FRACTIONAL DISTORTION IN HYPERBOLIC GROUPS

TIMOTHY RILEY

WORK WITH PALLAVI DANI

BOB GILMAN BIRTHDAY CONFERENCE

STEVENS INTITUTE, 14 JUNE 2023

GROUPS $H \leq G$, BOTH FINITELY GENERATED.

$$\mathrm{DIST}_H^G(n) = \max\{|h|_H \mid h \in H,\ |h|_G \leq n\}$$

INDEPENDENT OF GENERATING SETS UP TO —

$$f \precsim g \iff \exists C > 0,\ \forall n,\ f(n) \leq Cg(Cn+C) + Cn + C.$$

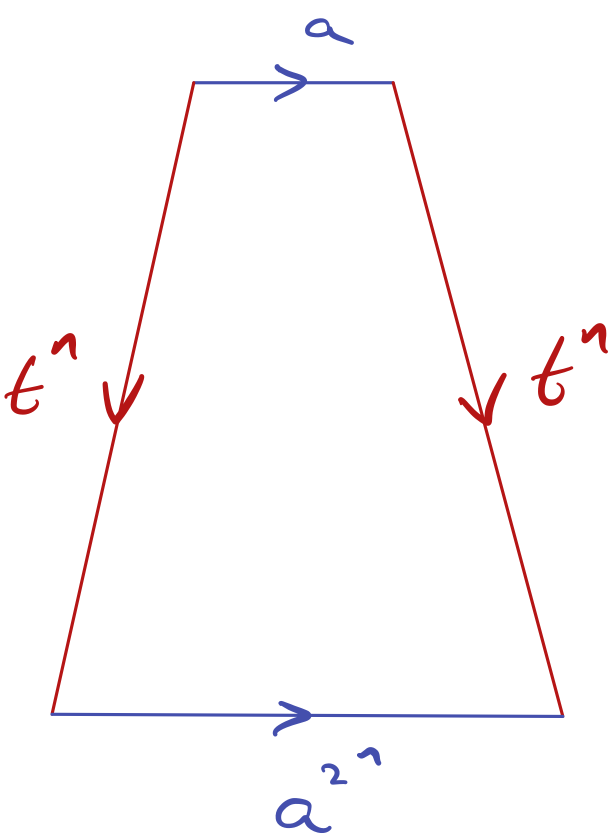
$$f \simeq g \iff f \precsim g \text{ AND } g \precsim f.$$

- $2^n \simeq \lambda^n \quad \forall \lambda > 1$
- For $\alpha, \beta \geq 1$, $n^\alpha \simeq n^\beta \iff \alpha = \beta$

$$2^{n^\alpha} \simeq 2^{n^\beta} \iff \alpha = \beta$$

Example

$$H = \langle a \rangle \leq \langle a, t \mid t^{-1} a t = a^2 \rangle = G$$



$$\mathrm{Dist}_H^G(n) \simeq 2^n$$

$$G_2 = \langle a, t, s \mid t^{-1} a t = a^2,\ s^{-1} t s = t^2 \rangle$$

$$\mathrm{Dist}_H^{G_2}(n) \simeq 2^{2^n}$$

OLSHANSKII – SAPIR 1998 THE SET OF DIST. FUNCTIONS FOR $H \leq F_2 \times F_2$ IS (UP TO $\simeq$) THE SET OF DEHN FUNCTIONS OF F.P. GROUPS.

MIKHAILOVA TRICK

- BRADY – BRIDSON 2000
- SAPIR – BIRGET – RIPS 2002
- BIRGET – OLSHANSKII – RIPS – SAPIR 2002

WHAT DISTORTION FUNCTIONS OCCUR FOR

$$\left\{ \begin{array}{l} \text{FINITELY GENERATED} \\ \text{FINITELY PRESENTED} \\ \text{HYPERBOLIC} \\ \text{FINITE-RANK FREE} \end{array} \right\}$$

SUBGROUPS OF HYPERBOLIC GROUPS ?

I. KAPOVICH (1999) AND FOLLOWING

F_2 SUBGROUPS $\rightsquigarrow$ TORSION-FREE NON-ELEMENTARY HYPERBOLIC SUBGROUPS

$\mathrm{DIST}_H^G(n)$ IN HYPERBOLIC GROUPS G —

n = QUASI-CONVEX = UNDISTORTED = SUBEXPONENTIALLY DISTORTED

$\mathbb{Z}$ - SUBGROUPS ARE UNDISTORTED

I. KAPOVICH 2001, GROMOV

GAP

$\exp(n)$ F_k IN HYPERBOLIC $F_k \rtimes \mathbb{Z}$

$\exp^k(n)$, $k = 1, 2, 3, \dots$

$\exp^n(1)$

MJ 1998, BARNARD–BRADY–DANI 2007

Akermann $A_k(n)$, $k = 1, 2, 3, \dots$ HYPERBOLIC HYDRA BRADY–DISON–R. 2013

NON-RECURSIVE — SELA 1993

RIPS CONSTRUCTION: $1 \to H \to G \to Q \to 1$

THEOREM (DANI–R.)

$\forall$ integers $p > q \geq 1$, $\exists\, F_3 = H \leq G$ hyperbolic with $\mathrm{Dist}_H^G(n) \simeq 2^{n^{p/q}}$.

Also

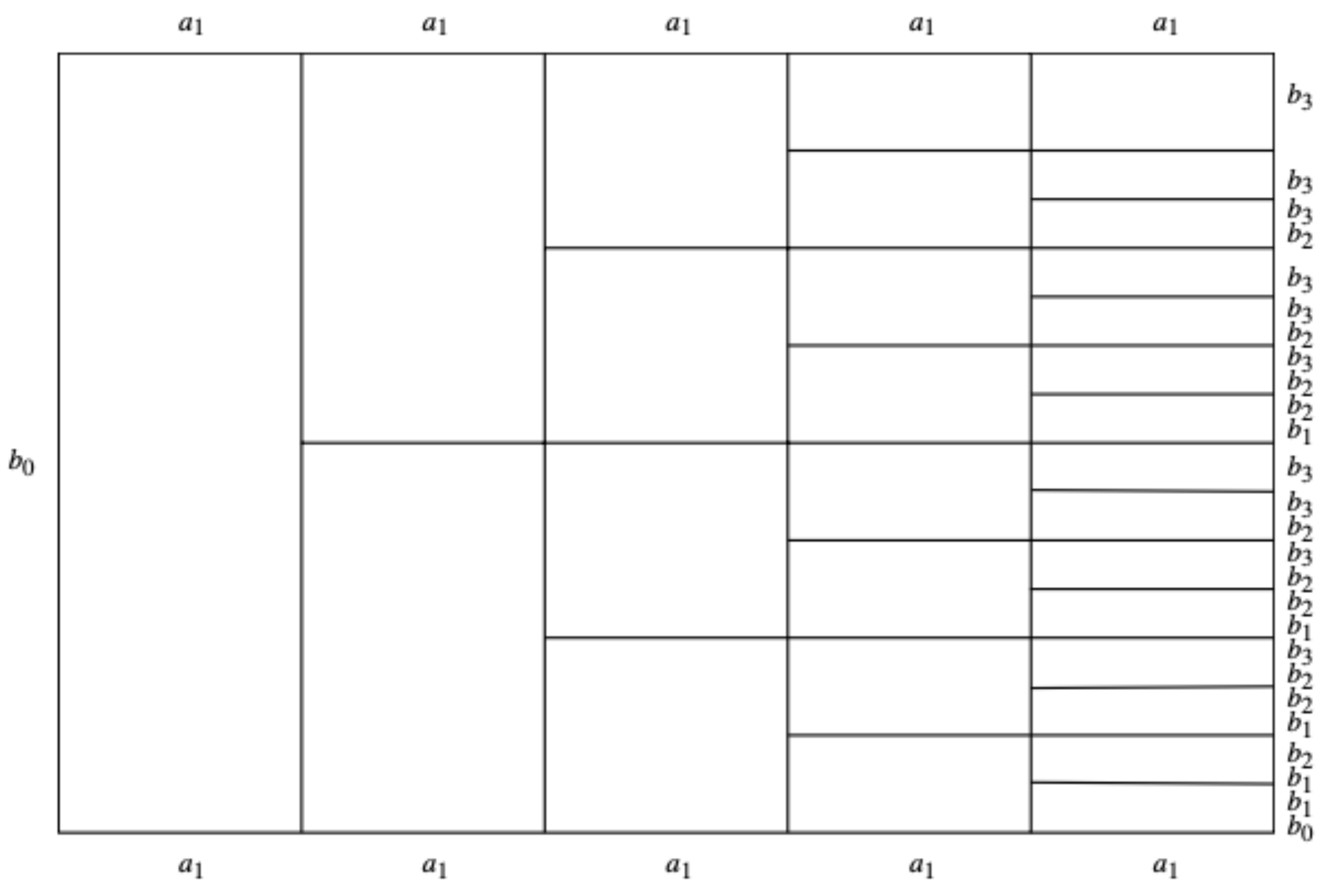
- uniform $C'(1/6)$
- $CAT(-1)$
- residually finite
- infinite height
- virtually special.

$F(b_0,\dots,b_p) \rtimes_\varphi \langle a_1 \rangle$

Target: $\mathrm{Dist}_H^G(n) \simeq 2^{n^{p/q}}$

$$\varphi: \begin{aligned} b_0 &\mapsto b_1 b_0 \\ &\vdots \\ b_{p-1} &\mapsto b_p b_{p-1} \\ b_p &\mapsto b_p \end{aligned}$$

$$|a_1^{-n} b_0 a_1^n|_F \simeq n^p$$

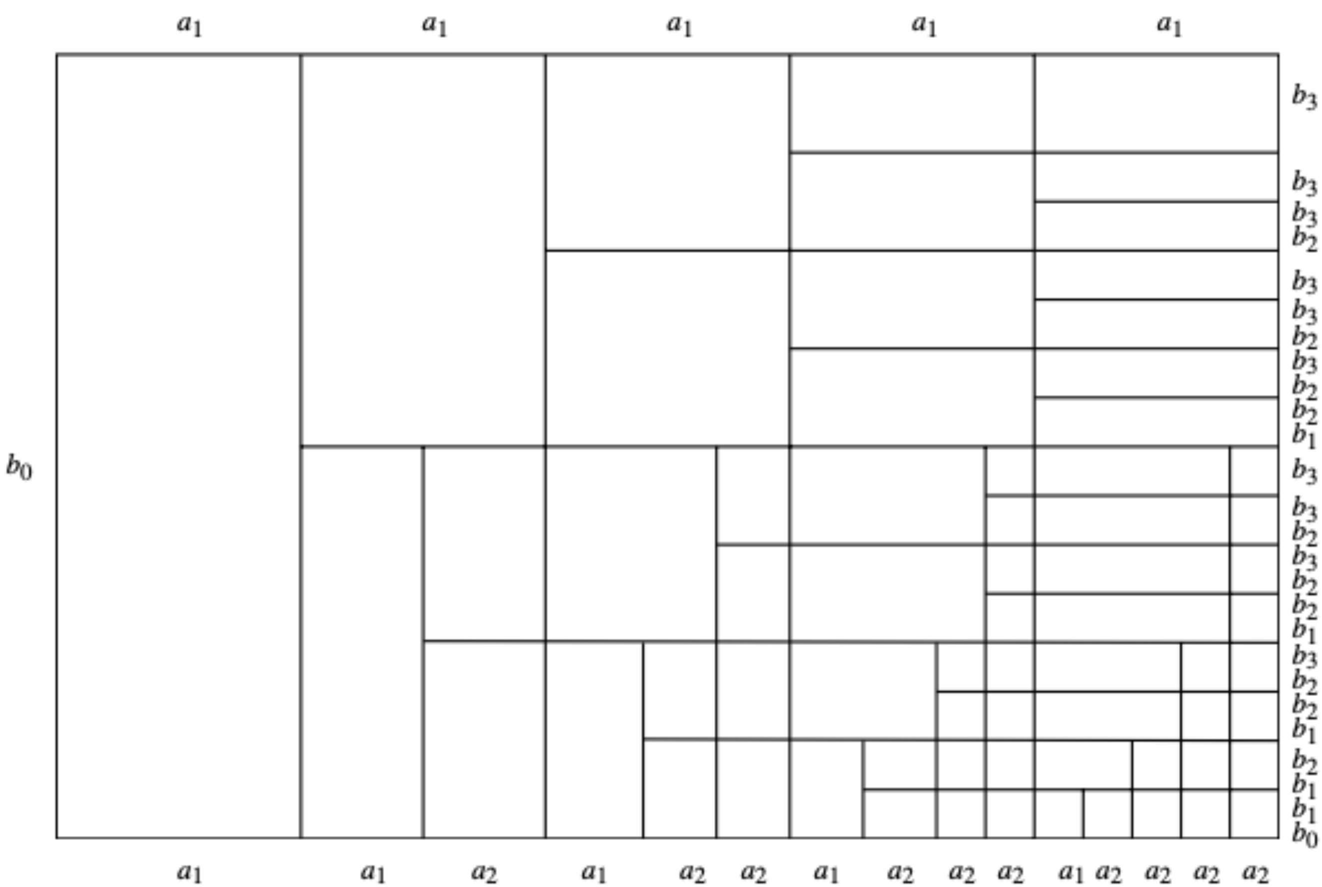


Add a_2 such that

$$a_1^{-1} b_{q-1} a_1 a_2 = b_q b_{q-1}$$

$$[b_i, a_2] = 1 \quad \forall i$$

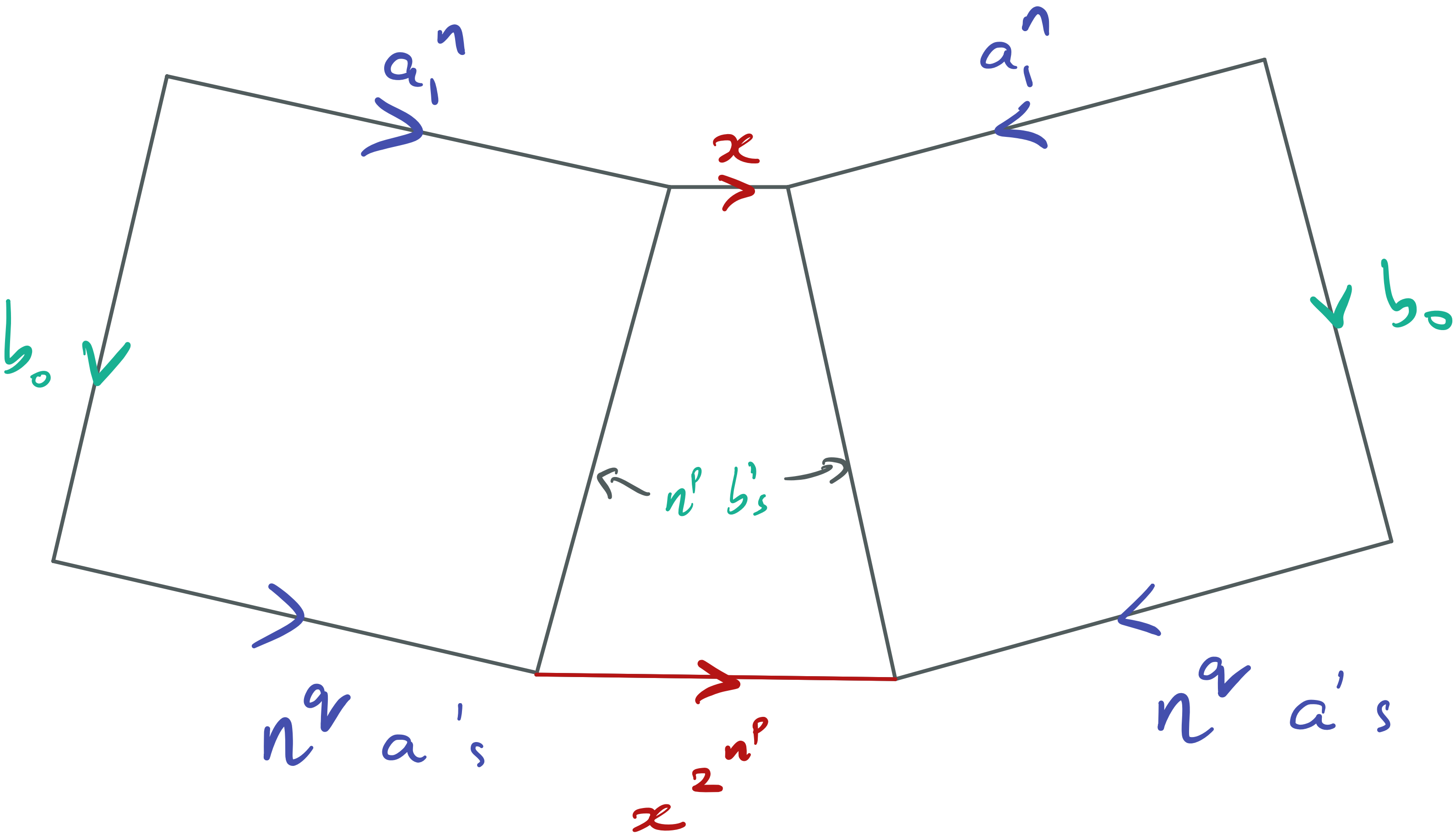
Target: $\mathrm{Dist}_H^G(n) \simeq 2^{n^{p/q}}$



Add x such that

$b_i^{-1} x b_i = x^2 \quad \forall i$

Target: $\mathrm{Dist}_H^G(n) \simeq 2^{n^{p/q}}$

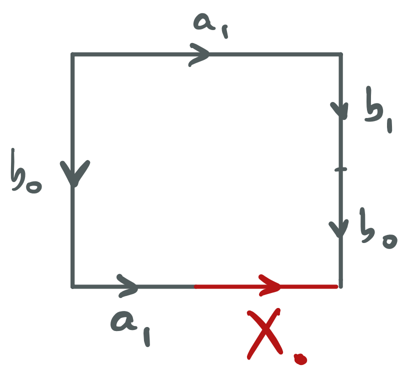


$$n^q \longmapsto 2^{n^p} \equiv n \longmapsto 2^{n^{p/q}}$$

"Hyperbolize":

Target: $\mathrm{Dist}_H^G(n) \simeq 2^{n^{p/q}}$

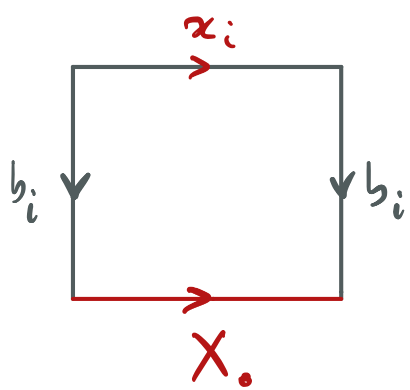
Add "noise" to achieve $C'(1/6)$.



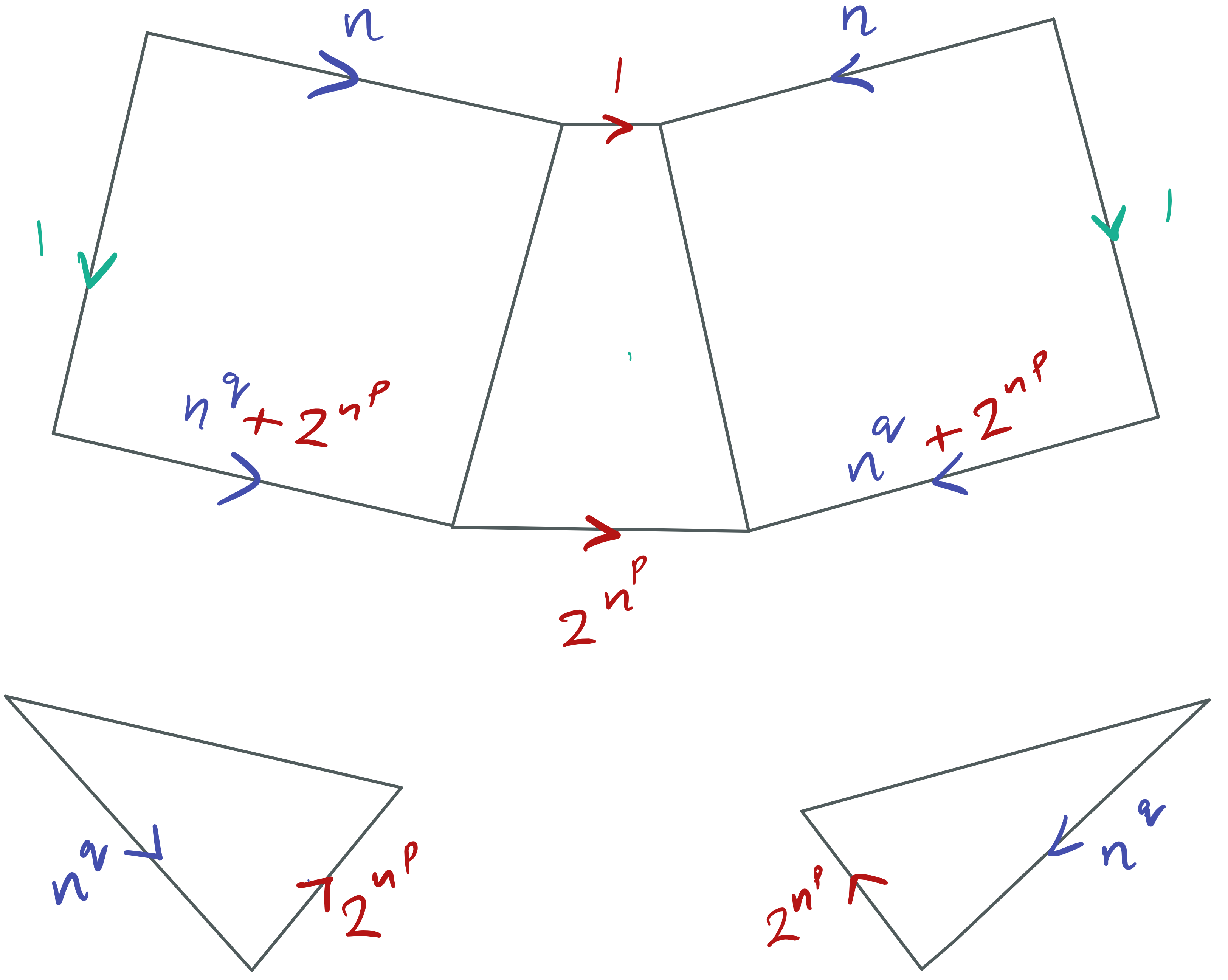
X_0 = long word on x_1, x_2

Rips words: subwords of

$$x_1 x_2 \, x_1 x_2^2 \, x_1 x_2^3 \, x_1 x_2^4 \ldots\ldots$$

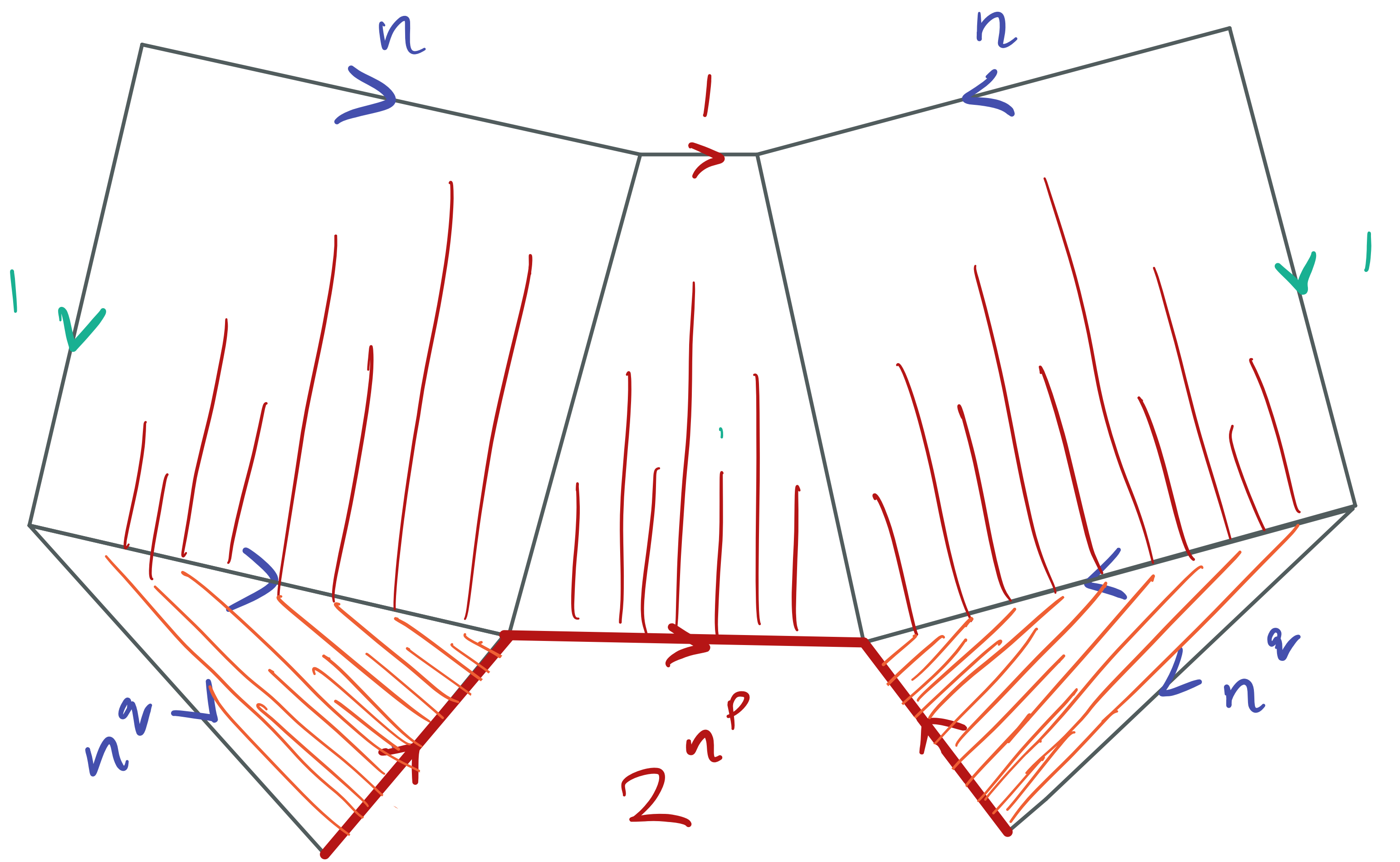


Target: $\mathrm{Dist}_H^G(n) \simeq 2^{n^{p/q}}$



Separate a's from noise.

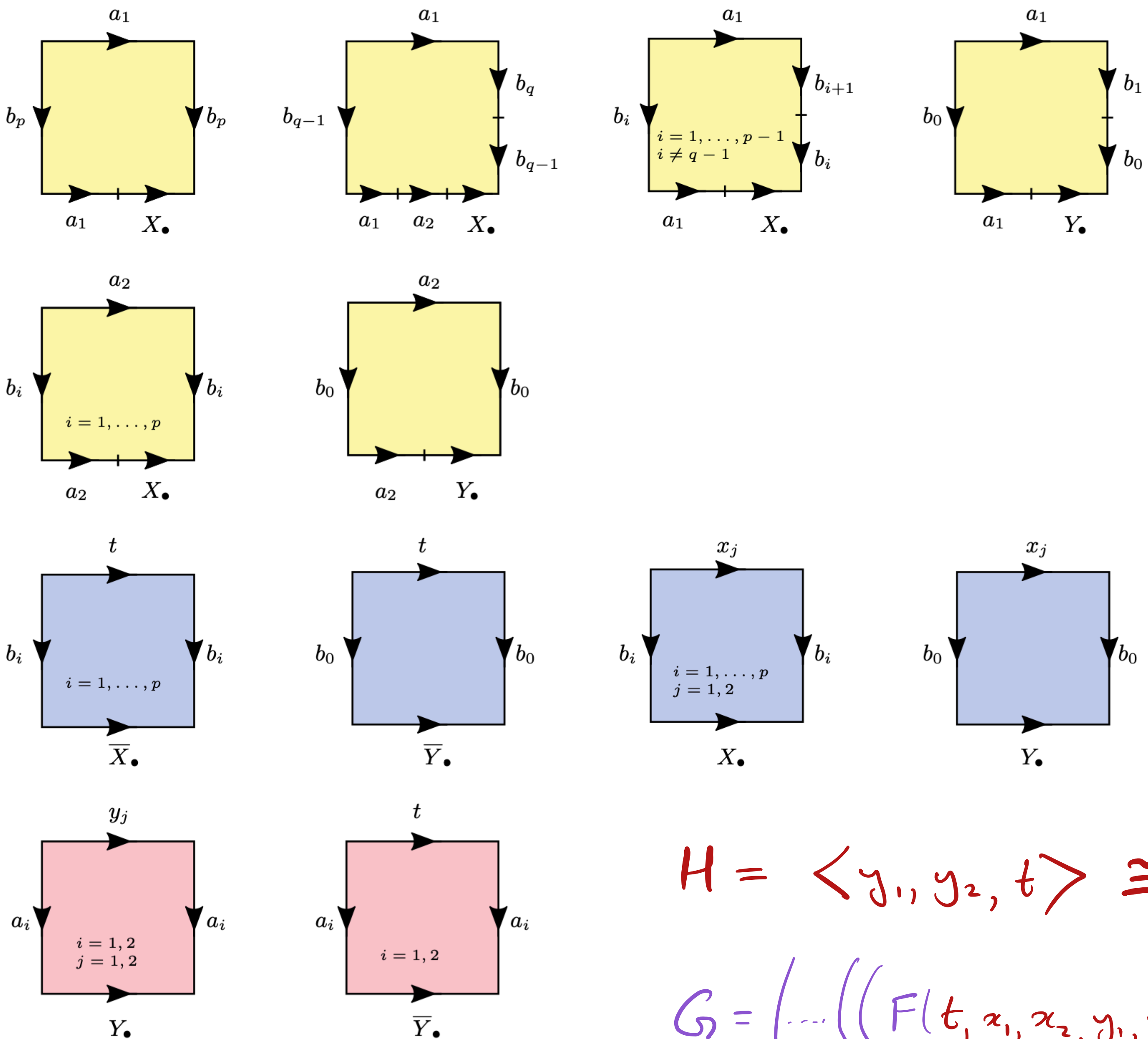
Target: $\mathrm{Dist}_H^G(n) \simeq 2^{n^{p/q}}$



$\frac{p}{q} \lneq \frac{p-1}{q-1}$, so $2^{n^{p/q}} \lneq 2^{n^{\frac{p-1}{q-1}}}$

Fix: Use 2 types of noise.

Target: $\mathrm{Dist}_H^G(n) \simeq 2^{n^{p/q}}$



Rips – Wise

$G = F *_t$

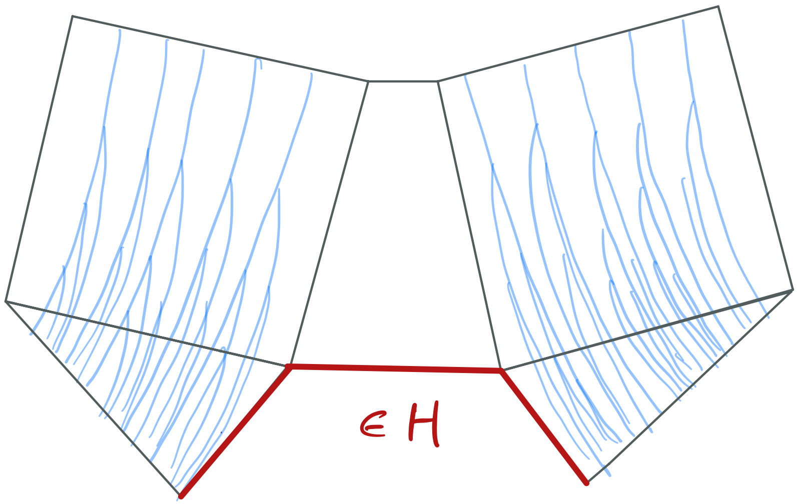
$H = \langle y_1, y_2, t \rangle \cong F_3$ because

$G = \left(\cdots \left(\left(F(t, x_1, x_2, y_1, y_2) *_{a_1, a_2} \right) *_{b_p} \right) \cdots \right) *_{b_0}$

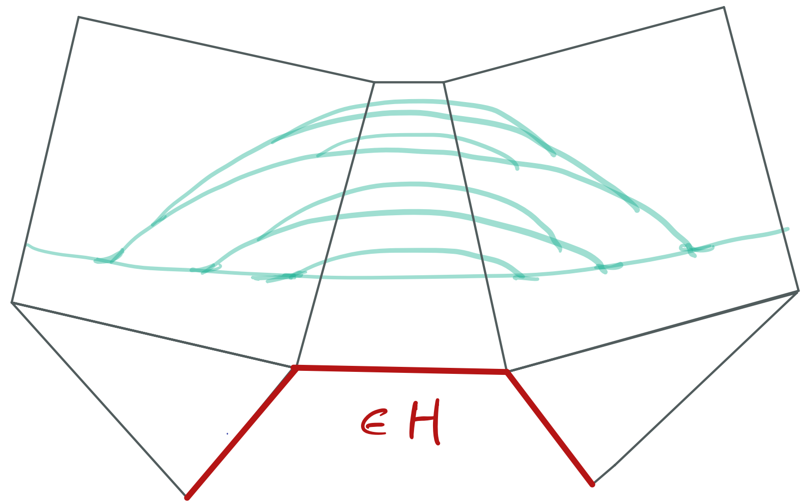
"Tracks" ⇝ Diagram rigidity

Target: $\mathrm{Dist}_H^G(n) \simeq 2^{n^{p/q}}$

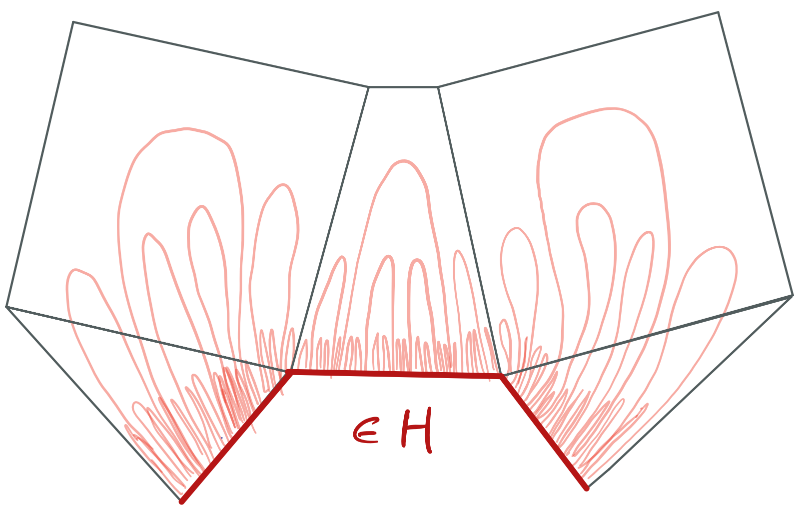
a-tracks

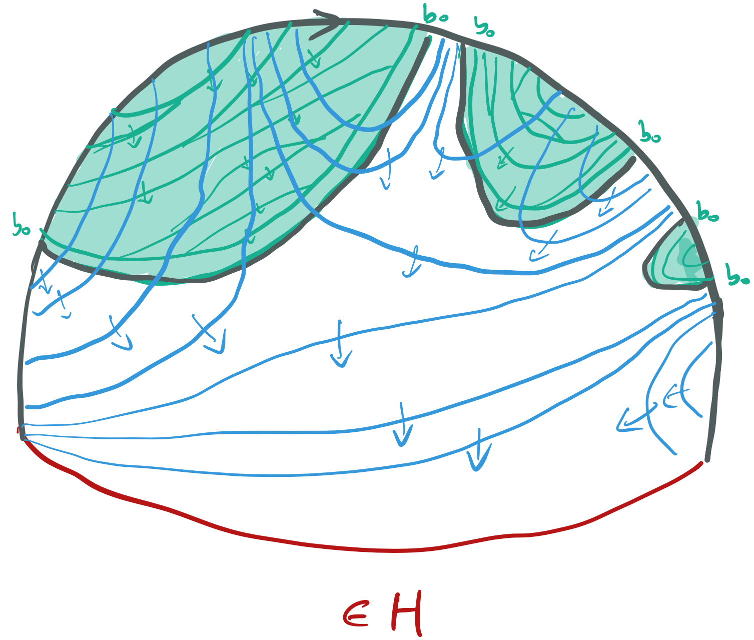


b-tracks



t-tracks





Thank you

+

Happy

birthday

Bob!