Outstanding Problems in Low-Dimensional Topology and Group Theory

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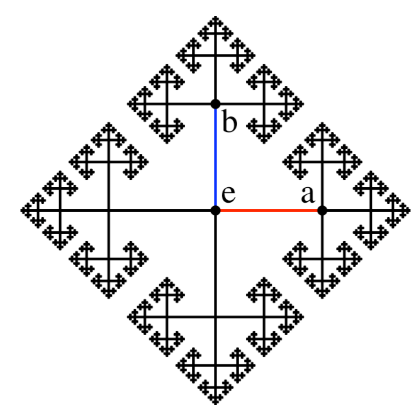


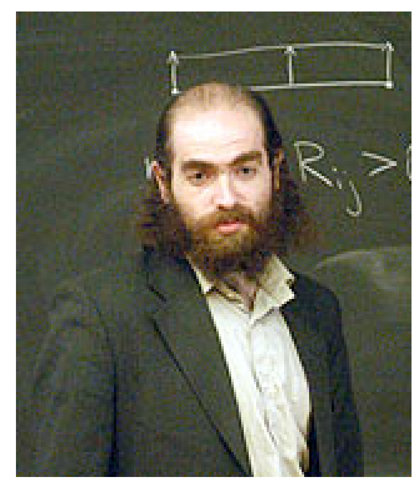
The Poincaré Conjecture

J.H. Poincaré

Every closed simply connected 3-manifold is homeomorphic to the 3-sphere.

W.P. Thurston

R. Hamilton

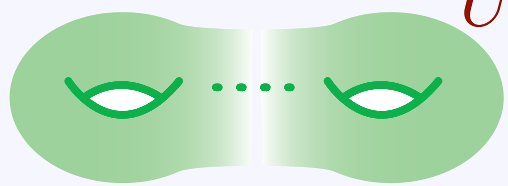
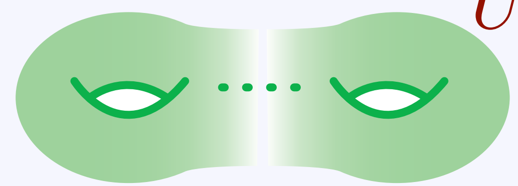


G. Perelman

“How not to prove the Poincaré Conjecture”

M a closed 3-manifold

A *Heegaard decomposition* of M:

$M =$  $\cup$ 

U_1 $\qquad$ U_2

$\uparrow$

$$\Sigma_g = U_1 \cap U_2 = \partial U_1 = \partial U_2$$

Let $\Gamma_g := \pi_1 \Sigma_g$ and $F_g = \pi_1 U_1 = \pi_1 U_2$.

$\Sigma_g \begin{array}{l} \hookrightarrow U_1 \\ \hookrightarrow U_2 \end{array}$ induces $\Gamma_g \begin{array}{l} \overset{\phi}{\twoheadrightarrow} F_g \\ \underset{\psi}{\twoheadrightarrow} F_g \end{array}$ and $\phi \times \psi : \Gamma_g \to F_g \times F_g$

that is surjective on each factor — a *splitting homomorphism*.

$\phi \times \psi$ is surjective iff $\pi_1 M = 1$.

Splitting homomorphisms $\phi_1 \times \psi_1$ and $\phi_2 \times \psi_2$ are *equivalent* when there exist automorphisms $\theta : \Gamma_g \circlearrowleft$ and $\alpha, \beta : F_g \circlearrowleft$ such that:

$$\begin{array}{ccc} \Gamma_g & \xrightarrow{\phi_1 \times \psi_1} & F_g \times F_g \\ \theta \downarrow & \circ & \downarrow \alpha \times \beta \\ \Gamma_g & \xrightarrow[\phi_2 \times \psi_2]{} & F_g \times F_g \end{array}$$

J.R. Stallings

Stallings invoked results of Jaco, Papakyriakopoulus, and Waldhausen. In fact, this formulation is due to Hempel.

The Poincaré Conjecture is true iff every epimorphism $\Gamma_g \twoheadrightarrow F_g \times F_g$ is equivalent to the standard one:

$$\langle a_1, b_1, \ldots, a_g, b_g \,|\, \textstyle\prod_{i=1}^g [a_i, b_i] \,\rangle$$
$$\twoheadrightarrow \langle a_1, \ldots, a_g \rangle \times \langle b_1, \ldots, b_g \rangle.$$

The Andrews–Curtis Conjecture

Every balanced presentation

$$\langle\, a_1, \ldots, a_m \;\mid\; r_1, \ldots, r_m \rangle$$

of the trivial group can be converted to

$$\langle\, a_1, \ldots, a_m \;\mid\; a_1, \ldots, a_m \rangle$$

using the moves

- $r_i,\ r_j \mapsto r_i r_j,\ r_j, \qquad i \neq j$
- $r_i \mapsto {r_i}^{-1}$
- $r_i \mapsto {a_k}^{\mp 1} r_i {a_k}^{\pm 1}.$

Stable Version. Also allow the move

$$\langle\, a_1, \ldots, a_m \;\mid\; r_1, \ldots, r_m \rangle \leftrightarrow \langle\, a_1, \ldots, a_m, a_{m+1} \;\mid\; r_1, \ldots, r_m, a_{m+1} \rangle$$

Some Candidate Counterexamples

— all are known to present the trivial group

Akbulut–Kirby (open for $n \geq 3$)

$$\langle\ a, b \ \mid\ aba = bab, a^{n+1} = b^n\ \rangle$$

Miller–Schupp

$$\langle\ a, b \ \mid\ a^2b = ba^3, w = 1\ \rangle$$

where the exponent sum of the b's in w is ± 1.

B.H. Neumann (proposed as a counterexample by Rapaport)

$$\langle\ a, b, c \ \mid\ c^{-1}bc = b^2, a^{-1}ca = c^2, b^{-1}ab = a^2\ \rangle$$

Example trivialization (*A.J. Casson*)

(A.D. Miasnikov, A.G. Miasnikov and V. Shpilrain were the first to find some trivialization)

$$\langle\ a, b\ \mid\ aba = bab, a^3 = b^2\ \rangle \rightsquigarrow \langle\ a, b\ \mid\ a, b\ \rangle$$

$abab^{-1}a^{-1}b^{-1},\ a^3b^{-2}$

$a^3b^{-2},\ a^2b^{-1}aba^{-1}b^{-1}$

$a^2b^{-1}aba^{-1}b^{-1},\ ab^{-1}ab^{-1}a^{-1}b$

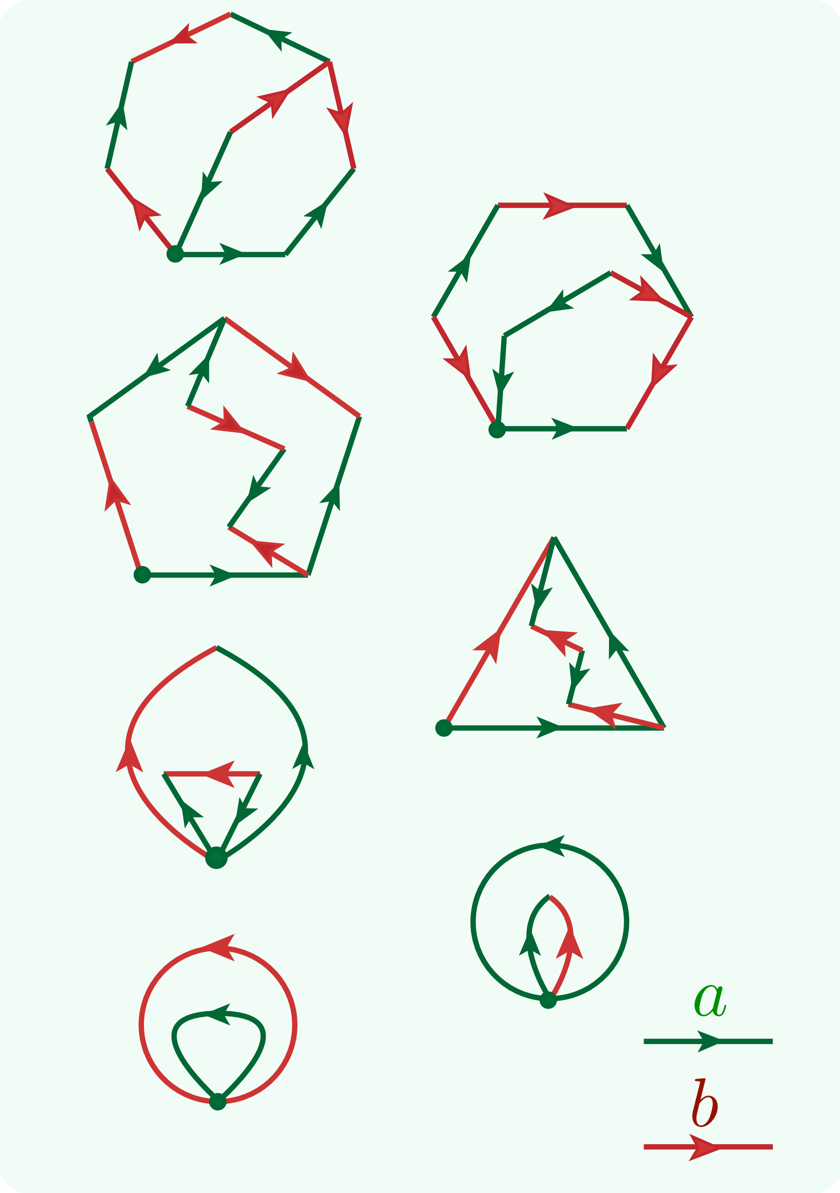
$ab^{-1}ab^{-1}a^{-1},\ a^2b^{-1}ab^{-1}$

$a^2b^{-1}ab^{-1},\ a^2b^{-1}$

$a^2b^{-1},\ ab^{-1}$

$ab^{-1},\ a$

$a,\ b$



The Grigorchuk–Kurchanov Conjecture

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$. Every $2n$-tuple

$$(r_1, \ldots, r_n, q_1, \ldots, q_n)$$

of words on $\mathcal{A}^{\pm 1}$ such that

$$\{\ t^{-1} r_i t \ \mid \ i = 1, \ldots, n;\ t \in F(q_1, \ldots, q_n)\ \}$$

generates $F(\mathcal{A})$, can be converted to the $2n$-tuple

$$(a_1, \ldots, a_n, 1, \ldots, 1)$$

using the moves:

- $r_i,\ r_j \mapsto r_i r_j,\ r_j, \quad (i \neq j)$
- $r_i \mapsto r_i{}^{-1}$
- $r_i \mapsto q_k{}^{\mp 1} r_i q_k{}^{\pm 1}$
- $q_i \mapsto q_i r_j$
- $q_i \mapsto q_i{}^{-1}$
- $q_i,\ q_j \mapsto q_i q_j,\ q_j, \quad (i \neq j)$.

R. Grigorchuk

P.F. Kurchanov

The **GK-Conjecture** implies the **AC-Conjecture**.

— take $(a_1, \ldots, a_n) = (q_1, \ldots, q_n)$.

The GK-Conjecture is true iff every epimorphism $F_{2g} \twoheadrightarrow F_g \times F_g$ is equivalent to the standard one:

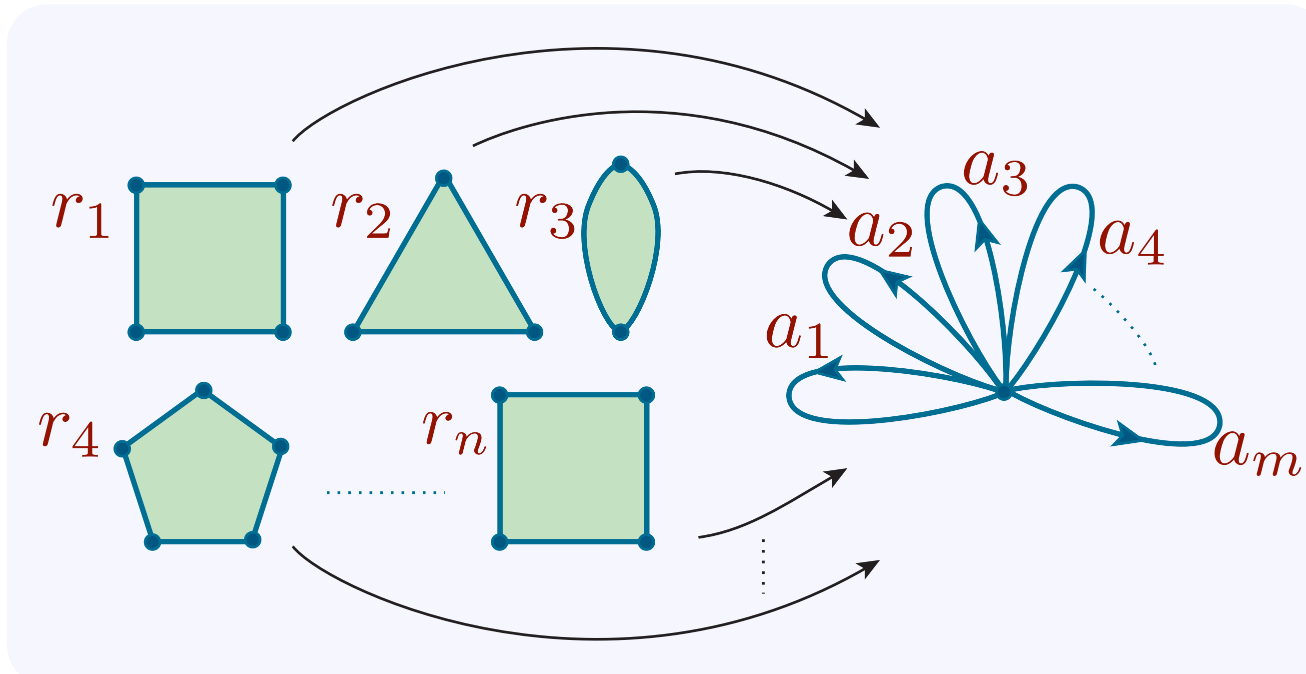
$$\langle a_1, b_1, \ldots, a_g, b_g \rangle \twoheadrightarrow \langle a_1, \ldots, a_g \rangle \times \langle b_1, \ldots, b_g \rangle.$$

(This theorem is established via results of Grigorchuk, Kurchanov, Lysenok. Cf. work of Craggs.)

From presentations to 2-complexes

$$\Gamma = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$$

$K =$



- $\pi_1(K) = \Gamma$ by the Seifert–van Kampen Theorem.
- If $m = n$ and Γ is trivial, then $\chi(K) = 1$ and K is contractible.

The Smooth 4-Dimensional Poincaré Conjecture

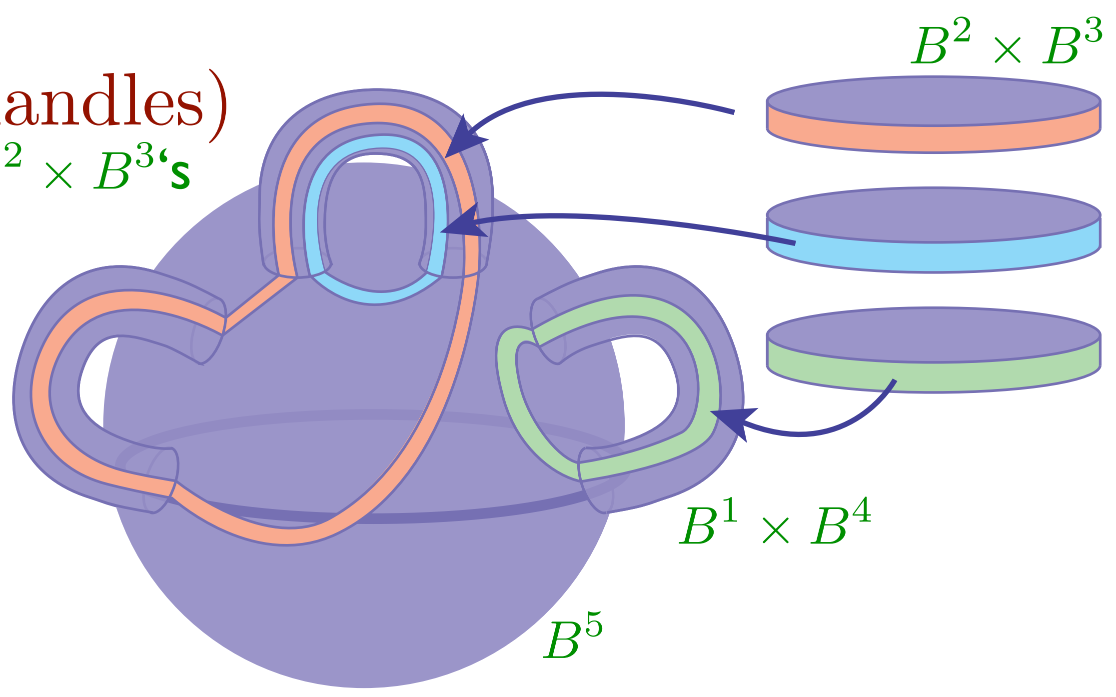
Every closed C^∞ 4-manifold M homotopy equivalent to the 4-sphere is diffeomorphic to the 4-sphere.

$K =$ the pres. 2-complex of a balanced pres. $\mathcal{P}$ of the trivial group.

Embed K in $\mathbb{R}^5$.

$W =$ a regular nbhd of K in $\mathbb{R}^5$

$= B^5 \cup \underset{B^1 \times B^4\text{'s}}{(1\text{-handles})} \cup \underset{B^2 \times B^3\text{'s}}{(2\text{-handles})}$



$M = \partial W$ is a closed 4-manifold with

$$\chi(M) = \chi(S^4) = 2$$

$$\pi_1 M = \pi_1(W \setminus K) = \pi_1 W = \pi_1 K = 1.$$

So $M \simeq_{\text{h.e.}} S^4$.

Unstable AC-moves	$\longleftrightarrow$	handle slides on W
Stabilization	$\longleftrightarrow$	adding a cancelling 1-handle, 2-handle pair to W

So if $\mathcal{P}$ is AC-trivializable, then M is diffeomorphic to S^4.

So potential counter-examples to the AC-Conjecture generate potential counterexamples to the C^∞ 4-diml Poincaré Conjecture.

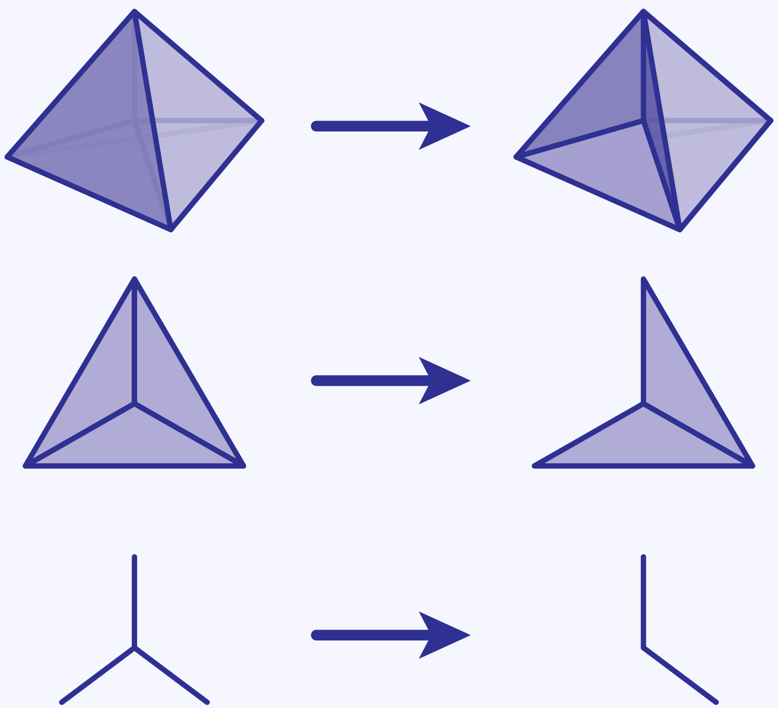
But —

(*Gompf*) The Akbulut–Kirby presentations all give standard 4-spheres.

Simple Homotopy and Collapsibility

K, L CW-complexes

Elementary collapses $K \overset{e}{\searrow} L$:



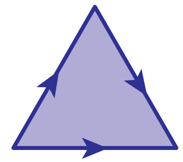
Elementary expansions $L \overset{e}{\nearrow} K$ are the inverses of elementary collapses.

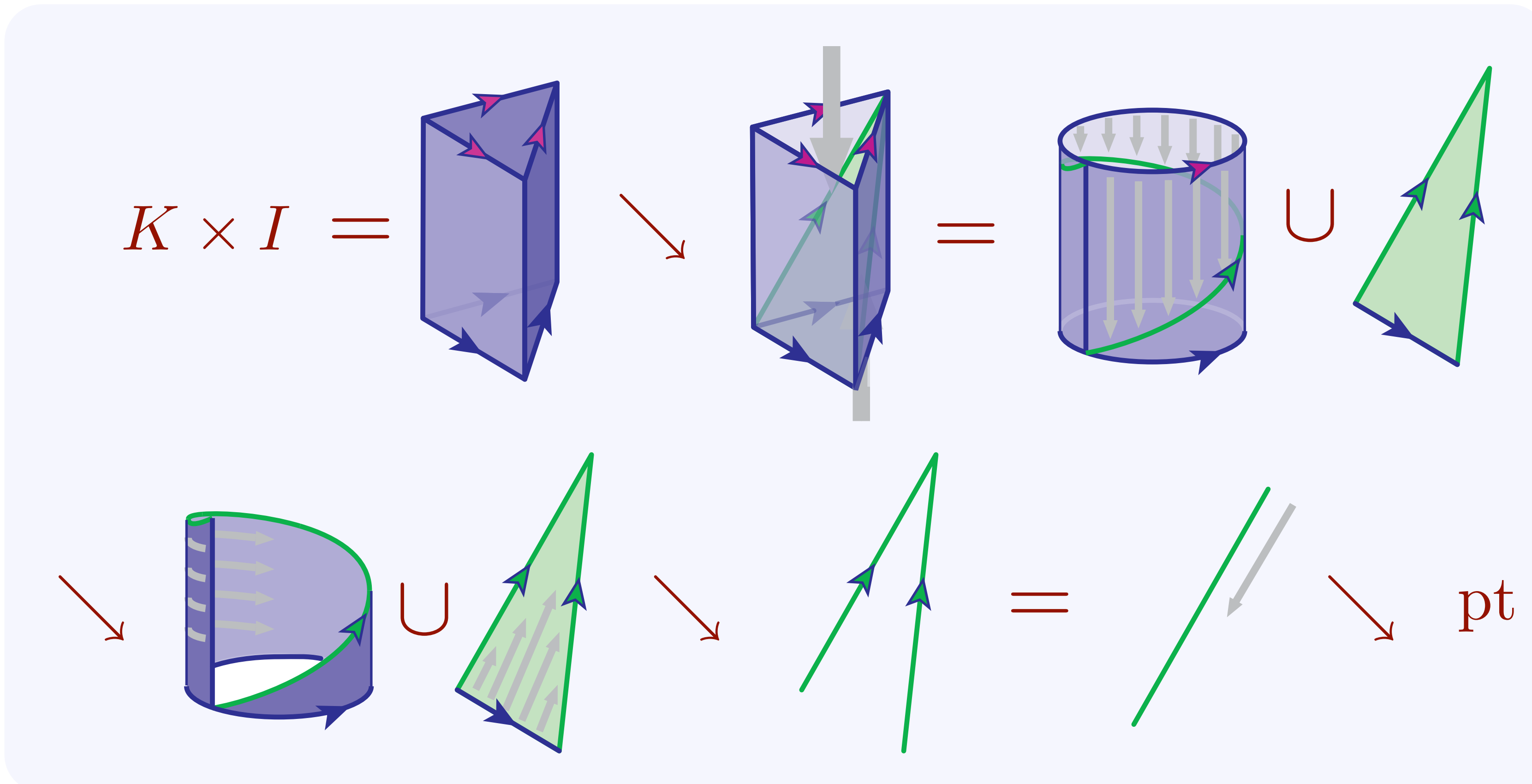
Write $K \diagup\!\!\searrow L$ (or $K \overset{n}{\diagup\!\!\searrow} L$) when K and L are related by a sequence of elementary collapses and expansions (involving cells of dim $\leq n$).

Spaces X, Y are *simple-homotopy equivalent* when they are homeomorphic to K, L such that $K \diagup\!\!\searrow L$.

X is *collapsible* ($X \searrow \mathrm{pt}$) when it is homeomorphic to a complex that can be reduced to a point via a sequence of elementary collapses.

Examples

1. The Dunce Hat: $K =$  — contractible, but not collapsible



2. M^n a triangulated manifold with boundary. Then $M^n \searrow N^{n-1}$ for some $(n-1)$-complex N^{n-1} (a spine of M^n).

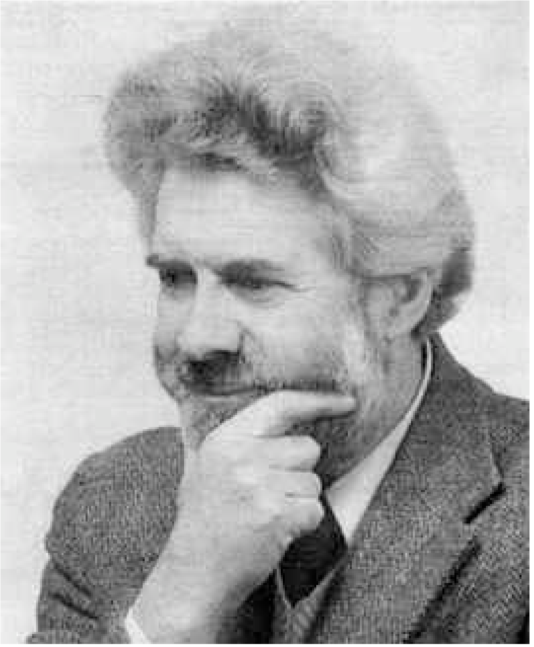
3. M^n a triangulated manifold. $M^n \searrow \mathrm{pt}$ iff $M^n \cong B^n$.

Folk-theorem (*P. Wright, J.R. Stallings*)**.** The Stable AC-Conjecture is true iff $K \overset{3}{\nearrow\!\searrow} \mathrm{pt}$ for every finite contractible 2-complex K.

The Zeeman Conjecture

If K^2 is a finite contractible 2-complex

$$K^2 \times I \searrow \mathrm{pt}\,.$$



E.C. Zeeman

The Zeeman Conjecture

The Stable AC-Conjecture

K a finite contractible 2-complex

$$K \nearrow K \times I$$

Then by Zeeman,

$$K \times I \searrow \mathrm{pt}.$$

So

$$K \overset{3}{\nearrow\!\searrow} \mathrm{pt}.$$

The Poincaré Conjecture

M a closed simply connected 3-manifold

Assume M is simplicially triangulated. Let N be M with the interior of a 3-simplex removed and K be a spine of N.

$$\left.\begin{array}{l} \chi(K) = \chi(N) = \chi(M) + 1 = 1 \\ \pi_1(K) = \pi_1(N) = \pi_1(M) = \{1\} \end{array}\right\} \Longrightarrow K \text{ is contractible}$$

So $N \times I \searrow \underline{K \times I \searrow \mathrm{pt}}$ by Zeeman.

So $N \times I \cong B^4$ and $N \times \{0\} \subseteq \partial(N \times I) \cong S^3$.

But $\partial N \cong S^2$ and in the PL-category an S^2 in an S^3 bounds a B^3 by the Schönflies Theorem.

So $N \cong B^3$ and $M \cong S^3$.

Whitehead's Asphericity Question

J.H.C.Whitehead and friend

Is every subcomplex of an aspherical 2-complex, itself aspherical?

An *Eilenberg-Maclane Space* (a $K(\Gamma,1)$) for a group Γ is an aspherical CW-complex with $\pi_1 K(\Gamma,1) = \Gamma$.

The geometric dimension, $\mathrm{gd}(\Gamma)$, of Γ is the minimal n such that there is a $K(\Gamma,1)$ for Γ of dimension n.

The cohomological dimension, $\mathrm{cd}(\Gamma)$, of Γ is the minimal n such that $\mathbb{Z}$ admits a resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

by projective $\mathbb{Z}\Gamma$-modules.

Note. $\mathrm{cd}(\Gamma) \leq \mathrm{gd}(\Gamma)$— the cellular chain complex of a $K(\Gamma,1)$ yields a projective resolution.

The Eilenberg–Ganea Conjecture

Theorem (Stallings–Swan)

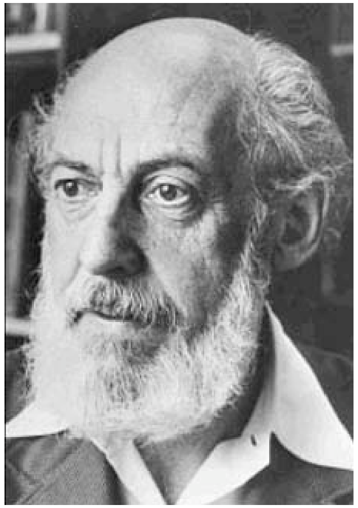
$$\mathrm{cd}(\Gamma) = 1 \iff \mathrm{gd}(\Gamma) = 1 \iff \Gamma \text{ is free.}$$

Eilenberg–Ganea Theorem

$$\mathrm{cd}(\Gamma) \geq 3 \implies \mathrm{cd}(\Gamma) = \mathrm{gd}(\Gamma)$$

Eilenberg–Ganea Conjecture

$$\mathrm{cd}(\Gamma) = 2 \implies \mathrm{gd}(\Gamma) = 2$$



S. Eilenberg

T. Ganea

(I am unaware of anywhere Eilenberg and Ganea actually stated this as conjecture.)

Γ is of *type* F_n if it admits a $K(\Gamma, 1)$ with finitely many n-cells.

Note. Type F_2 is *finite presentability*.

Γ is of *type* FP_n if $\mathbb{Z}$ admits a partial projective resolution

$$P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

by *finitely generated* projective $\mathbb{Z}\Gamma$-modules.

Note. Type F_n implies type FP_n.

Γ is of *type* FP if $\mathbb{Z}$ admits a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

by *finitely generated* projective $\mathbb{Z}\Gamma$-modules.

Bestvina–Brady Groups

A finite graph G with vertices $\mathcal{A}$ and edges $\mathcal{E}$ determines a *right-angled Artin group*

$$A = \langle \mathcal{A} \mid [a_i, a_j] = 1, \ \forall (a_i, a_j) \in \mathcal{E} \rangle.$$

Define Γ to be the kernel of the homomophism $A \to \mathbb{Z}$ in which $a \mapsto 1, \forall a \in \mathcal{A}$.

N is the flag complex with 1-skeleton G.

Theorem Γ is

- of type FP_n iff $H_i(N) = 0, \forall i < n$.
- of type FP iff N is acyclic.
- finitely presentable (of type F_2) iff N is simply connected.

M. Bestvina

N. Brady

Corollary

Suppose N is a spine of the Poincaré Homology Sphere.
Then Γ is FP but not finitely presentable. (In fact, $\mathrm{cd}(\Gamma) = 2$.)

Moreover,

- either Γ is a counterexample to the Eilenberg–Ganea Conjecture (i.e. has $\mathrm{gd}(\Gamma) = 3$),
- or Whitehead's Asphericity question has a negative answer. (The universal cover of a 2-dimensional $K(\Gamma, 1)$ would have a non-contractible subcomplex.)

Main tools –
- Morse Theory
- CAT(0) geometry

The Relation Gap Problem

$$\Gamma = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle = F/R$$

where $F = F(a_1, \ldots, a_m)$ and $R = \langle\langle r_1, \ldots, r_n \rangle\rangle$.

F acts on R by conjugation, so induces an action of Γ on $R^{\mathrm{ab}} = \dfrac{R}{[R, R]}$.

Rank of R^{ab} as a $\mathbb{Z}\Gamma$-module $\leq \min \left\{ k \;\middle|\; \begin{array}{l} \exists\ s_1, \ldots, s_k \in F, \\ R = \langle\langle s_1, \ldots s_k \rangle\rangle \end{array} \right\}$.

The difference is the *relation gap*.

Open question. Is there a presentation with (finite) non-zero relation gap?

If Γ is of type FP_2 then R^{ab} is finitely generated as a $\mathbb{Z}\Gamma$-module.

So the Bestvina–Brady example has infinite relation gap.

Bridson–Tweedale example

$$\Gamma = \left\langle x, y, s, t \;\middle|\; \begin{array}{ll} x^m = 1, & x^s x x^{-s} = x^{m+1}, \\ y^n = 1, & y^t y y^{-t} = y^{n+1} \end{array} \right\rangle$$

where $m, n > 1$, and $(m+1)^m - 1$ and $(n+1)^n - 1$ are coprime.

M. Bridson

The $\mathbb{Z}\Gamma$-module R^{ab} is generated by

$$x^s x x^{-s} x^{-m-1},\ y^t y y^{-t} y^{-n-1},\ x^m y^n.$$

Express Γ as $F(x, y, s, t)/R$.

M. Tweedale

Conjecture. R is not the normal closure of 3 elements.

(Cf. examples of K. Gruenberg & P. Linnell, and also some Bestvina–Brady-style examples of Brisdon & Tweedale.)

The $D(2)$ Conjecture

X a space with universal cover $\widetilde{X}$.

X enjoys the $D(n)$ property when

- $H_i(\widetilde{X}) = 0$ for all $i > n$, and
- $H^{n+1}(X, \mathcal{M}) = 0$ for all local coefficient systems $\mathcal{M}$ on X.

C.T.C. Wall

Theorem. For $n \neq 2$, a finite CW complex X is homotopic to a finite n-dimensional CW complex iff X enjoys the $D(n)$ property.

Folk Conjecture. The same is true when $n = 2$.

Theorem (M. Dyer). If there is a group Γ with $H^3(\Gamma, \mathbb{Z}\Gamma) = 0$ and a presentation that

- has a relation gap, and
- realises the *deficiency* of Γ,

then the $D(2)$-Conjecture is false.

J. Harlander wrote up a proof.
See also M. Tweedale's thesis.

The Bridson–Tweedale examples would satisfy these conditions if they have a relation gap.

Magnus Problem

Given a balanced presentation

$$\langle\ a_1, \ldots, a_m \ \mid\ r_1, \ldots, r_m \rangle$$

of the trivial group, can some r_i always be replaced by a primitive element of $F(a_1, \ldots, a_m)$ whilst triviality of the group is preserved?

W. Magnus

S.V. Ivanov gave an explicit negative example (with $m = 3$).

Kervaire–Laudenbach Conjecture

If Γ is non-trivial and $r \in \Gamma * \mathbb{Z}$,

then $\dfrac{\Gamma * \mathbb{Z}}{\langle\langle r \rangle\rangle}$ is non-trivial.

M. Kervaire

F. Laudenbach

A. Klyachko

(Klyachko.) The conjecture is true for all torsion free Γ.

There is a good account by R. Fenn.

Remark (*S.V. Ivanov*).

Special case of the Kervaire–Laudenbach Conjecture –

If $\langle\, a_1, \ldots, a_m \;\mid\; r_1, \ldots, r_m \rangle$ is a balanced presentation of the trivial group and ${a_m}^{\pm 1}$ does not occur in $r_1, \ldots, r_{m-1}$, then $\Gamma = \langle\, a_1, \ldots, a_{m-1} \;\mid\; r_1, \ldots, r_{m-1} \rangle$ is also trivial.

By Klyachko, in a counterexample, Γ would contain torsion elements and so its presentation 2-complex would not be aspherical, answering Whitehead's problem negatively.

"Toy" Problems

1.) Can $\langle\ a, b \ \mid \ aba = bab, a^4 = b^3\ \rangle$ be converted to $\langle\ a, b \ \mid \ a, b\ \rangle$ using Andrews–Curtis moves?

2.) Do there exist r_1, r_2, r_3 such that

$$\left\langle x, y, s, t \,\middle|\, \begin{array}{ll} x^2 = 1, & x^s x x^{-s} = x^3, \\ y^3 = 1, & y^t y y^{-t} = y^4 \end{array} \right\rangle$$

$$= \langle\ x, y, s, t \mid r_1,\ r_2,\ r_3\ \rangle?$$

3.) Does the Bestvina–Brady example have $\mathrm{gd}(\Gamma) = 3$?

Triumphs of Geometry

Poincaré Conjecture — Geometrization + Ricci Flow

Bestvina–Brady — CAT(0) geometry and Morse Theory

Magnus Problem (Ivanov) — small cancellation

A (far from complete) list of references / sources of further reading –

- S. Akbulut and R. Kirby, *A potential smooth counterexample in dimension 4 to the Poincaré conjecture, the Schonflies conjecture, and the Andrew-Curtis conjecture*, Topology 24, no. 4 (1985), 375-390.
- J.J. Andrews and M.L. Curtis, *Free Groups and Handlebodies*, Proc. AMS, Vol. 16, No. 2., pp. 192-195, 1965
- M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math., Vol.129, pp. 445-470, 1997
- K. Brown, *Cohomology of Groups*
- *M.M. Cohen, A course in simple-homotopy theory*, Springer, Graduate texts in mathematics, 1973
- D.J.Collins, R.I.Grigorchuk, P.F.Kurchanov and H.Zieschang, *Combinatorial Group Theory and Applications to Geometry*, Springer
- R. Craggs, *Free Heegaard diagrams and extended Nielsen Transformations I, Michigan Math. J., 26, 1979*
- R. Craggs, *Free Heegaard diagrams and extended Nielsen Transformations II, Ill. J. Math., 23, 1979*
- S. Eilenberg, T. Ganea, *On the Lusternik-Schnirelmann category of abstract groups*, Annals of Mathematics, 2nd Ser., 65 (1957), no. 3, 517 – 518
- J. Harlander, C. Hog-Angeloni, W. Metzler, S. Rosebrock, *Problems in Low-dimensional topology*, Encyclopaedia of Mathematics, Springer
- S.V. Ivanov, *On balanced presentations of the trivial group*, Inventiones, Volume 165, Number 3 / September, 2006
- C. Hog-Angeloni, W.Metzler, A.J. Sieradski (eds.), *Two-dimensional Homotopy and Combinatorial Group Theory*, LMS Lecture Note Series No. 197, C.U.P., 1993
- A.D.Myasnikov, A.G.Myasnikov and V.Shpilrain, On the Andrews-Curtis equivalence, Contemp. Math., Amer. Math. Soc. 296 (2002), 183-198.
- C.F. Miller III and P.E. Schupp, *Some presentations of the trivial group*
- E.S. Rapaport, *Remarks on groups of order 1*, Amer. Math. Monthly, 75, (1968), 714–720
- D. Rolfsen, *Cousins of the Poincaré Conjecture* (lecture notes), http://www.math.ubc.ca/~rolfsen/
- J.R. Stallings, *How not to prove the Poincaré Conjecture*, http://math.berkeley.edu/~stall/
- The World of Groups - *open problems list*, http://www.grouptheory.info/
- P. Wright, *Group presentations and formal deformations,* Trans. Amer. Math. Soc. 208, (1975), 161–169
- E.C. Zeeman, *On the dunce hat*, Topology 2, 1964, 341–358.