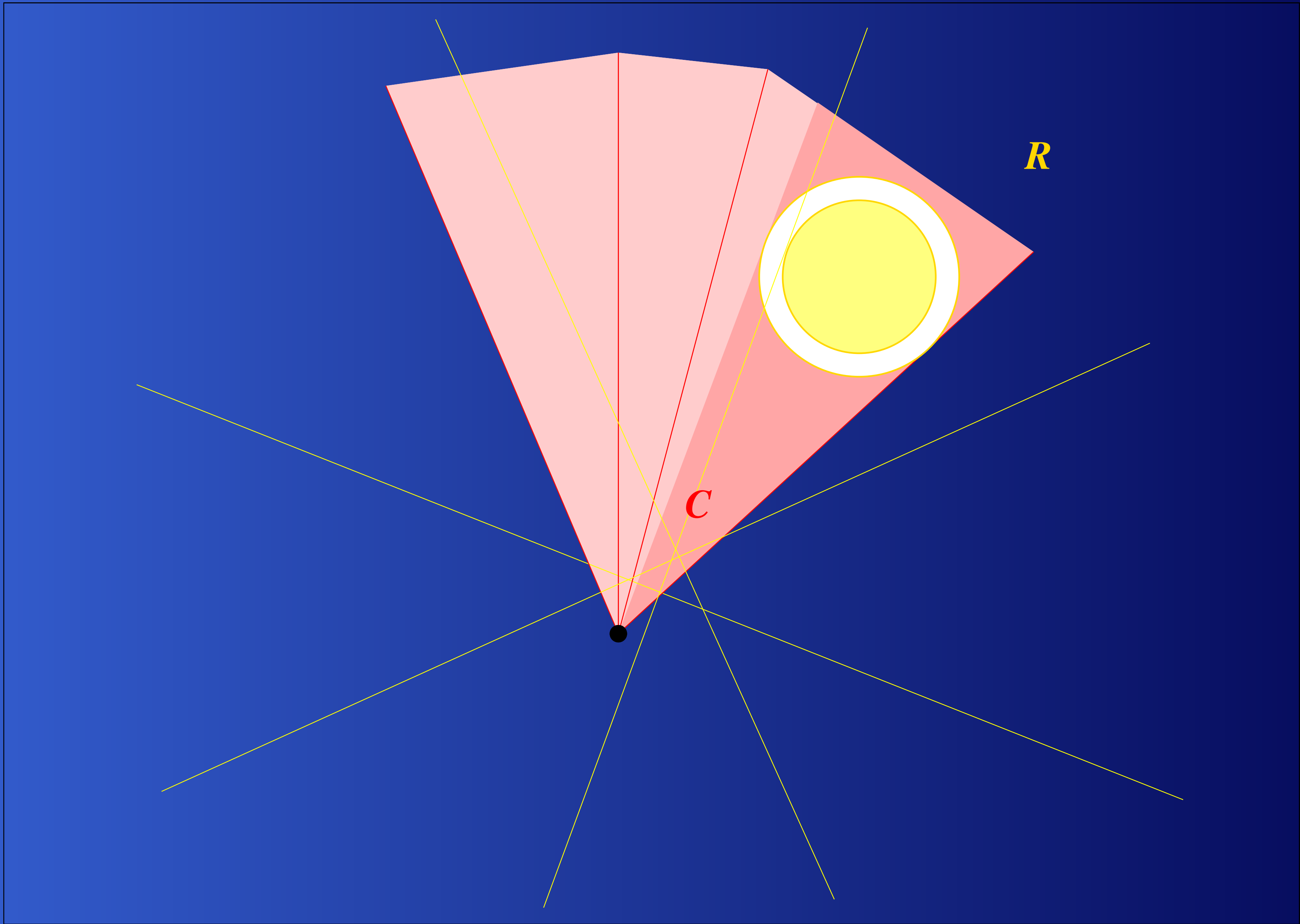

Scaling (or stretching)

Corollary

For any $\lambda, \beta \in \Lambda_W$ *with* $\lambda - \beta \in \Lambda_R$*, the function*

$$N \in \mathbb{N} \quad \longmapsto \quad K_{N\lambda \, N\beta}$$

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- This function is the Ehrhart polynomial of the Gelfand-Tsetlin polytope $\mathrm{GT}_{\lambda\mu}$. (Kirillov)
- $\mathrm{GT}_{\lambda\mu}$ is not an integral polytope in general (Clifford, King-Tollu-Toumazet, DeLoera-McAllister).

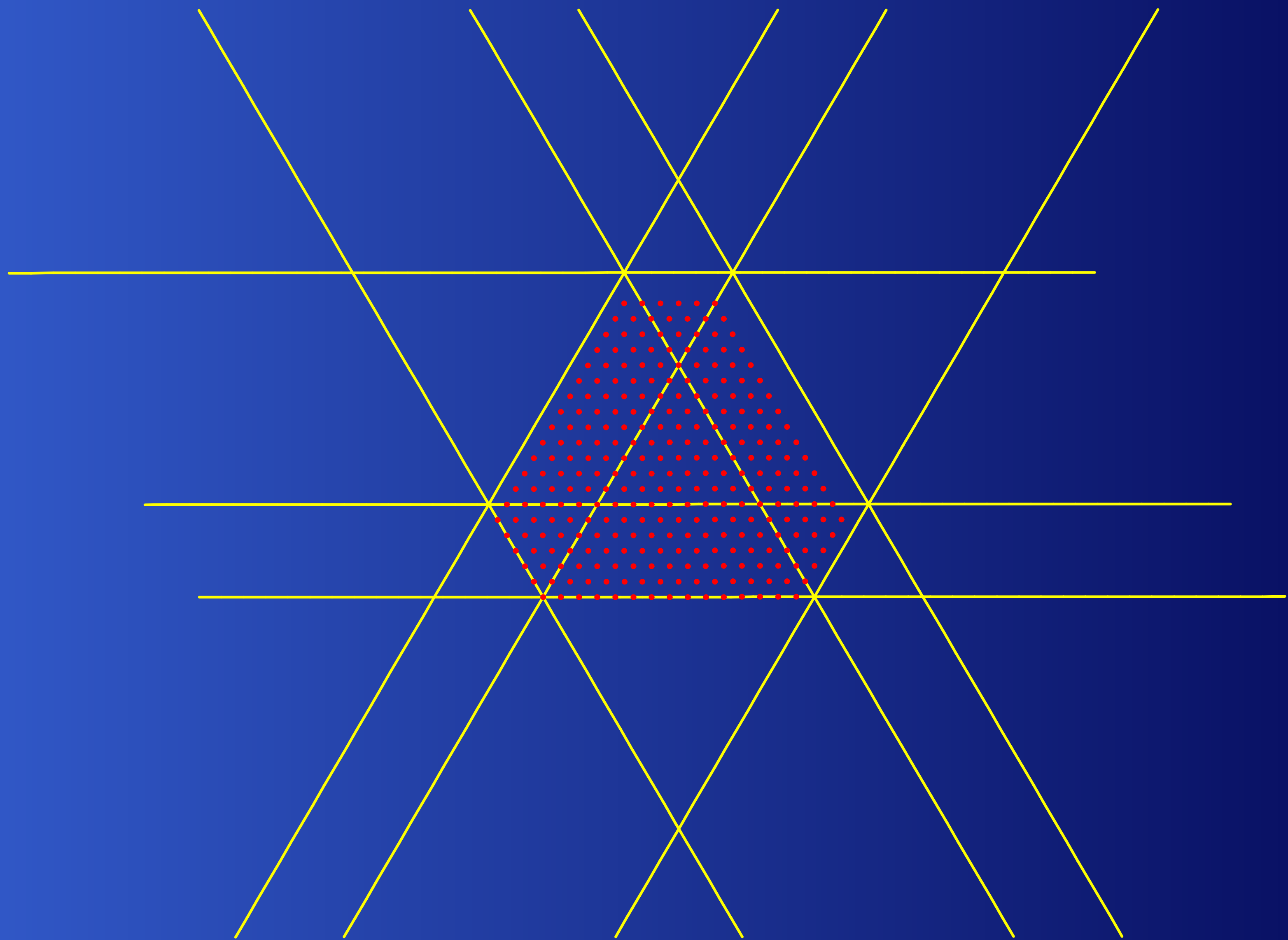
Factorization patterns

Theorem C

Suppose that H is the hyperplane supporting a facet of the permutahedron with normal $\theta(\omega_j)$.

Then the polynomials giving the Kostka numbers in all the domains of the permutahedron with a facet on H are divisible by $j(k-j)-1$ linear factors.

Idea of proof



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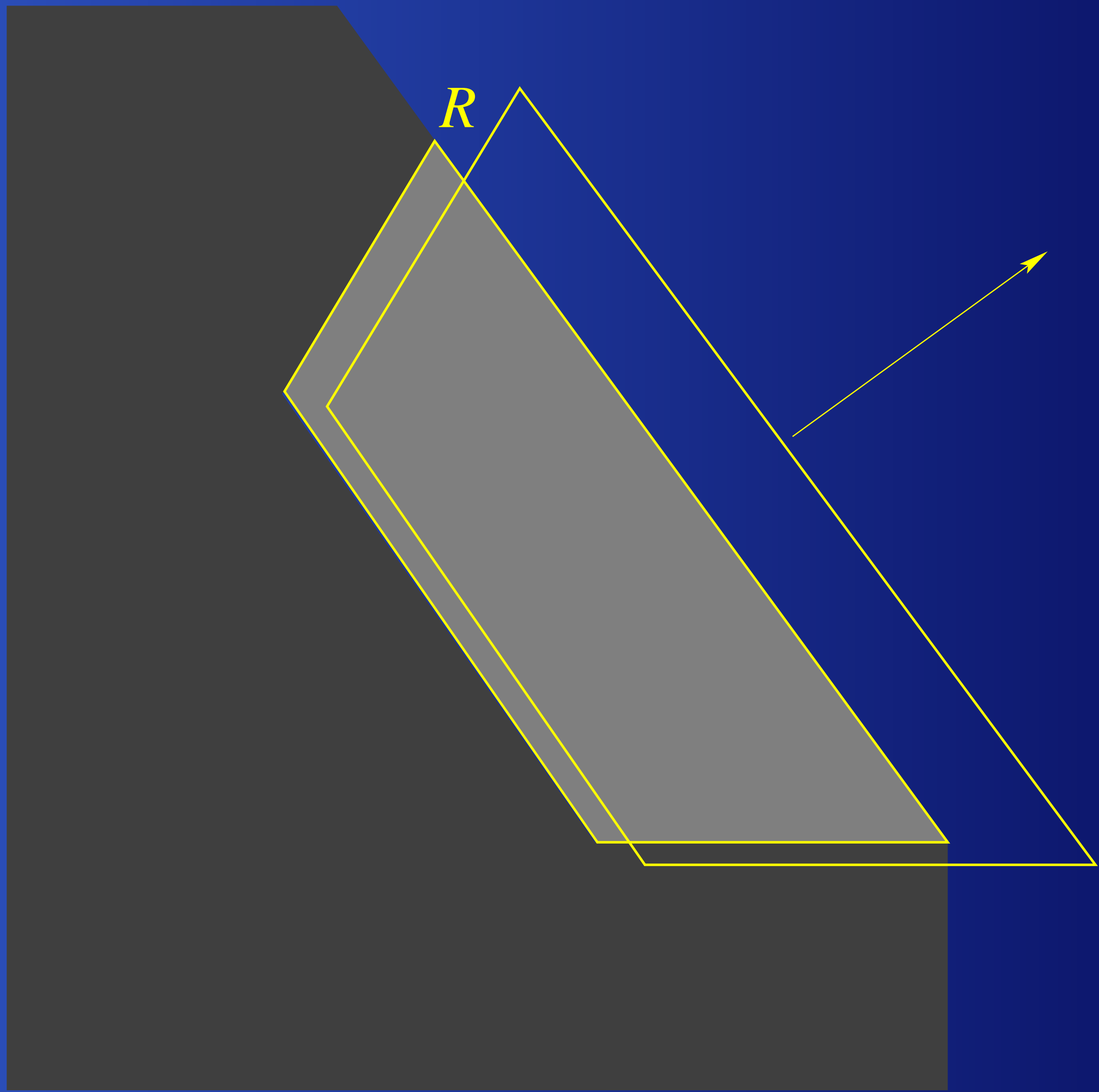

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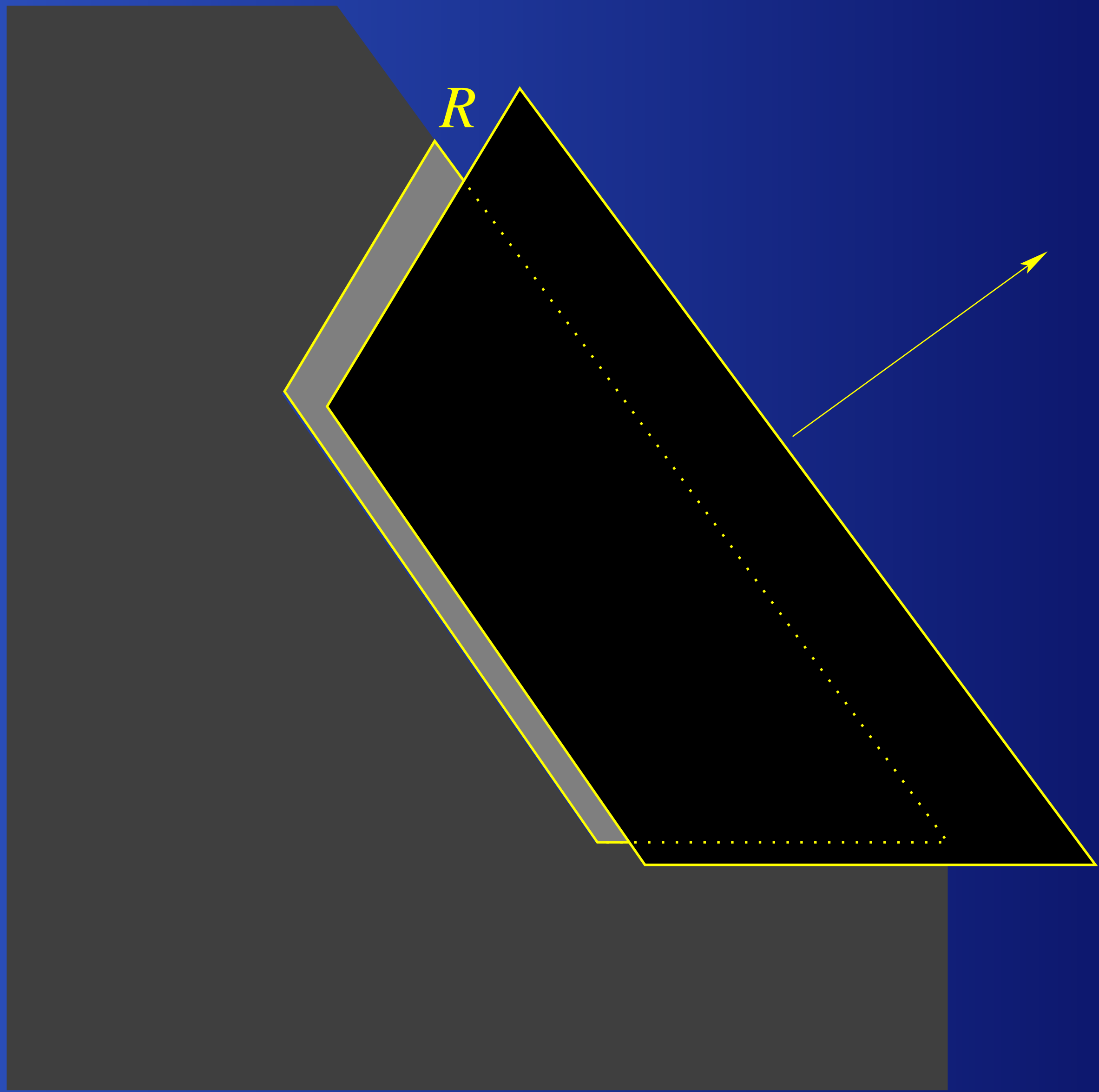

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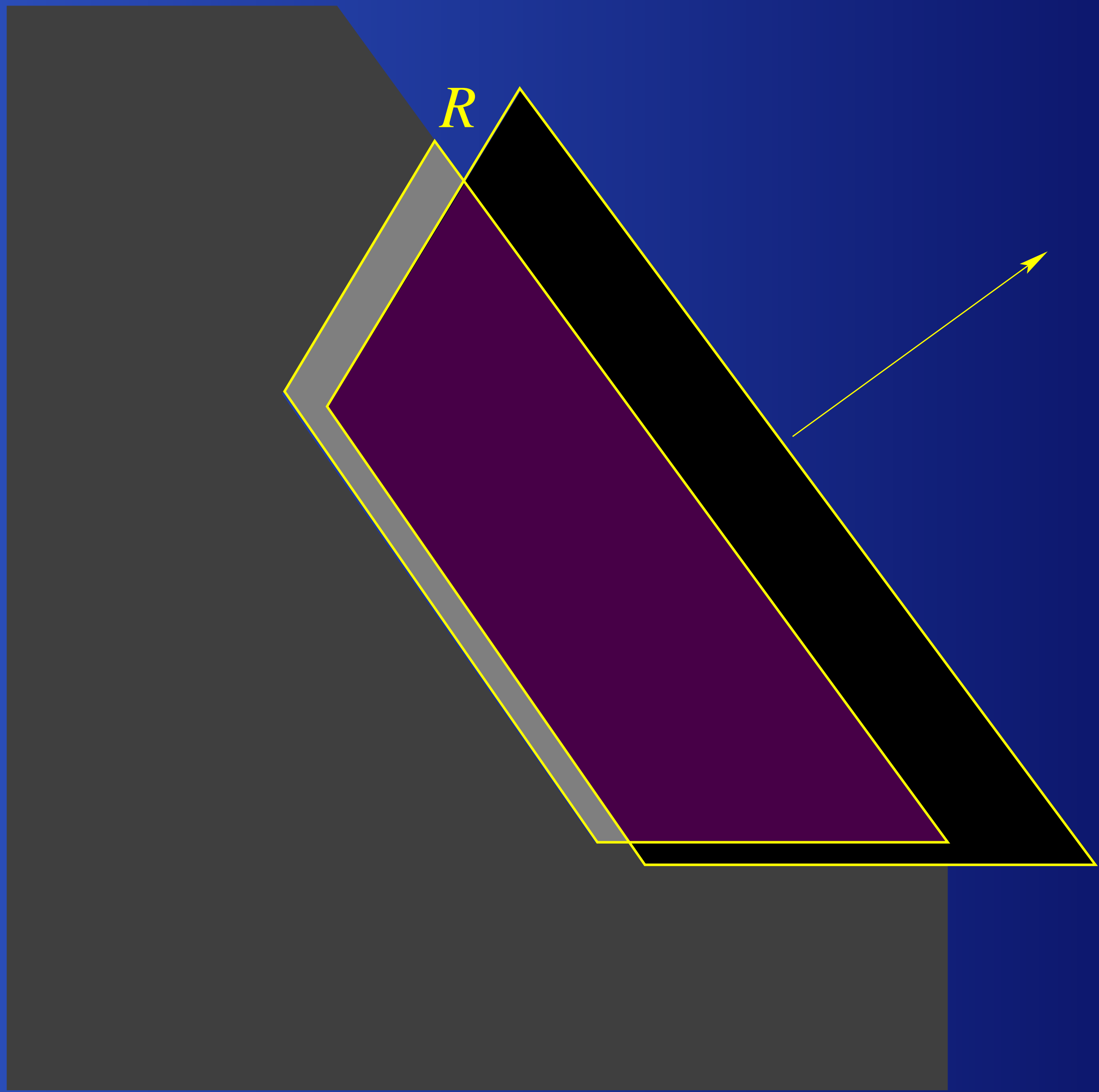

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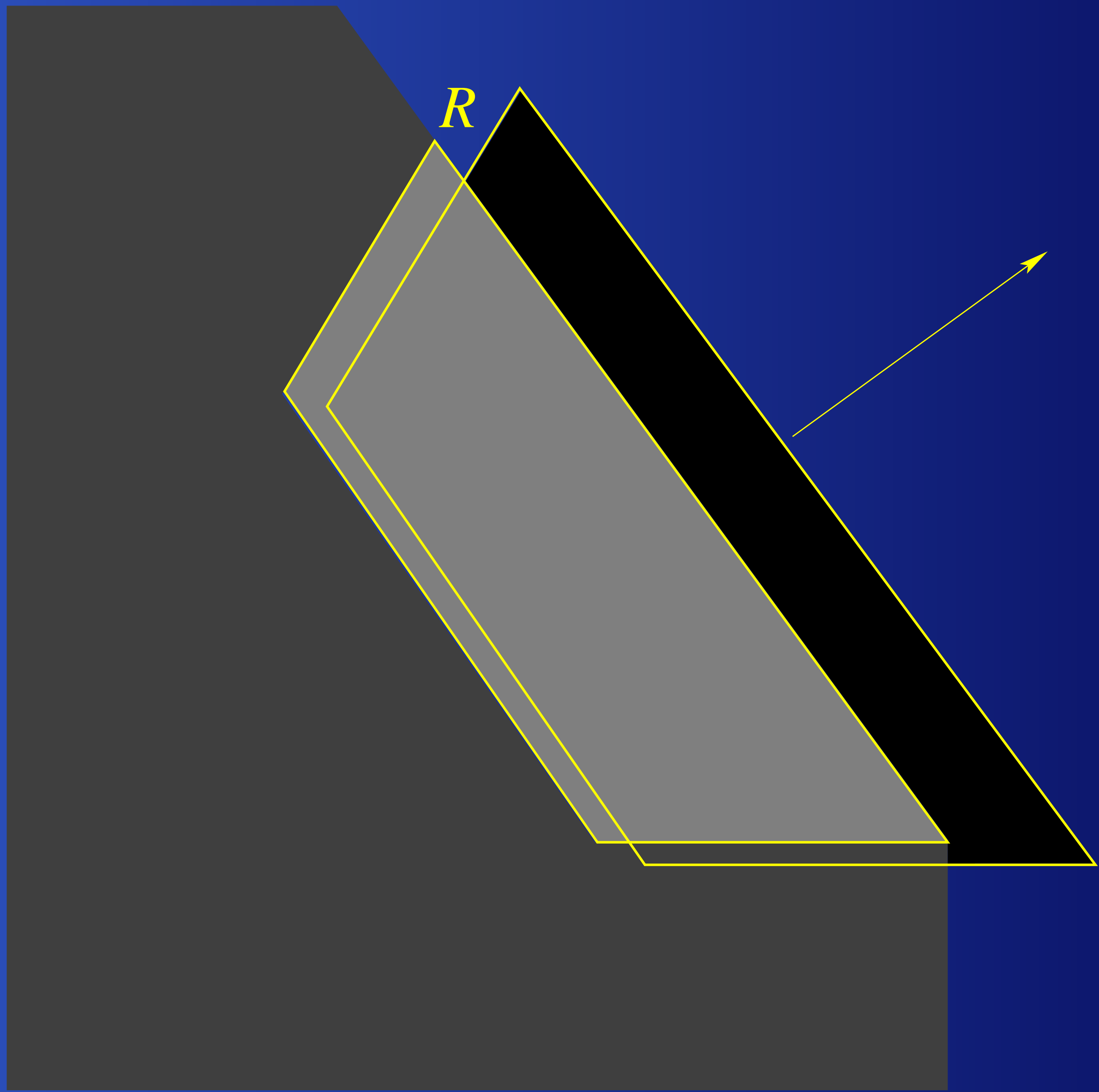

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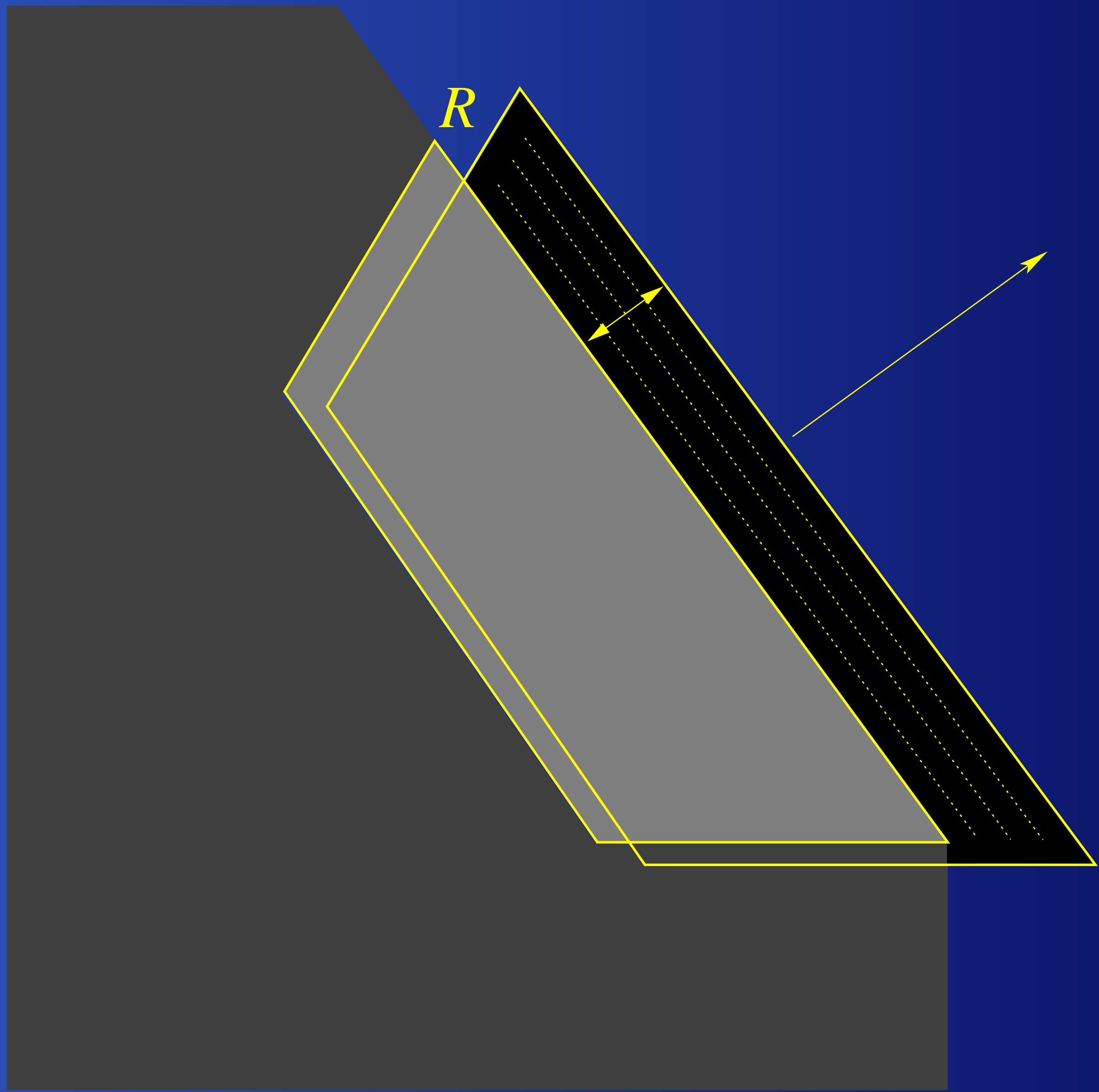


Similar factorization phenomena were recently observed to hold for general vector partition functions by Szenes and Vergne.

The A_3 picture

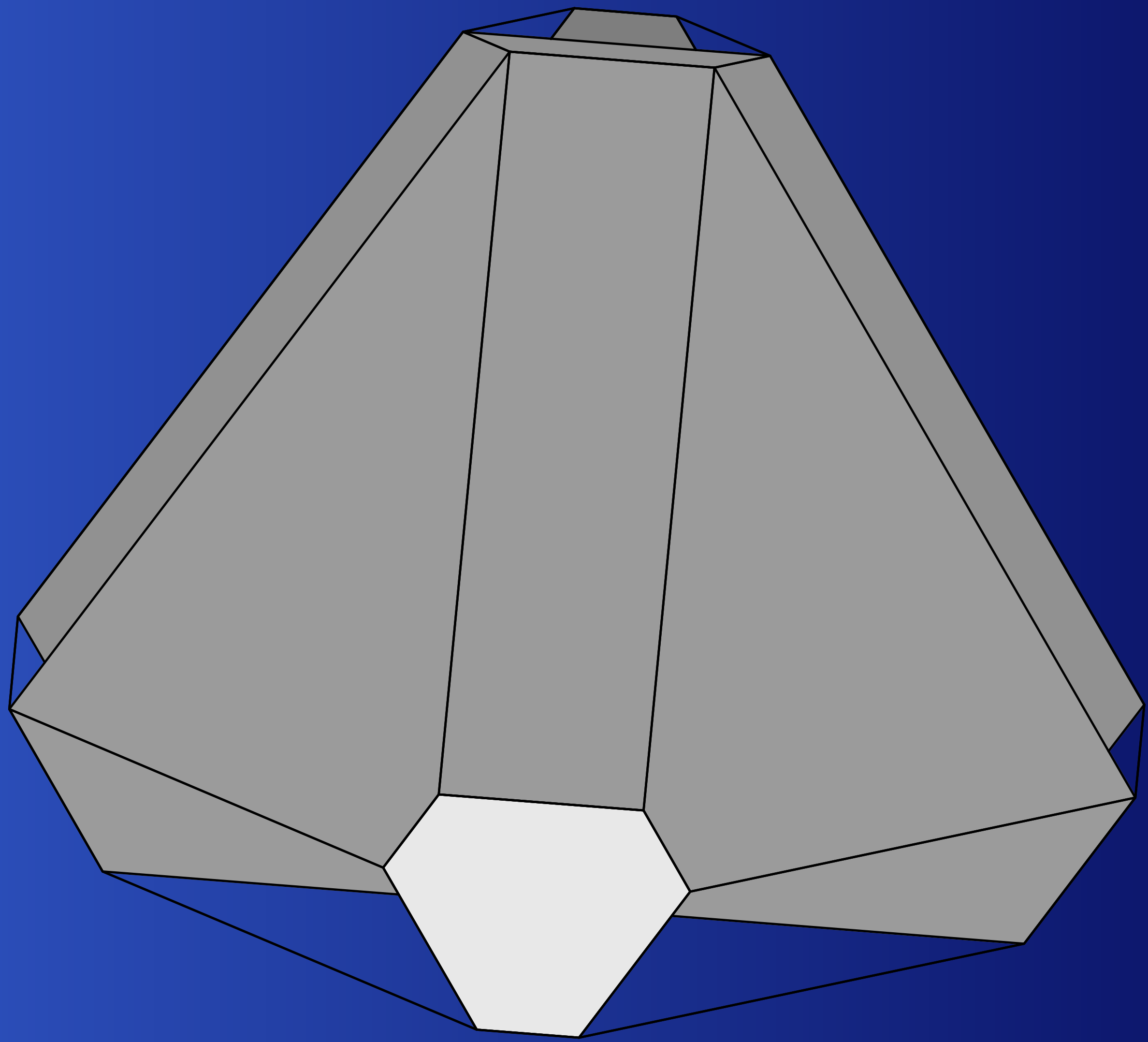
The Duistermaat-Heckman function

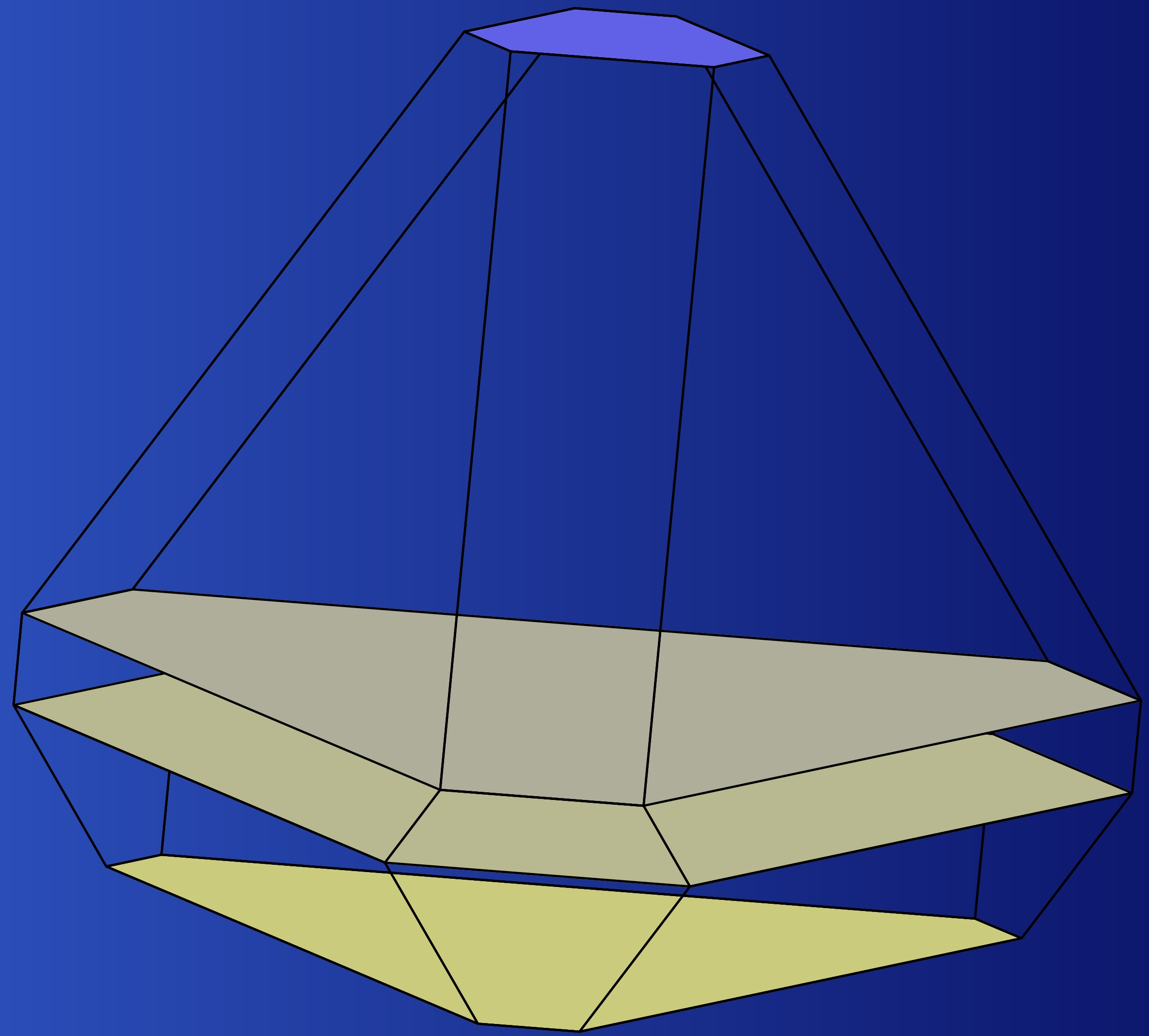
- For every λ there is a function, the Duistermaat-Heckman function, that is piecewise polynomial on $\mathrm{conv}(\mathfrak{S}_k \cdot \lambda)$.

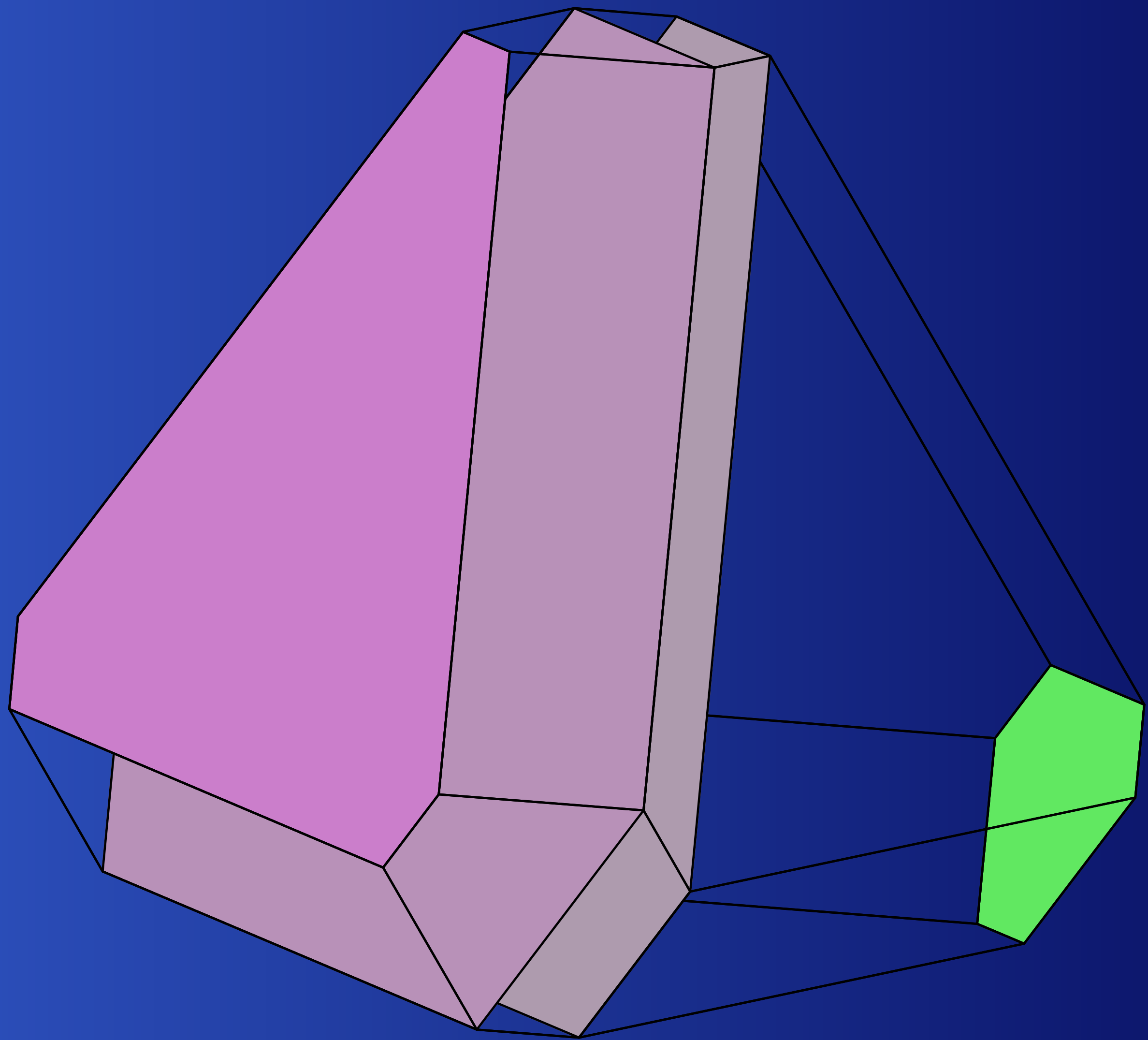
- It approximates the Kostka numbers.

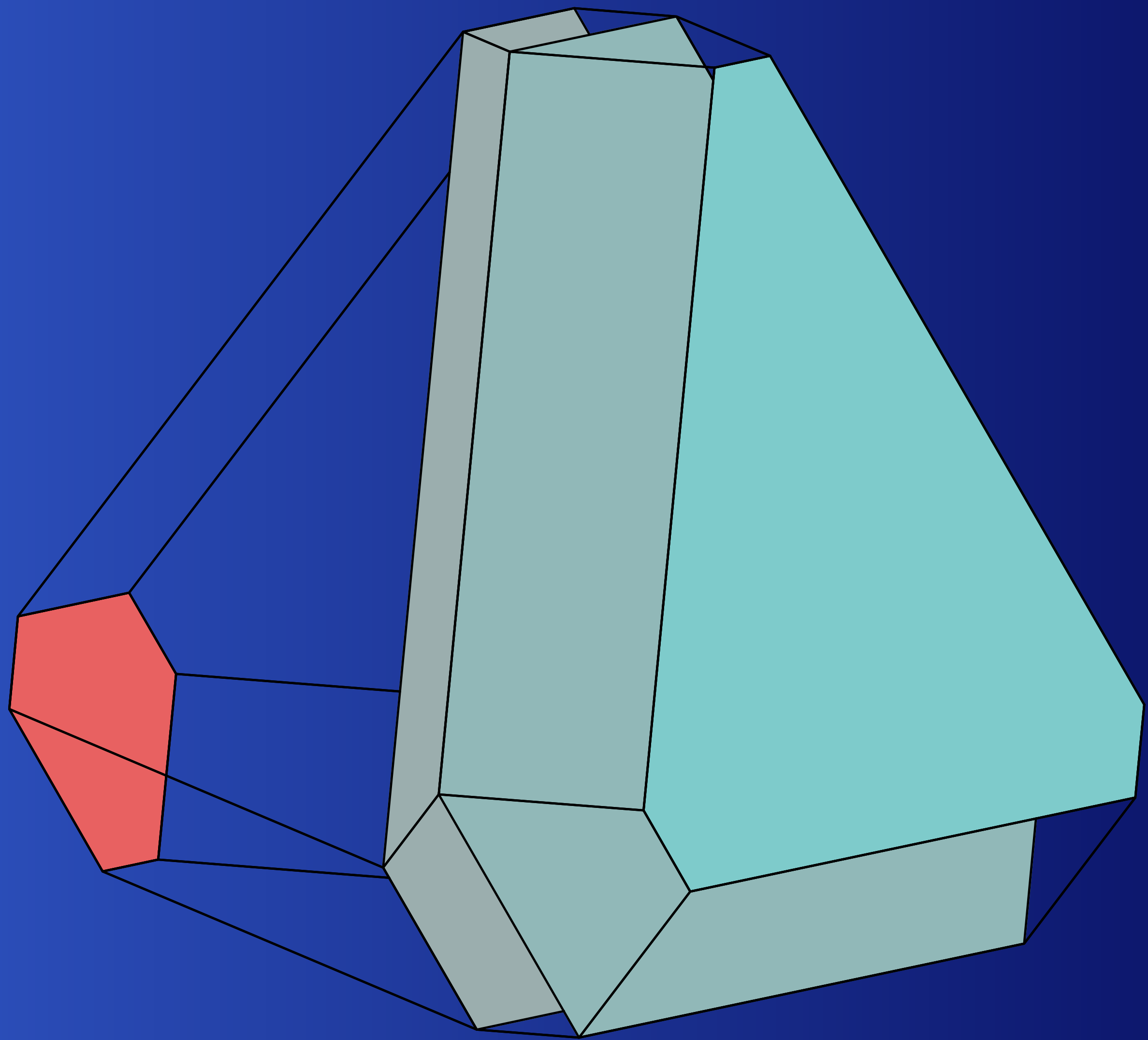
$$K_{\lambda\beta} = \sum_{\sigma\in\mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma)} K(\sigma(\lambda+\delta) - (\beta+\delta)) \,.$$

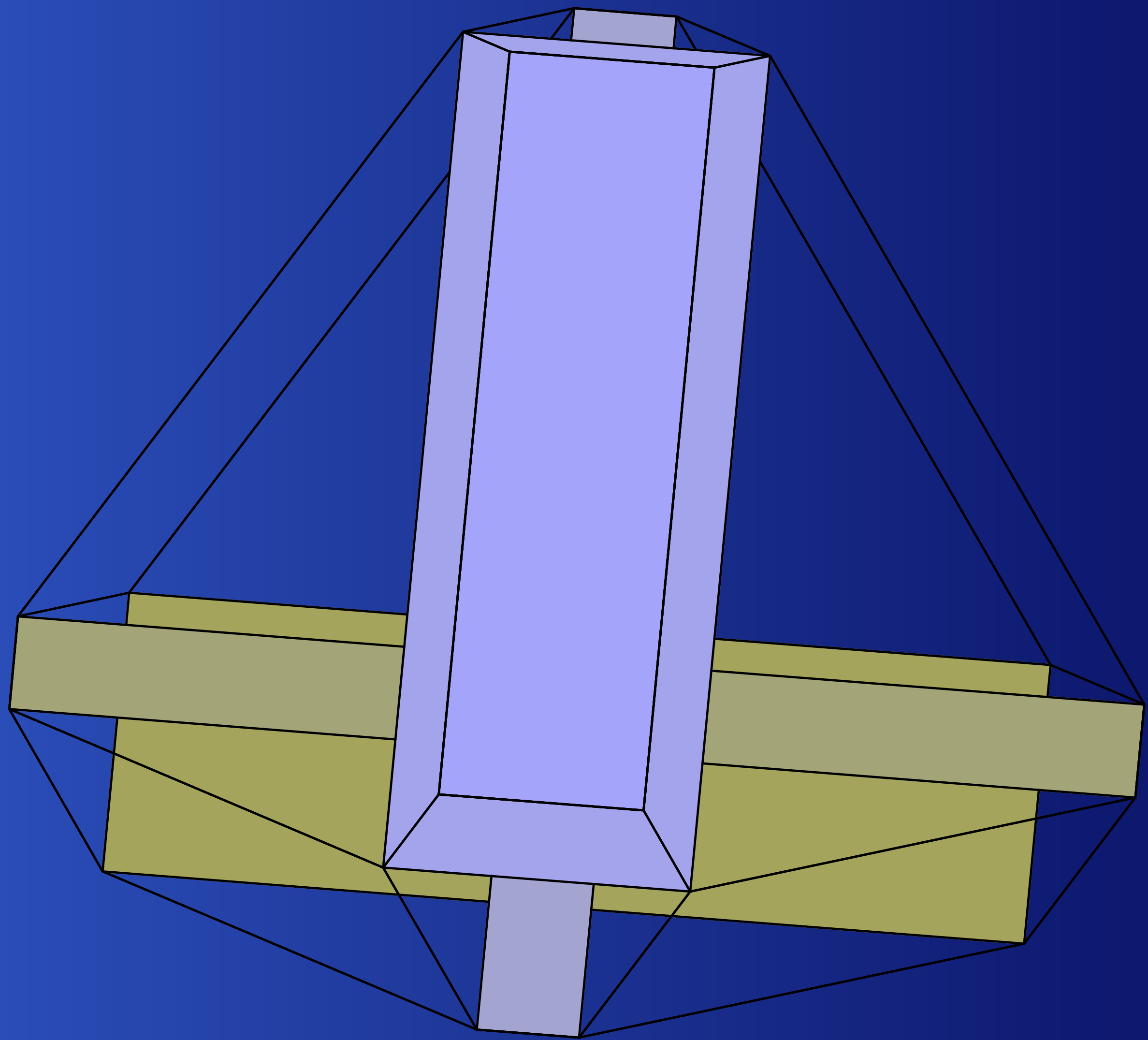
$$f_\lambda^{\mathrm{DH}}(\beta) = \sum_{\sigma\in\mathfrak{S}_k} (-1)^{\mathrm{inv}(\sigma)} \tilde{K}(\sigma(\lambda) - \beta) \,.$$

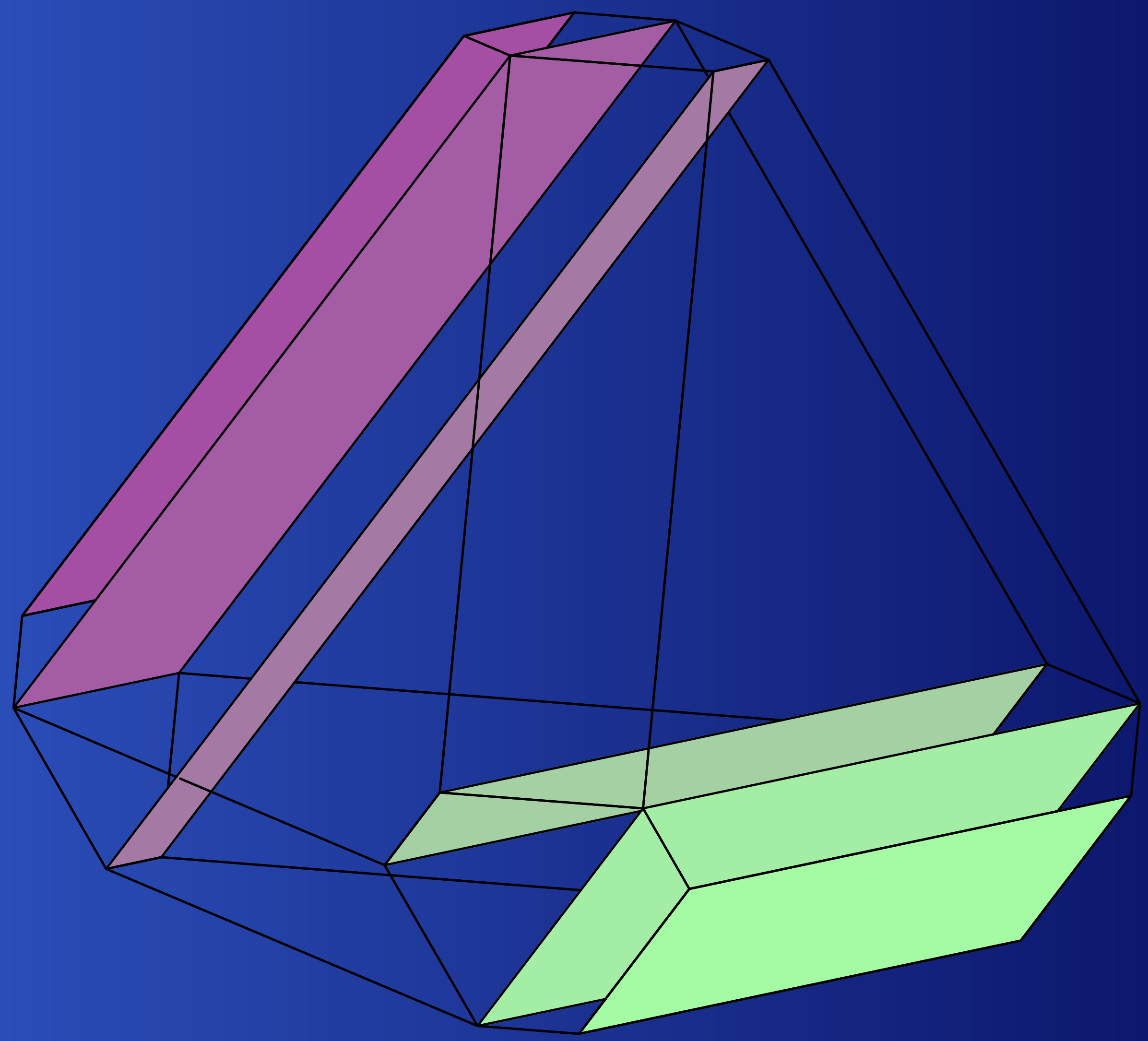


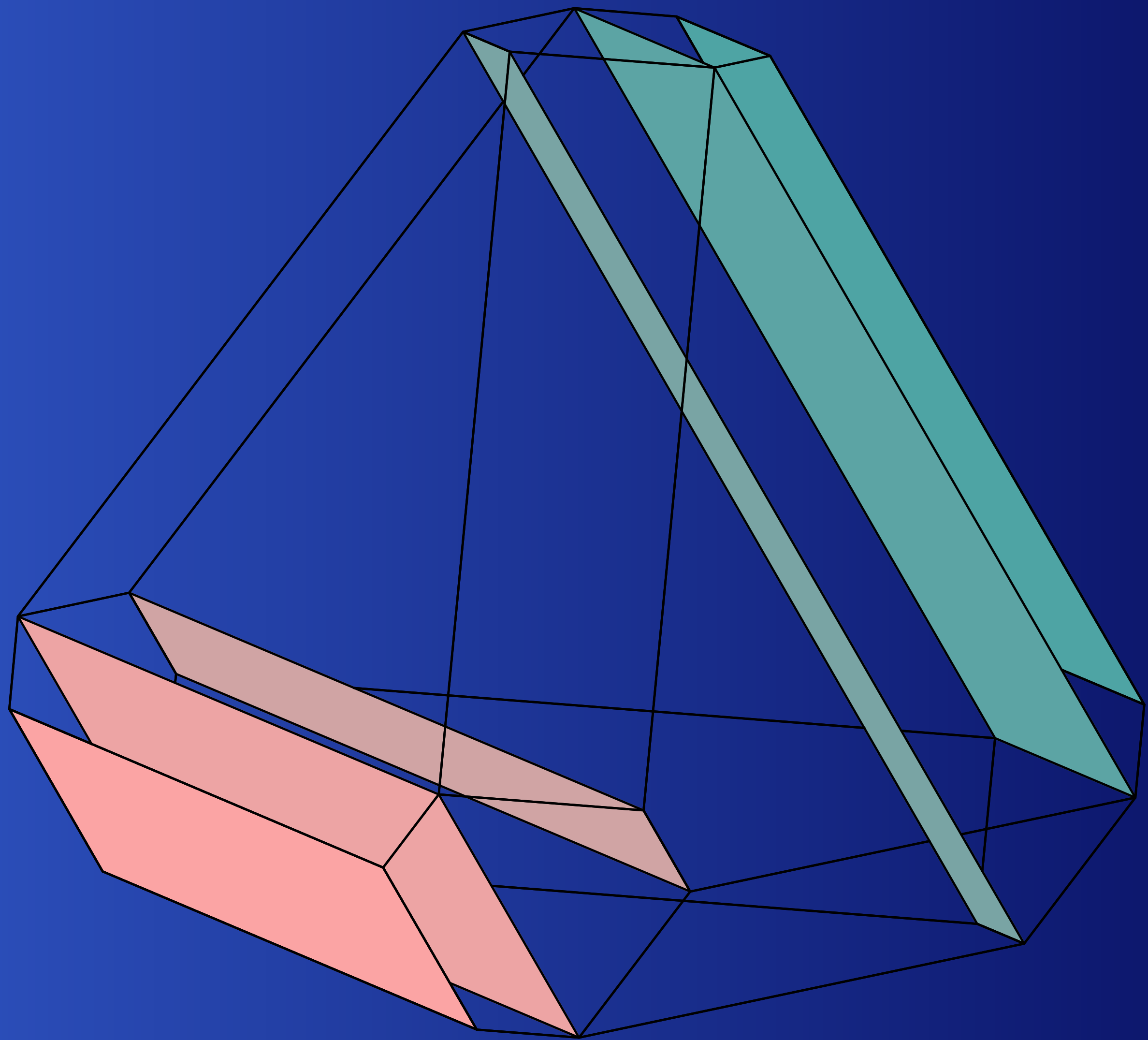


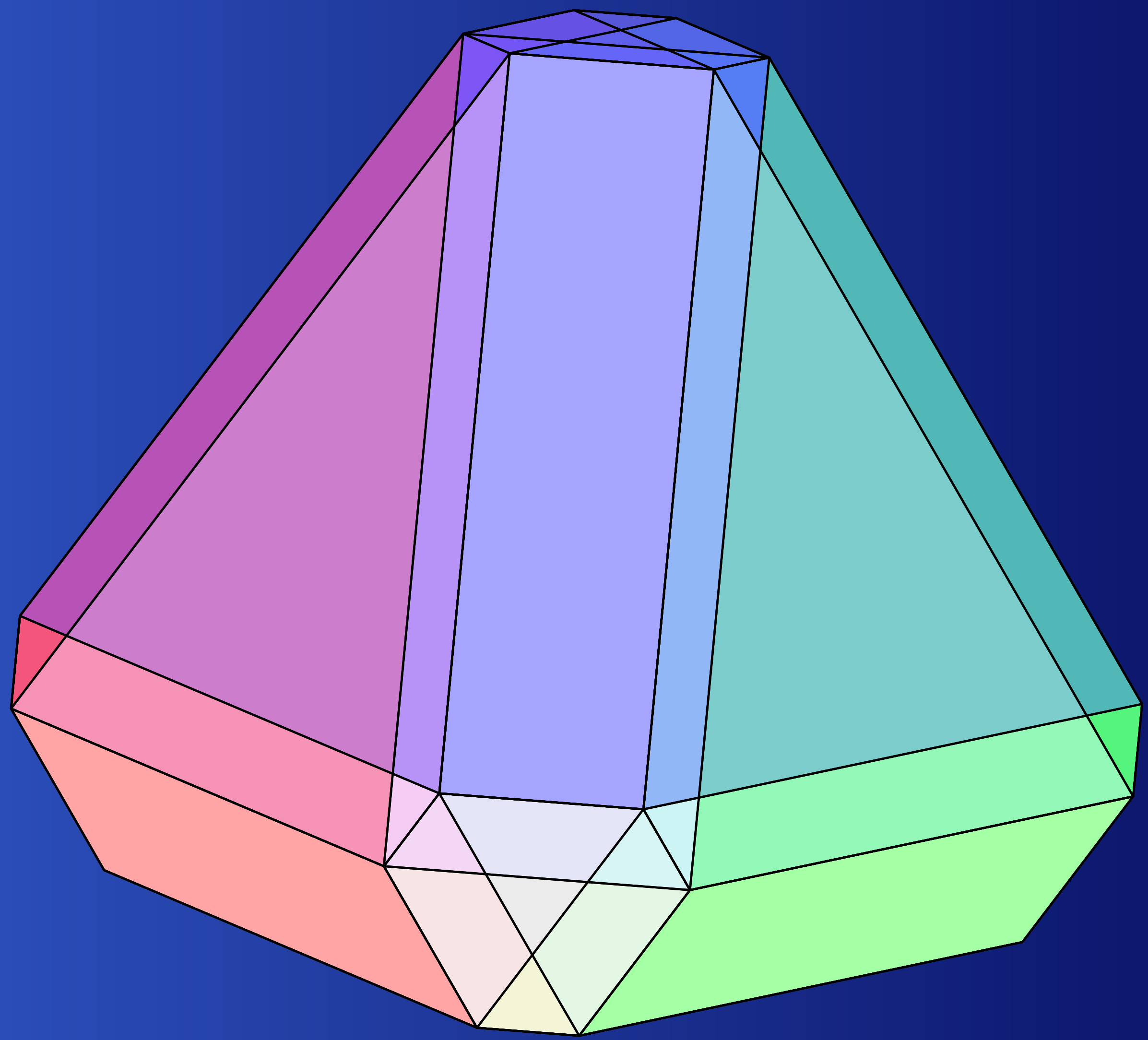












The DH function and the $K_{\lambda\beta}$

Theorem D

The partition of the permutahedron into its domains of polynomiality for the Duistermaat-Heckman function (for fixed λ) also partitions the permutahedron into domains of polynomiality for the Kostka numbers.

Partition function for A_3-multiplicities

$$m_\lambda(\beta) = \phi_{E_4}\left(B_4 \begin{pmatrix} \lambda \\ \beta \end{pmatrix} \right)$$

Partition function for A_3-multiplicities

$$\text{with} \quad E_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Partition function for A_3-multiplicities

and $$B_4\binom{\lambda}{\beta} = \begin{pmatrix} \lambda_1+\beta_1+\beta_2+\beta_3 \\ \lambda_2-\lambda_3 \\ \lambda_1-\lambda_2 \\ \lambda_2-\lambda_3 \\ \lambda_2+2\lambda_3-\beta_1-\beta_2-\beta_3 \\ 2\lambda_3-\beta_1-\beta_2 \\ \lambda_3-\beta_1 \\ \lambda_3-\beta_2 \\ \lambda_2-\beta_3 \end{pmatrix}.$$

Chamber complex for A_3-multiplicities

- E_4 is not unimodular.

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Chamber complex for A_3-multiplicities

- E_4 is not unimodular.
- The chamber complex $\mathcal{C}^{(4)}$ in (λ, β)-space with cones has 1202 top-dimensional cones.
- However, it is not invariant under the action of the symmetric group $\mathfrak{S}_4$ on the β-coordinates.

The glued complex $\mathcal{G}$

Proposition

If we group together the top-dimensional cones from $\mathcal{C}^{(4)}$ with a particular polynomial, their union is always a convex polyhedral cone again.

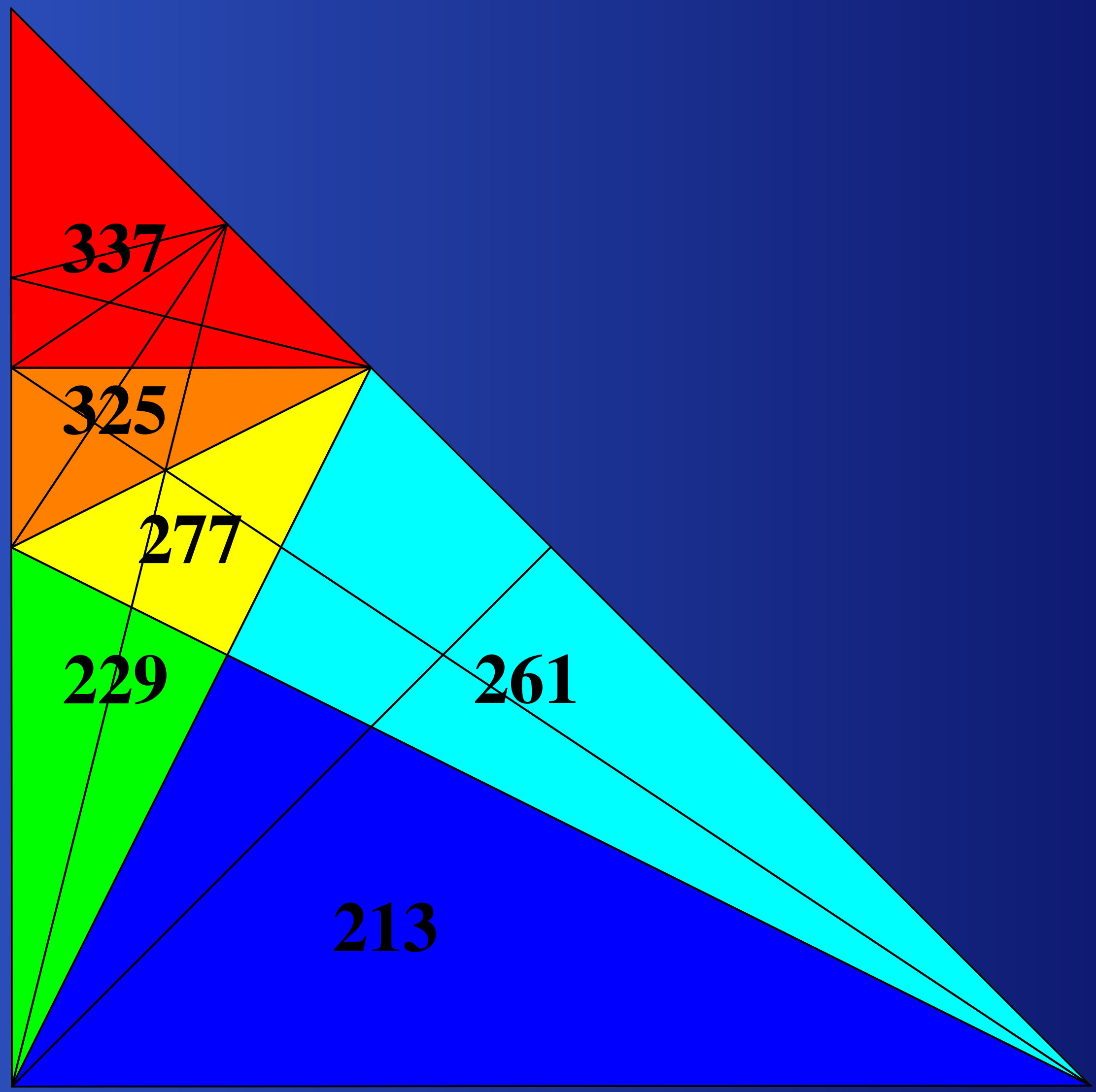
Grouping cones this way yields a glued chamber complex $\mathcal{G}$ in (λ, β)-space with 612 cones. These cones form 64 orbits under the action of $\mathfrak{S}_4$ on the β-coordinates.

The glued complex $\mathcal{G}$

Theorem E

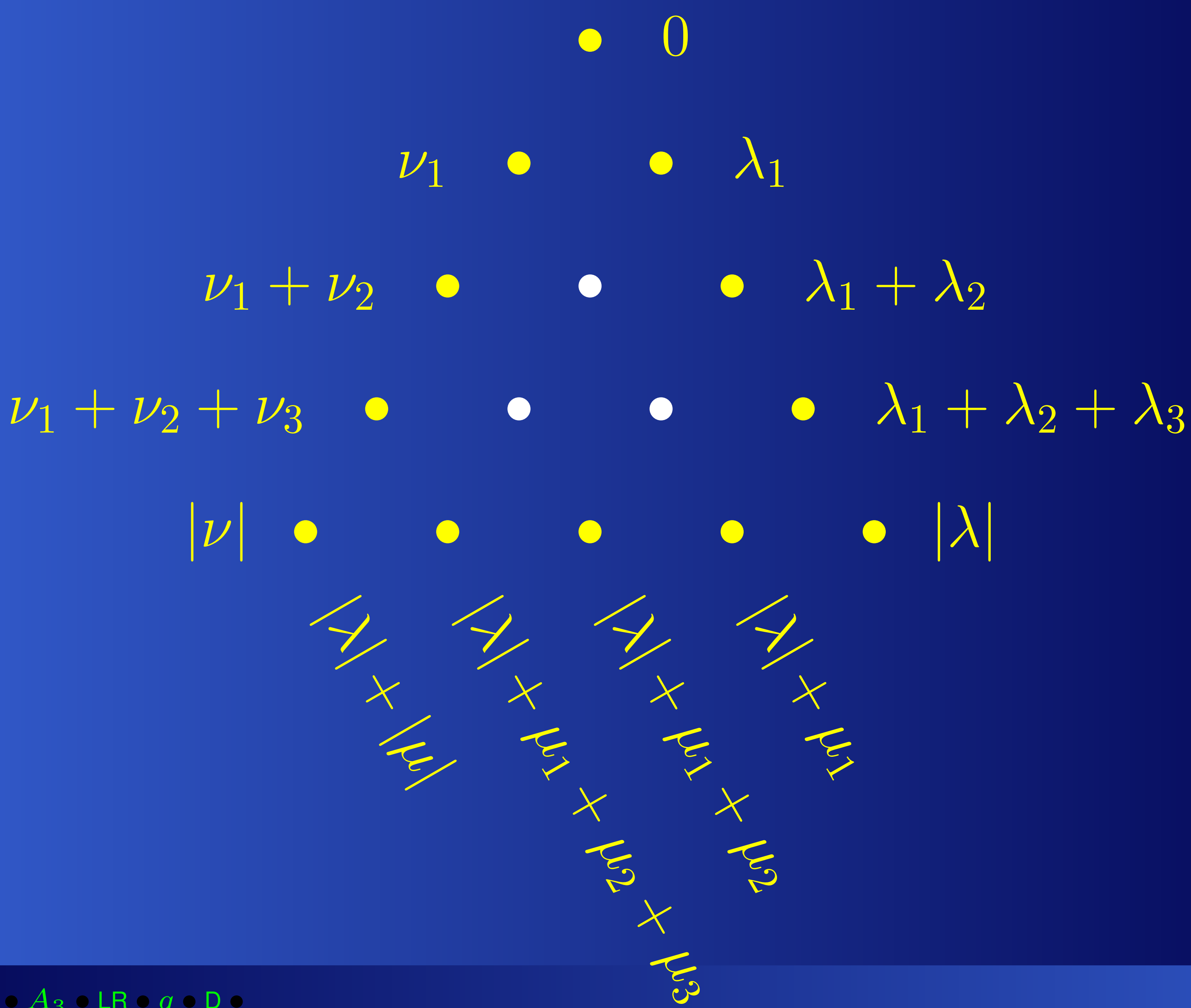
For A_3, the optimal partition of the permutahedron into domains of polynomiality for the weight multiplicities coincides with the partition of the permutahedron into domains of polynomiality for the Duistermaat-Heckman measure.

The complex $\mathcal{G}_\Lambda$



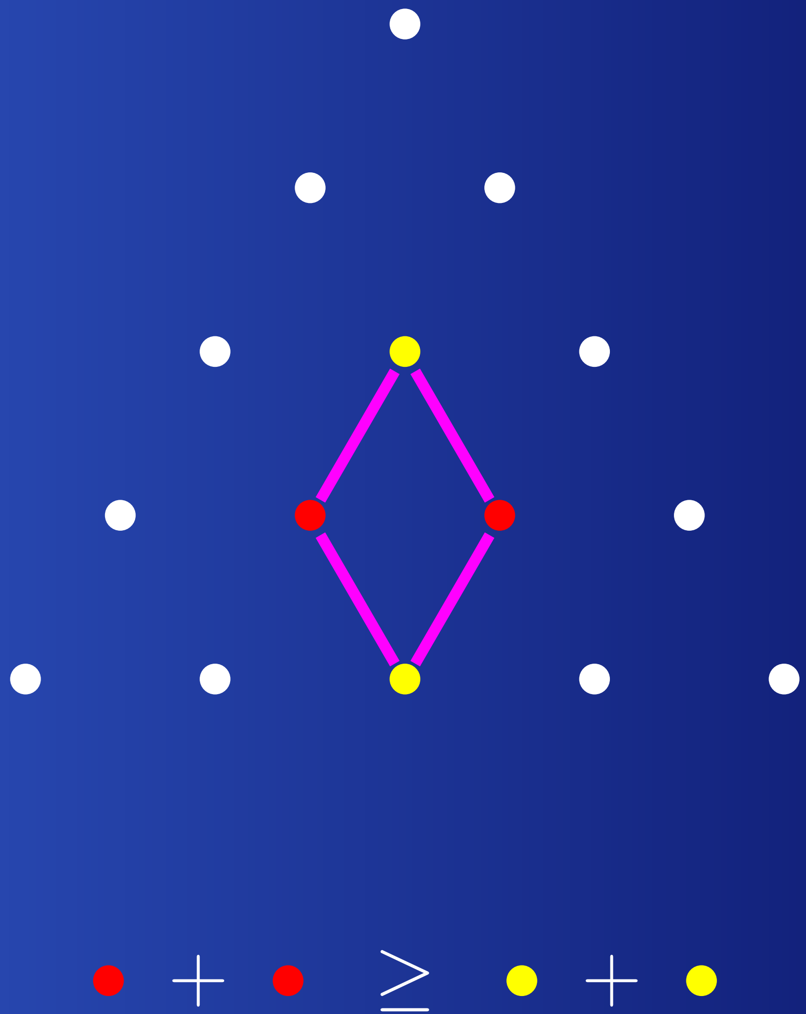
Littlewood-Richardson coefficients

Hives

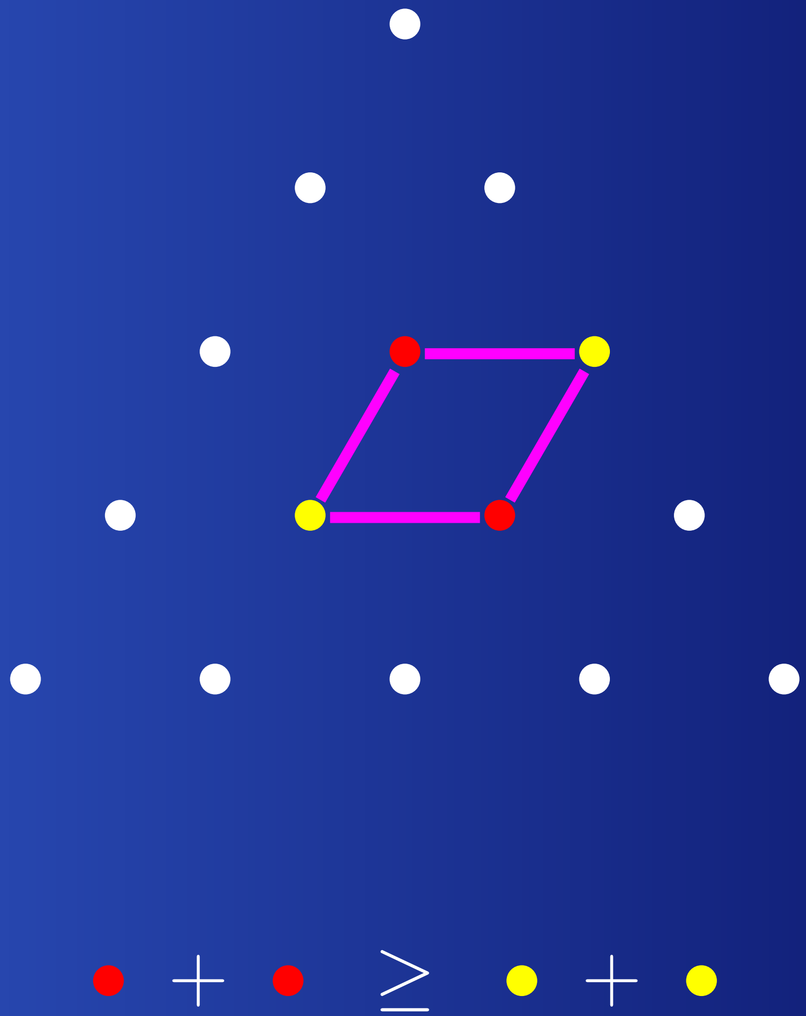


0
ν_1
λ_1
$\nu_1 + \nu_2$
$\lambda_1 + \lambda_2$
$\nu_1 + \nu_2 + \nu_3$
$\lambda_1 + \lambda_2 + \lambda_3$
$|\nu|$
$|\lambda|$
$|\lambda| + |\mu|$
$|\lambda| + \mu_1 + \mu_2 + \mu_3$
$|\lambda| + \mu_1 + \mu_2$
$|\lambda| + \mu_1$

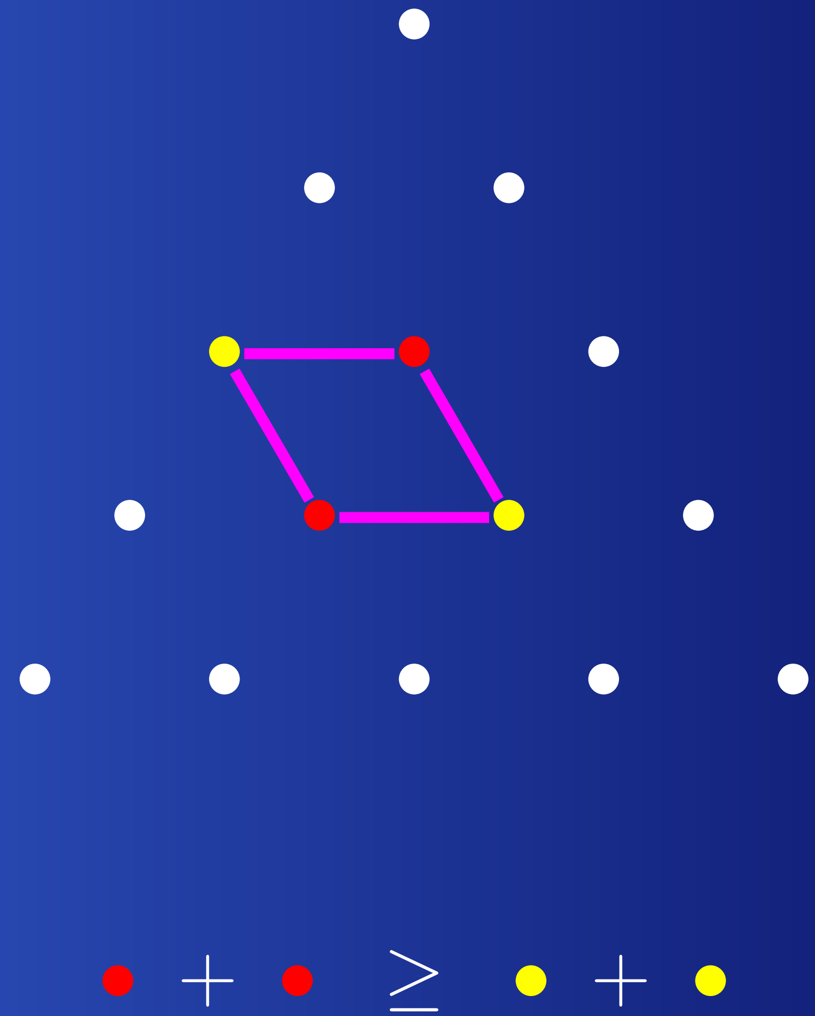
Hive conditions



Hive conditions



Hive conditions



Theorem **(Knutson-Tao, Fulton)**

Let λ, μ and ν be partitions with at most k parts such that $|\lambda| + |\mu| = |\nu|$.

The Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is the number of integral k-hives satisfying the boundary conditions and the hive conditions.

Steinberg's formula

$$c^{\nu}_{\lambda\mu} = \sum_{\sigma\in\mathfrak{S}_k}\sum_{\tau\in\mathfrak{S}_k}(-1)^{\mathrm{inv}(\sigma\tau)}K(\sigma(\lambda+\delta)+\tau(\mu+\delta)-(\nu+2\delta))\,.$$

Stretching for LR coefficients

- This shows in particular that the function

$$N \in \mathbb{N} \quad \longmapsto \quad c_{N\lambda\ N\mu}^{N\nu}$$

  is polynomial in N.

  This was known previously (Derksen-Weyman, Knutson).

- This function is the Ehrhart polynomial of the hive polytope for λ, μ and ν.

A q-analogue of the Kostant partition function

The Kostant partition function

- The Kostant partition function :

$$K(v) = \left|\left\{(k_\alpha)_{\alpha\in\Delta_+} \in \mathbb{N}^{|\Delta_+|} \ : \ \sum_{\alpha\in\Delta_+} k_\alpha \alpha = v\right\}\right|,$$

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- $K(\mu)$ is the number of integer points inside the polytope

$$Q_\mu = \left\{ (k_\alpha)_{\alpha \in \Delta_+} \in \mathbb{R}_{\geq 0}^{|\Delta_+|} \ : \ \sum_{\alpha \in \Delta_+} k_\alpha \alpha = \mu \right\} .$$

Generating function for the $K(\mu)$

$$\sum_{\mu} K(\mu)\, e^{\mu} = \prod_{\alpha \in \Delta_+} \frac{1}{1 - e^{\alpha}}$$

The classical q-analogue

- Lusztig :

$$\tilde{K}_q(\mu) = \sum_{(k_\alpha)_\alpha \in Q_\mu} q^{\sum k_\alpha} \,.$$

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- Lusztig :

$$\tilde{K}_q(\mu) = \sum_{(k_\alpha)_\alpha \in Q_\mu} q^{\sum k_\alpha} .$$

- Generating function :

$$\sum_\mu \tilde{K}_q(\mu)\, e^\mu = \prod_{\alpha\in\Delta_+} \left(1 + qe^\alpha + q^2 e^{2\alpha} + \cdots\right)$$

$$= \prod_{\alpha\in\Delta_+} \frac{1}{1 - qe^\alpha} .$$

Our q-analogue

$$K_q(\mu) = \sum_{(k_\alpha)_\alpha \in Q_\mu} q^{|\{k_\alpha > 0\}|}.$$

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- Generating function :

$$\begin{aligned}
\sum_\mu K_q(\mu)\, e^\mu &= \prod_{\alpha\in\Delta_+} \left(1 + qe^\alpha + qe^{2\alpha} + \cdots\right) \\
&= \prod_{\alpha\in\Delta_+} \left(1 + \frac{qe^\alpha}{1-e^\alpha}\right) \\
&= \prod_{\alpha\in\Delta_+} \frac{1+(q-1)e^\alpha}{1-e^\alpha}.
\end{aligned}$$

Theorem **(Guillemin-Sternberg-Weitsman)**

Let λ be a dominant weight such that $\lambda - \delta$ is also dominant.

Then the multiplicity of the weight ν in the tensor product $V_{\lambda-\delta} \otimes V_\delta$ is given by

$$\dim \left(V_{\lambda-\delta} \otimes V_\delta\right)_\nu = \sum_{\omega \in \mathcal{W}} (-1)^{|\omega|} K_2(\omega(\lambda) - \nu)\,.$$

Theorem F

For the root system A_n, the q-analogue $K_q(\mu)$ is given by polynomials of degree $\binom{n}{2}$ with coefficients in $\mathbb{Q}[q]$ of degree $\binom{n+1}{2}$ over the relative interior of the cells of the chamber complex for the usual Kostant partition function.

An example: A_2

Two top-dimensional cones:

$$\tau_1 = \{a_1\alpha_1 + a_2\alpha_2 \ : \ a_1, a_2 > 0 \text{ and } a_1 > a_2\},$$
$$\tau_2 = \{a_1\alpha_1 + a_2\alpha_2 \ : \ a_1, a_2 > 0 \text{ and } a_1 < a_2\},$$

Three 1-dimensional cones:

$$\tau_3 = \{a(\alpha_1 + \alpha_2) \ : \ a > 0\},$$
$$\tau_4 = \{a_1\alpha_1 \ : \ a_1 > 0\},$$
$$\tau_5 = \{a_2\alpha_2 \ : \ a_2 > 0\},$$

One 0-dimensional cone

$$\tau_6 = \{0\}.$$

An example: A_2

For $\mu = (\mu_1, \mu_2, \mu_3)$ in the root lattice

(in particular, $\mu_1 + \mu_2 + \mu_3 = 0$)

$$K_q(\mu) = \begin{cases} (\mu_1 + \mu_2 - 1)q^3 + 2q^2 & \text{if } \mu \in \tau_1\,, \\ (\mu_1 - 1)q^3 + 2q^2 & \text{if } \mu \in \tau_2\,, \\ (\mu_1 - 1)q^3 + q^2 + q & \text{if } \mu \in \tau_3\,, \\ q & \text{if } \mu \in \tau_4 \text{ or } \mu \in \tau_5\,, \\ 1 & \text{if } \mu \in \tau_6\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Directions