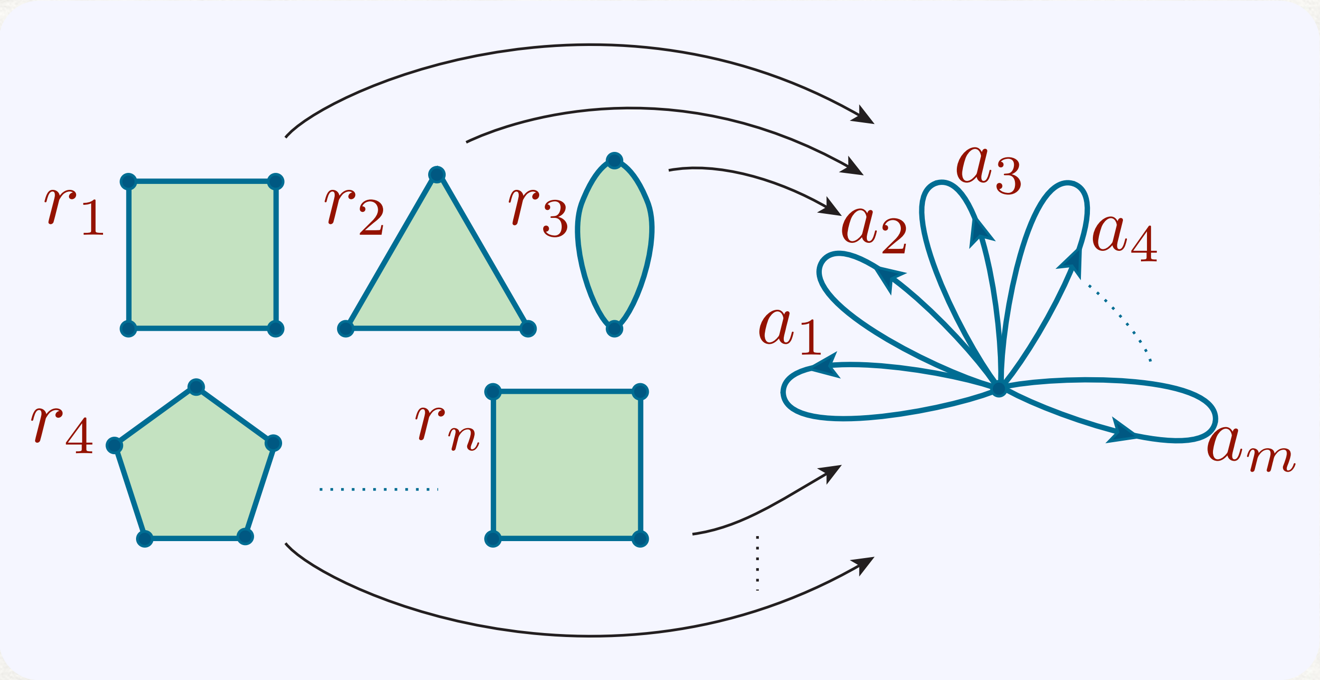
Cornell University

# SCREWBALL CONJECTURES IN GROUP THEORY AND LOW-DIMENSIONAL TOPOLOGY

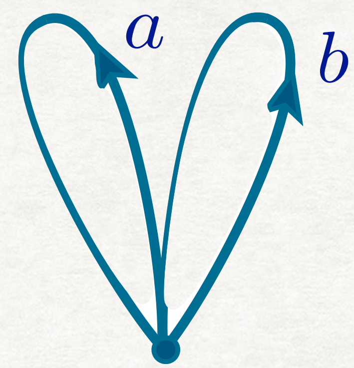
TIMOTHY RILEY

$$\Gamma = \langle\, a_1, \ldots, a_m \;\mid\; r_1, \ldots, r_n \rangle$$
$$= F(a_1, \ldots, a_m) / \langle\!\langle r_1, \ldots, r_n \rangle\!\rangle$$

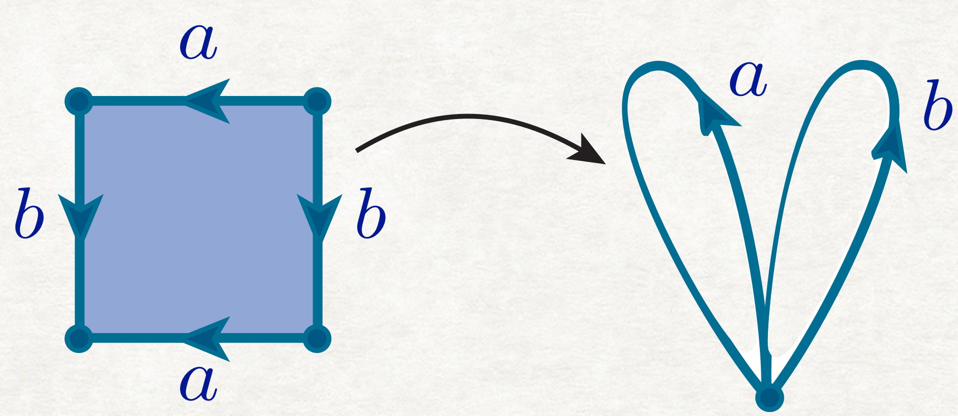
$K =$ 

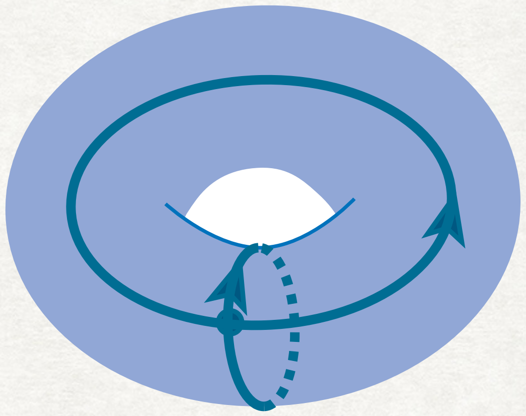

$\pi_1(K) = \Gamma$ by Seifert–van Kampen

$$\langle a, b \mid \; \rangle$$

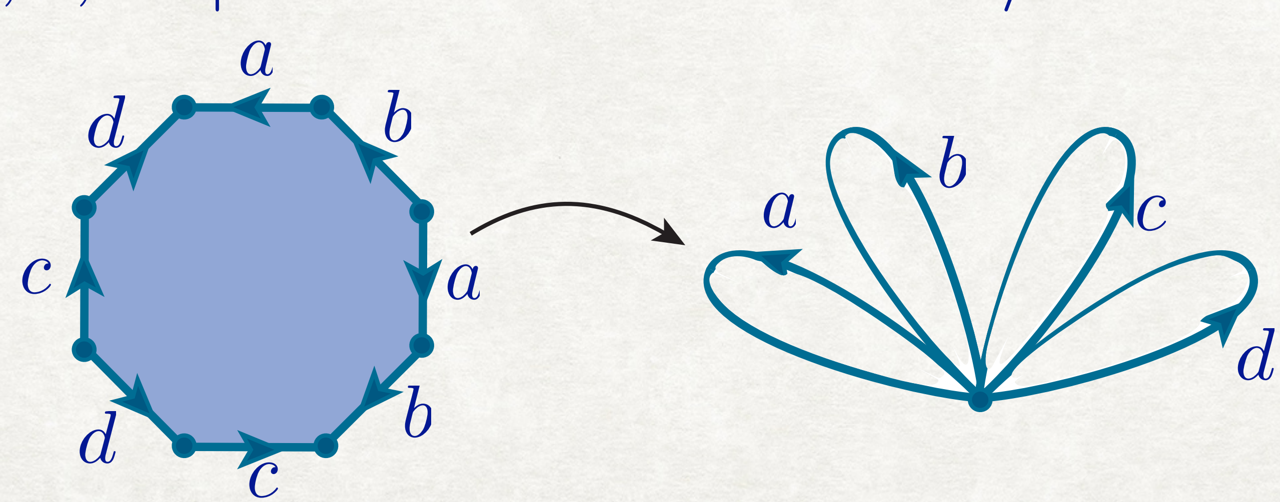


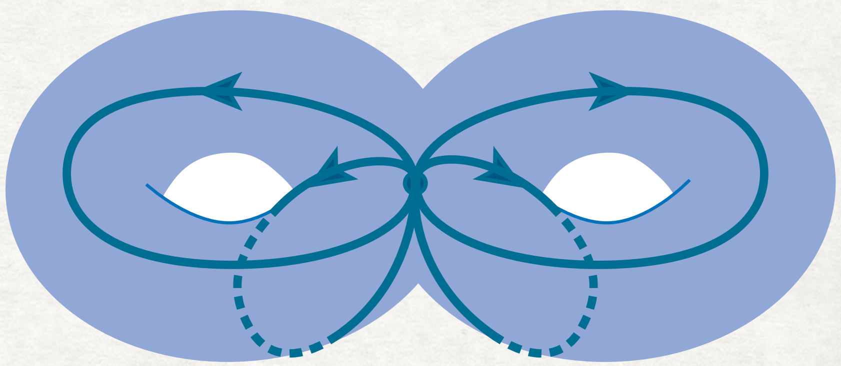

$$\langle a, b \mid a^{-1}b^{-1}ab \rangle$$



$$\langle \, a, b, c, d \;\; \mid \;\; a^{-1}b^{-1}ab\, c^{-1}d^{-1}cd \, \rangle$$



# THE ANDREWS–CURTIS CONJECTURE

**FREE GROUPS AND HANDLEBODIES**

J. J. ANDREWS[1] AND M. L. CURTIS[1]

In this note we state a conjecture about free groups and give some topological consequences which would follow if the conjecture is true.

Proc AMS 1965

# THE ANDREWS–CURTIS CONJECTURE

Every balanced presentation

$$\langle\, a_1, \ldots, a_m \;\mid\; r_1, \ldots, r_m \rangle$$

of the trivial group can be converted to

$$\langle\, a_1, \ldots, a_m \;\mid\; a_1, \ldots, a_m \rangle$$

using the moves

$$r_i,\ r_j \mapsto r_i r_j,\ r_j$$

$$r_i \mapsto r_i^{-1}$$

$$r_i \mapsto a_k^{\mp 1} r_i a_k^{\pm 1}$$

$$a^{-1}b^{-1}ab^{-1}a^{-1},\ aba^{-1}$$

$$a^{-1}b^{-1},\ aba^{-1}$$

$$a^{-1}b^{-1},\ b$$

$$a^{-1},\ b$$

$$a,\ b$$

$abab^{-1}a^{-1}b^{-1},\ a^3b^{-2}$

$a^3b^{-2},\ a^2b^{-1}aba^{-1}b^{-1}$

$a^2b^{-1}aba^{-1}b^{-1},\ ab^{-1}ab^{-1}a^{-1}b$

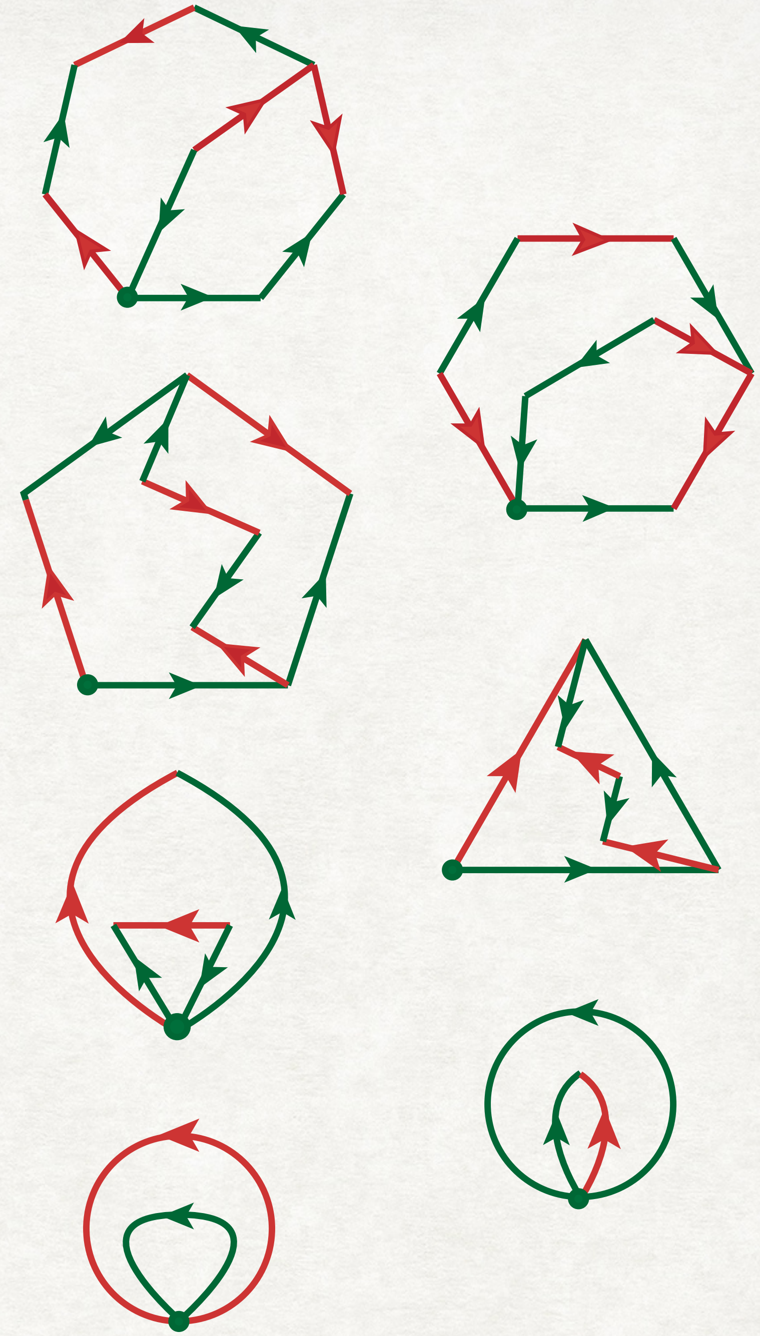
$ab^{-1}ab^{-1}a^{-1},\ a^2b^{-1}ab^{-1}$

$a^2b^{-1}ab^{-1},\ a^2b^{-1}$

$a^2b^{-1},\ ab^{-1}$

$ab^{-1},\ a$

$a,\ b$



Andrew Casson

Unknown:

$$\langle\, a, b \mid aba = bab, a^{n+1} = b^n \,\rangle \qquad n \geq 3$$

**Akbulut–Kirby**

$$\langle\, a, b \mid a^2 b = b a^3, w = 1 \,\rangle$$

where the exponent sum of $b$ in $w$ is $\pm 1$.

**Miller–Schupp**

$$\langle\, a, b, c \mid c^{-1} b c = b^2, a^{-1} c a = c^2, b^{-1} a b = a^2 \,\rangle$$

**B.Neumann–Rapaport–Higman**

## Computational searches

**Miasnikov** & **Myasnikov** 1999

**Casson** ~2003

**Havas** & **Ramsay** 2003

**Panteleev** & **Ushakov** 2016

**Lisitsa** 2019

No counterexamples with relations of total length $\leq 12$.

All examples with relations of total length $\leq 13$ trivializable or reducible to

$$\langle\ a, b \ \mid \ aba = bab, a^4 = b^3\ \rangle$$

**Bridson** (2015) cf. **Lishak** (2015)

Presentations of total length $\sim n$ which are trivialisable but require at least

$$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \Big\} \lfloor \log_2 n \rfloor$$

moves.

E.g.

$$\langle a, t, \alpha, \tau \mid [tat^{-1}, a]a^{-1}, \quad [\tau\alpha\tau^{-1}, \alpha]\alpha^{-1},$$
$$\alpha t^{-1}\alpha^{-1}[a,\ [t[t[ta^{20}t^{-1},\ a]t^{-1},\ a]t^{-1},\ a]],$$
$$a\tau^{-1}a^{-1}[\alpha,\ [\tau[\tau[\tau\alpha^{20}\tau^{-1},\ \alpha]\tau^{-1},\ \alpha]\tau^{-1},\ \alpha]]\rangle.$$

requires at least $10^{10,000}$ moves.

**Myasnikov** (1984)

No obstructions in solvable groups

**Borovik-Lubotzky-Myasnikov** (2003)

No obstructions in finite groups

$$\langle\, a_1, \ldots, a_m \;\mid\; r_1, \ldots, r_m \rangle$$

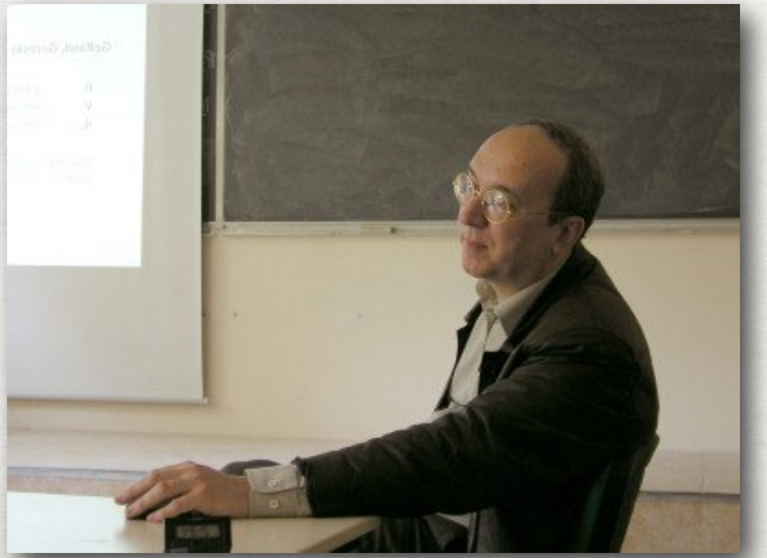
$$\rightsquigarrow$$

$$\langle\, a_1, \ldots, a_m \;\mid\; a_1, \ldots, a_m \rangle$$

$$r_i,\; r_j \mapsto r_i r_j,\; r_j$$

$$r_i \mapsto r_i^{-1}$$

$$r_i \mapsto a_k^{\mp 1} r_i a_k^{\pm 1}$$



$$P = \langle x, y \mid [x,y],\ 1 \rangle$$

$$Q = \langle x, y \mid [x,[x,y^{-1}]]^2 y [y^{-1},x] y^{-1},\ [x,[[y^{-1},x],x]] \rangle$$

$$\cong \mathbb{Z}^2$$

**Barmak** (2018)

$P$ and $Q$ are not AC-equivalent despite their 2-complexes being simple homotopy equivalent.

Invariant in $F_2/F_2''$.

Stabilization:

$$\langle\, a_1, \ldots, a_m \;\mid\; r_1, \ldots, r_m \rangle$$

$$\leftrightsquigarrow\; \langle\, a_1, \ldots, a_m, a_{m+1} \;\mid\; r_1, \ldots, r_m, a_{m+1} \rangle$$

# THE POINCARE CONJECTURE

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

Il y a donc deux cycles de $V$ qui ne sont pas équivalents à zéro; donc $V$ n'est pas simplement connexe.

En d'autres termes, le groupe fondamental de $V$ ne saurait se réduire à la substitution identique, puisqu'il contient comme sous-groupe le groupe icosaédrique.

Il resterait une question à traiter :

Est-il possible que le groupe fondamental de $V$ se réduise à la substitution identique, et que pourtant $V$ ne soit pas simplement connexe?

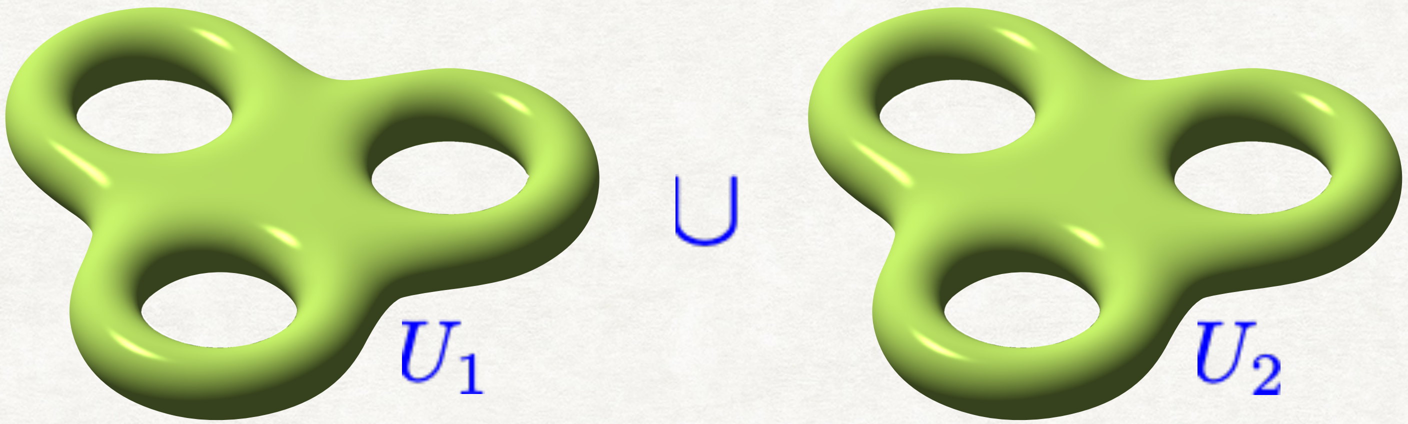
En d'autres termes, peut-on tracer les cycles $K''_1$ et $K''_2$ de telle façon qu'ils ne soient pas bouclés et ne se coupent pas; que les équivalences

$$K'_1 \equiv K'_2 \equiv 0, \quad K''_1 \equiv K''_2 \equiv 0$$

Cinquième Complément à L'Analysis Situs, 1904



$M$ a closed 3-manifold




$$\Sigma = U_1 \cap U_2 = \partial U_1 = \partial U_2$$

$$\Gamma = \pi_1 \Sigma = \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] \right\rangle$$

$$F = \pi_1 U_1 = \pi_1 U_2$$

$\Sigma \begin{array}{l} \hookrightarrow U_1 \\ \hookrightarrow U_2 \end{array}$ induces $\Gamma \begin{array}{l} \twoheadrightarrow F \\ \twoheadrightarrow F \end{array}$ i.e. $\Gamma \to F \times F$

which is surjective iff $\pi_1 M = 1$. A “splitting homomorphism.”

Equivalence of splitting homomorphisms:

$$\begin{array}{ccc} \Gamma & \twoheadrightarrow & F \times F \\ \downarrow\cong & \circ & \downarrow(\alpha,\beta) \\ \Gamma & \twoheadrightarrow & F \times F \end{array} \qquad \alpha, \beta \in \mathrm{Aut}(F)$$

**Stallings** (1965)

Poincaré holds iff every splitting homomorphism is equivalent to

$$\left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] \right\rangle \twoheadrightarrow \langle a_1, \ldots, a_g \rangle \times \langle b_1, \ldots, b_g \rangle$$

# GRIGORCHUK-KURCHANOV CONJECTURE

$A = \{a_1, \dots, a_n\}$. Every tuple of words

$$(r_1, \dots, r_n, q_1, \dots, q_n)$$

on $A^{\pm 1}$ such that

$$\{\ t^{-1} r_i t \ \mid\ i = 1, \dots, n;\ t \in F(q_1, \dots, q_n)\ \}$$

generates $F(A)$ can be converted to

$$(a_1, \dots, a_n, 1, \dots, 1)$$

using

$$r_i,\ r_j \mapsto r_i r_j,\ r_j \qquad q_i, q_j \mapsto q_i q_j, q_j$$

$$r_i \mapsto r_i^{-1} \qquad q_i \mapsto q_i^{-1}$$

$$r_i \mapsto a_k^{\mp 1} r_i a_k^{\pm 1} \qquad q_i \mapsto q_i r_j$$

G-K $\Longrightarrow$ A-C

(1993) G–K holds iff every splitting homomorphism

$$F_{2n} \twoheadrightarrow F_n \times F_n$$

is equivalent to

$$\langle a_1, b_1, \ldots, a_n, b_n \rangle \twoheadrightarrow \langle a_1, \ldots, a_g \rangle \times \langle b_1, \ldots, b_g \rangle$$

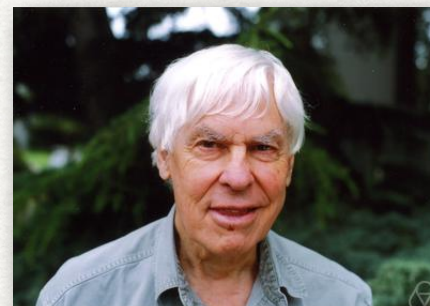
# THE GENERALIZED POINCARE CONJECTURE

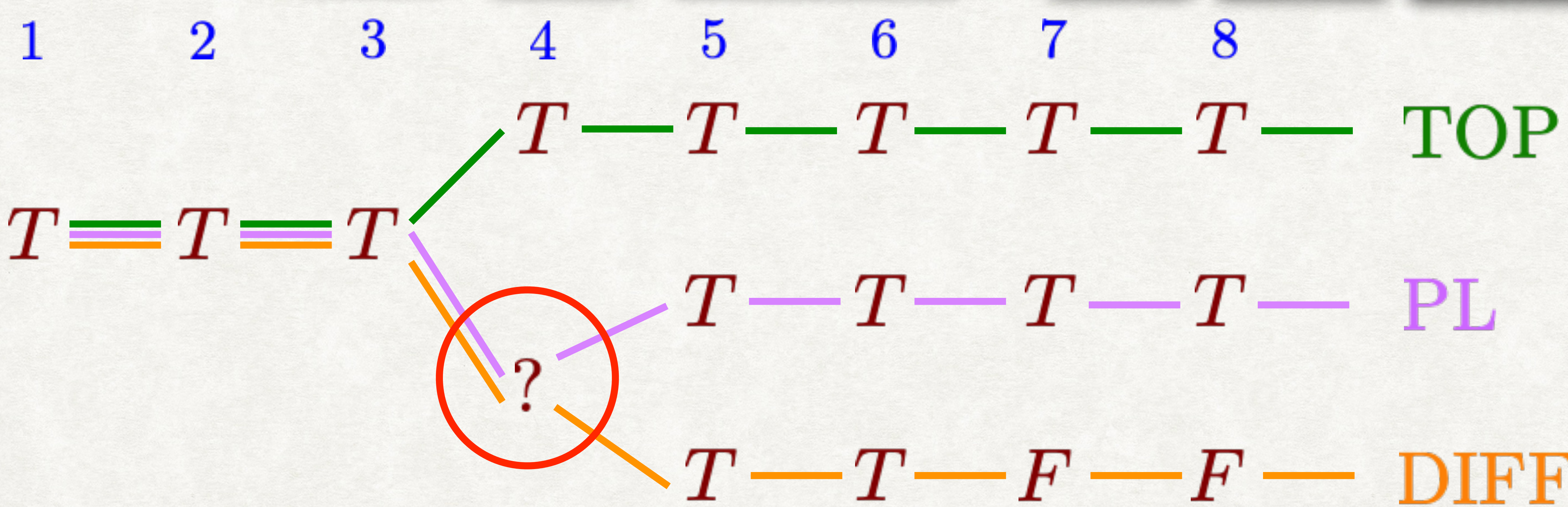
If a closed $n$-manifold $M$ is homotopic to an $n$-sphere, is it an $n$-sphere?

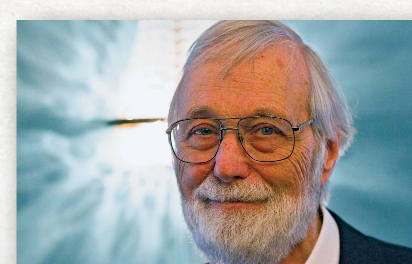
# THE GENERALIZED POINCARE CONJECTURE

If a closed $n$–manifold $M$ is homotopic to an $n$–sphere, is it an $n$–sphere?



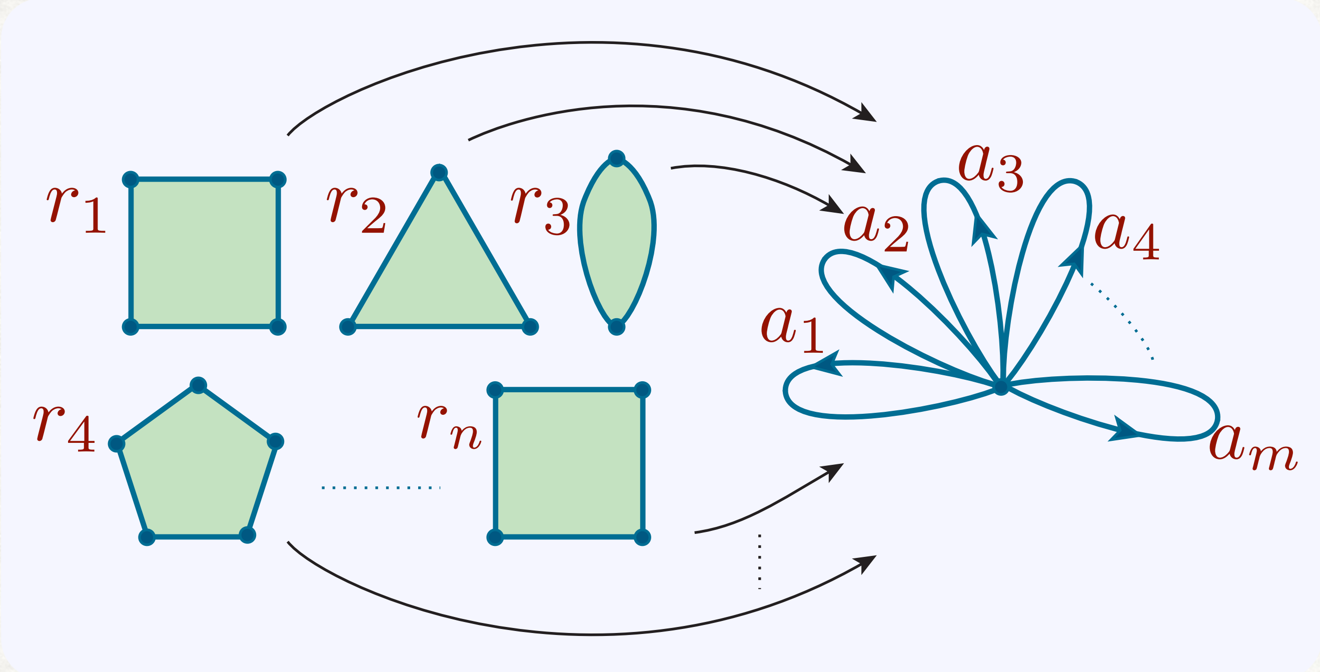



$$\Gamma = \langle\, a_1, \ldots, a_m \;\mid\; r_1, \ldots, r_n \rangle$$

$K =$




If $m = n$ and $\Gamma$ is trivial,

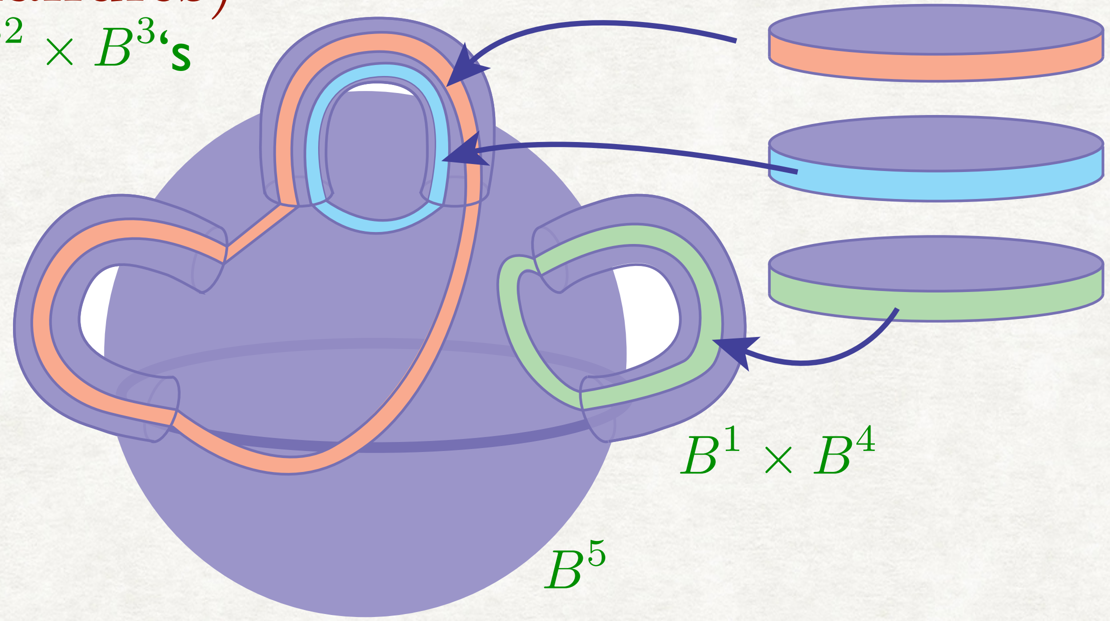
then $\chi(K) = 1$ and $K$ is contractible.

$K \hookrightarrow \mathbb{R}^5$

$N =$ regular neighbourhood of $K$ in $\mathbb{R}^5$

$= B^5 \cup (\text{1-handles}) \cup (\text{2-handles})$

$B^1 \times B^4$'s $\quad B^2 \times B^3$'s




$M = \partial N$ is a closed 4-manifold.

If the presentation is balanced, then $\chi(M) = \chi(S^4) = 2$.

$\pi_1 M = \pi_1 N = \pi_1 K$

So if $\pi_1 K = 1$, then $M$ is homotopy equivalent to $S^4$.

A-C-moves correspond to handle-slides.

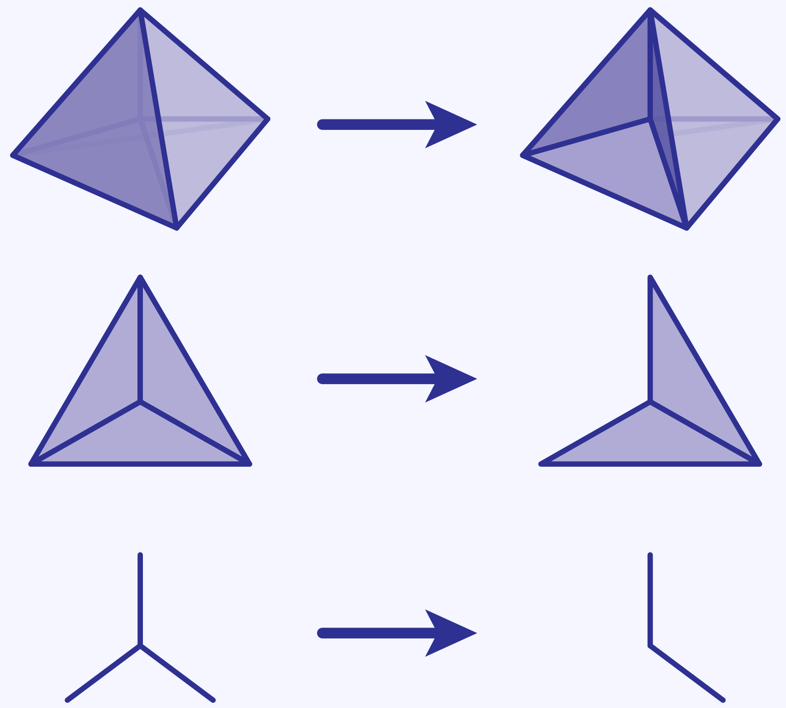
So potential counterexamples to A-C yield potential counterexamples to the Smooth 4-Dimensional Poincaré Conjecture.

**Gompf** (1991). $\langle\ a, b \ \mid \ aba = bab, a^5 = b^4\ \rangle$ yields a standard 4-sphere.

# SIMPLE HOMOTOPY AND COLLAPSIBILITY

$K$, $L$ CW-complexes

*Elementary collapses:*



Elementary expansions are their inverses.

$X$, $Y$ "simple homotopic" when homeomorphic to complexes related by a sequence of elementary collapses and expansions.

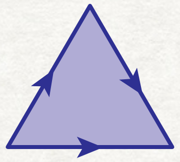
$X$ "collapsible" when homeomorphic to a complex reducible to a point through a sequence of elementary collapses.

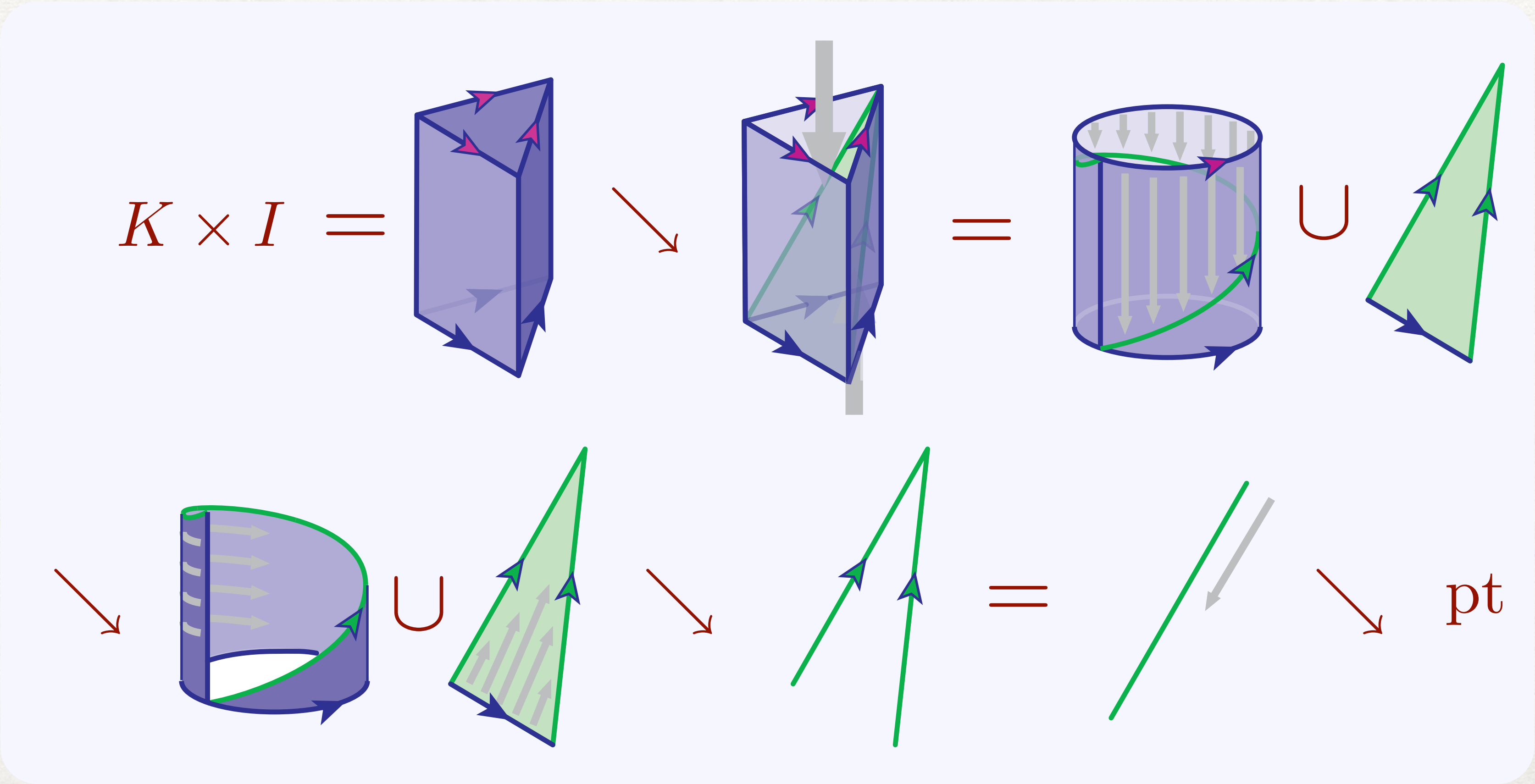
**Whitehead**: a PL-manifold is collapsible iff it is a ball.

# THE ZEEMAN CONJECTURE

1964. If $K$ is a finite contractible 2-complex then $K \times I$ is collapsible.

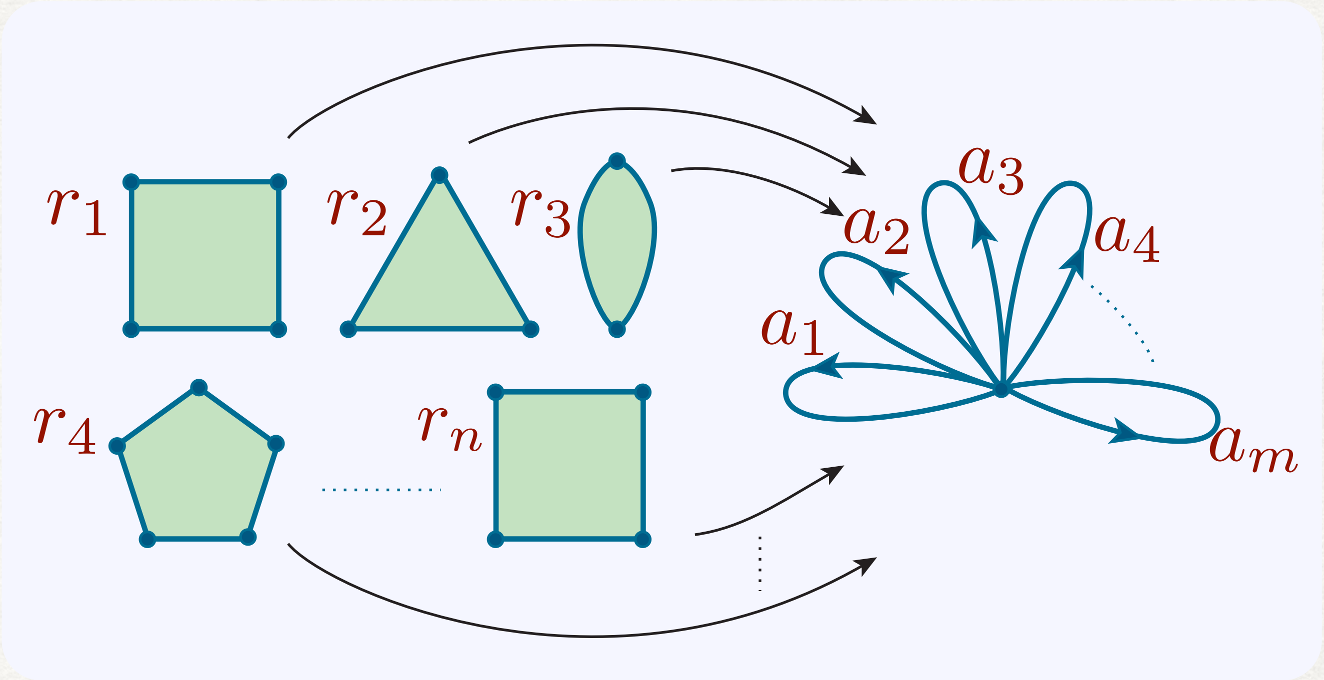
E.g. the dunce hat $\langle a \mid a^2 a^{-1} \rangle$ $K=$






## Folklore, P. Wright, J.R. Stallings

A-C with stabilization holds iff every finite contractible 2-complex is simple homotopic to a point via dimension at most 3.

$K =$ 


$$r_i,\ r_j \mapsto r_i r_j,\ r_j$$
$$r_i \mapsto r_i^{-1}$$
$$r_i \mapsto a_k^{\mp 1} r_i a_k^{\pm 1}$$

So Zeeman $\Longrightarrow$ A-C with stabilization.

Also, Zeeman $\Longrightarrow$ Poincaré!

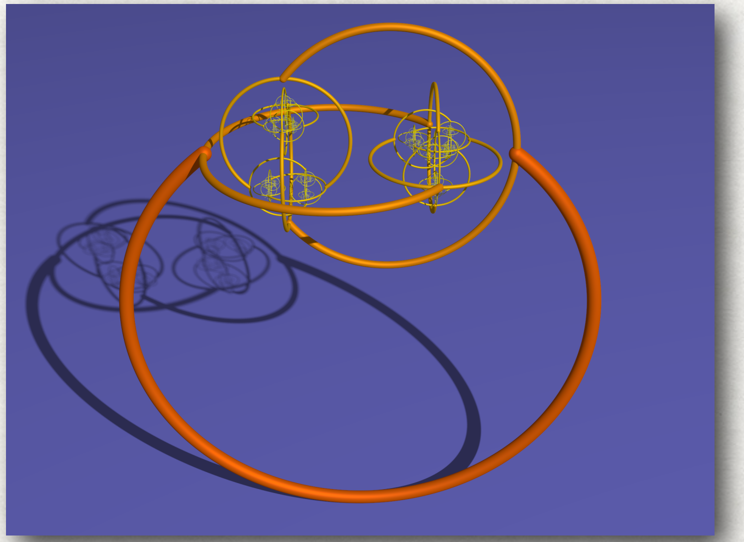
Suppose $M$ is a closed simply connected 3-manifold.

Assume $M$ is simplicially triangulated.

Let $N$ be $M$ with the interior of a 3-simplex removed.

Let $K$ be a spine of $N$.

$$\left.\begin{array}{l}\chi(K) = \chi(N) = \chi(M) + 1 = 1 \\ \pi_1 K = \pi_1 N = \pi_1 M = 1\end{array}\right\} \Longrightarrow K \text{ contractible}$$



$$N \times I \searrow K \times I \underset{\text{Zeeman}}{\searrow} \text{pt}$$

$$N \times I \underset{\text{Whitehead}}{\cong} B^4 \quad N \times \{0\} \subseteq \partial(N \times I) \cong S^3$$

$\partial N \cong S^2$ A PL-$S^2$ in an $S^3$ bounds a $B^3$ by Schönflies.

So $N \cong B^3$ and $M \cong S^3$.

**R. Lickorish** (1973)

$\langle a, b \mid a^2b^3, a^3b^4 \rangle$ does not give a counterexample to Zeeman.

**M. Cohen** (1975)

If $K$ is a finite contractible 2-complex then $K \times I^6$ is collapsible.

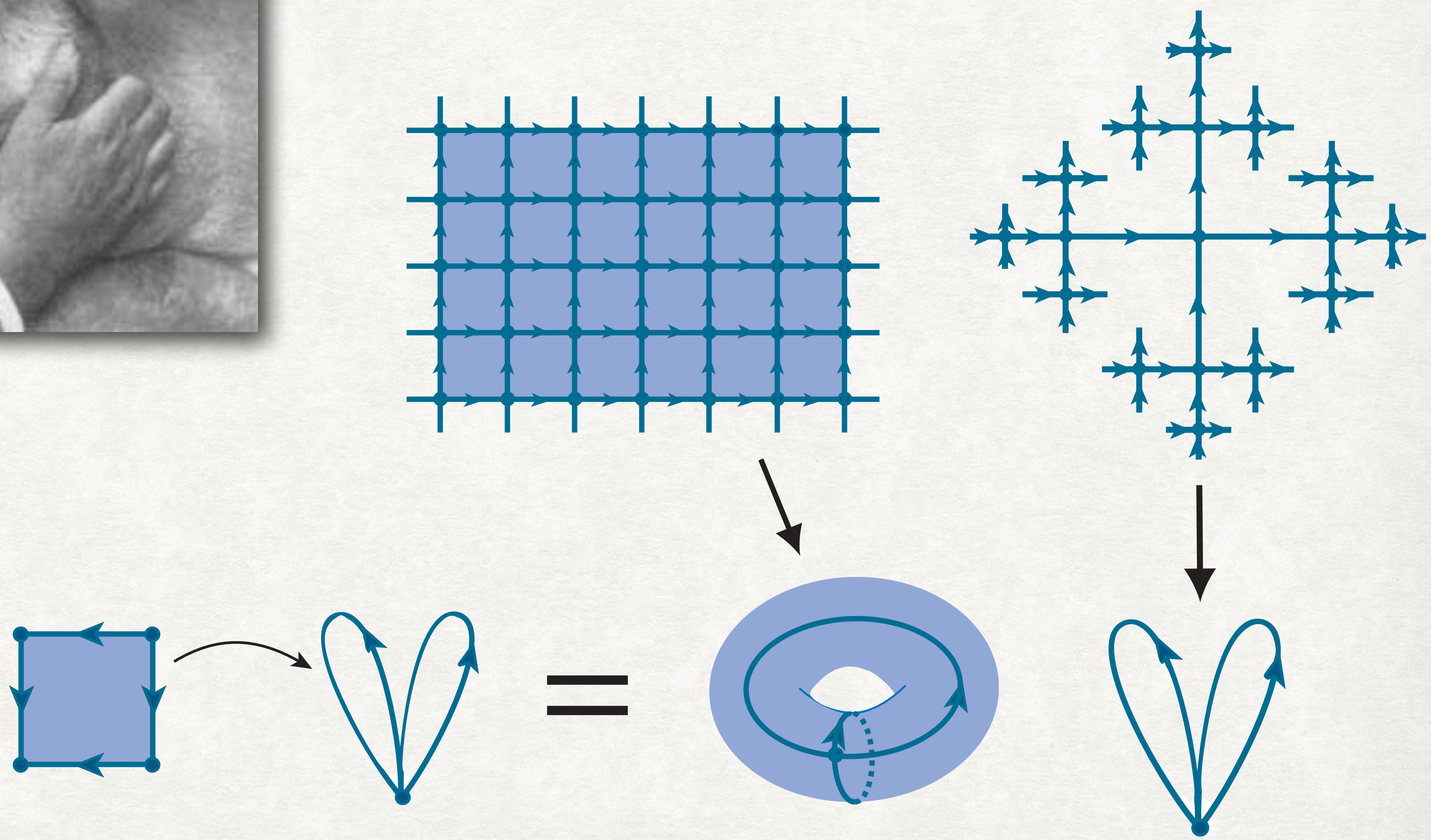
**M. Cohen** (1977)

For all $n \geq 3$, there is a finite contractible $n$-complex $K$ such that $K \times I$ is not collapsible.

Zeeman is true for 2-complexes arising from Poincaré.

# WHITEHEAD'S ASPHERICITY QUESTION

Is every connected subcomplex of an aspherical 2-complex itself aspherical?



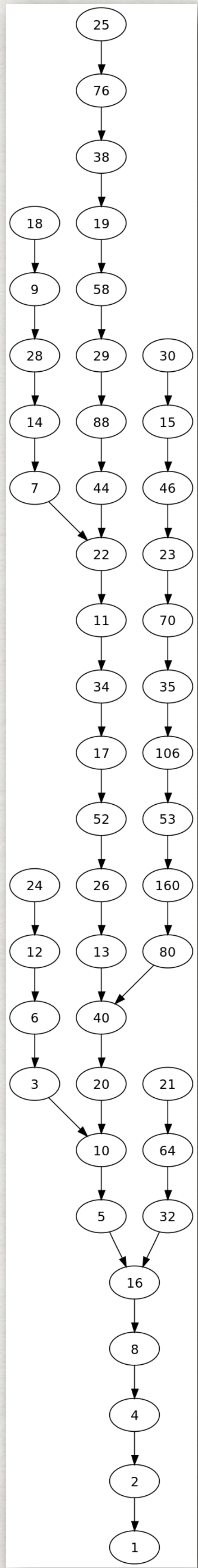
Why are these questions hard?

There is no algorithm to decide which finite 2-complexes have trivial $\pi_1$.

Open Question

Is there an algorithm to decide which finite simplicial 2-complexes are contractible?

…equivalently, which finite balanced presentations give the trivial group?




# Collatz map

$$n \mapsto \begin{cases} 3n+1 & \text{for } n \text{ odd} \\ n/2 & \text{for } n \text{ even} \end{cases}$$

**Leary** (2019)

There is no algorithm to decide which infinite recursively described acyclic aspherical 2-complexes are contractible.

**Leary** (2019)

There is an infinite recursively described acyclic aspherical 2-complex which is contractible iff the Collatz Conjecture is true.

# THANK YOU

## Further references / acknowledgements


D. Calegari, The 4-dimensional Poincaré Conjecture, unpublished notes available on his web page

A. Kupers, Zeeman's Conjecture, unpublished notes available on his web page

J. Harlander, C. Hog-Angeloni, W. Metzler, S. Rosebrock, Problems in Low-dimensional topology, Encyclopaedia of Mathematics, Springer

D. Rolfsen, Cousins of the Poincaré Conjecture, unpublished notes on his web site