SCREWBALL CONJECTURES IN GROUP THEORY
AND LOW-DIMENSIONAL TOPOLOGY

TIMOTHY RILEY
\[ \Gamma = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle \]
\[ = F(a_1, \ldots, a_m)/\langle r_1, \ldots, r_n \rangle \]

\[ K = \]

\[ \pi_1(K) = \Gamma \text{ by Seifert–van Kampen} \]
\[ \langle a, b \mid \rangle \]

\[ \langle a, b \mid a^{-1}b^{-1}ab \rangle \]

\[ \langle a, b, c, d \mid a^{-1}b^{-1}ab\, c^{-1}d^{-1}cd \rangle \]
THE ANDREWS–CURTIS CONJECTURE

FREE GROUPS AND HANDLEBODIES

J. J. ANDREWS\(^1\) AND M. L. CURTIS\(^1\)

In this note we state a conjecture about free groups and give some topological consequences which would follow if the conjecture is true.

Proc AMS 1965
Every balanced presentation
\[ \langle a_1, \ldots, a_m \mid r_1, \ldots, r_m \rangle \]
of the trivial group can be converted to
\[ \langle a_1, \ldots, a_m \mid a_1, \ldots, a_m \rangle \]
using the moves
\[
\begin{align*}
r_i, r_j &\leftrightarrow r_i r_j, r_j \\
r_i &\leftrightarrow r_i^{-1} \\
r_i &\leftrightarrow a_k^{\pm1} r_i a_k^{\pm 1}
\end{align*}
\]
\[
\begin{align*}
a^{-1}b^{-1}ab^{-1}a^{-1}, \ aba^{-1} \\
a^{-1}b^{-1}, \ aba^{-1} \\
a^{-1}b^{-1}, \ b \\
a^{-1}, \ b \\
a, \ b
\end{align*}
\]
$a b a b^{-1} a^{-1} b^{-1}, a^3 b^{-2}$
$a^3 b^{-2}, a^2 b^{-1} a b a^{-1} b^{-1}$
$a^2 b^{-1} a b a^{-1} b^{-1}, a b^{-1} a b^{-1} a^{-1} b$
$a b^{-1} a b^{-1} a^{-1}, a^2 b^{-1} a b^{-1}$
$a^2 b^{-1} a b^{-1}, a^2 b^{-1}$
$a^2 b^{-1}, a b^{-1}$
$a b^{-1}, a$
a, b

Andrew Casson
Unknown:

\[ \langle a, b \mid aba = bab, a^{n+1} = b^n \rangle \quad n \geq 3 \]

Akbulut–Kirby

\[ \langle a, b \mid a^2b = ba^3, w = 1 \rangle \]

where the exponent sum of \( b \) in \( w \) is \( \pm 1 \).

Miller–Schupp

\[ \langle a, b, c \mid c^{-1}bc = b^2, a^{-1}ca = c^2, b^{-1}ab = a^2 \rangle \]

B.Neumann–Rapaport–Higman
Computational searches

Miasnikov & Myasnikov 1999
Casson ~2003
Havas & Ramsay 2003
Panteleev & Ushakov 2016
Lisitsa 2019

No counterexamples with relations of total length $\leq 12$.

All examples with relations of total length $\leq 13$ trivializable or reducible to

$$\langle a, b \mid aba = bab, a^4 = b^3 \rangle$$

Presentations of total length $\sim n$ which are trivialisable but require at least

$$2^\ldots2^{[\log_2 n]}$$

moves.

E.g.

$$\langle a, t, \alpha, \tau \mid [tat^{-1}, a]a^{-1}, \ [\tau \alpha \tau^{-1}, \alpha]a^{-1},$$

$$\alpha t^{-1}a^{-1}[a, [t[t[a^{20}t^{-1}, a]t^{-1}, a]t^{-1}, a]],$$

$$a \tau^{-1}a^{-1}[\alpha, [\tau[\tau \alpha^{20} \tau^{-1}, \alpha]\tau^{-1}, \alpha]\tau^{-1}, \alpha]]\rangle.$$  

requires at least $10^{10,000}$ moves.
Myasnikov (1984)
No obstructions in solvable groups

No obstructions in finite groups

\[
\langle a_1, \ldots, a_m \mid r_1, \ldots, r_m \rangle
\]
\[
\Downarrow
\]
\[
\langle a_1, \ldots, a_m \mid a_1, \ldots, a_m \rangle
\]

\[
r_i, r_j \mapsto r_i r_j, r_j
\]
\[
r_i \mapsto r_i^{-1}
\]
\[
r_i \mapsto a_k^\pm r_i a_k^\pm 1
\]
\[ P = \langle x, y \mid [x, y], 1 \rangle \]
\[ Q = \langle x, y \mid [x, [x, y^{-1}]]^2 y[y^{-1}, x]y^{-1}, [x, [[y^{-1}, x], x]] \rangle \]
\[ \cong \mathbb{Z}^2 \]

**Barmak** (2018)

\( P \) and \( Q \) are not AC-equivalent despite their 2–complexes being simple homotopy equivalent.

Invariant in \( F_2/F_2'' \).
Stabilization:

\[
\langle a_1, \ldots, a_m \mid r_1, \ldots, r_m \rangle \\
\leftrightarrow \langle a_1, \ldots, a_m, a_{m+1} \mid r_1, \ldots, r_m, a_{m+1} \rangle
\]
Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

Il y a donc deux cycles de $\mathcal{V}$ qui ne sont pas équivalents à zéro; donc $\mathcal{V}$ n’est pas simplement connexe.

En d’autres termes, le groupe fondamental de $\mathcal{V}$ ne saurait se réduire à la substitution identique, puisqu’il contient comme sous-groupe le groupe icosahédrique.

Il resterait une question à traiter :

Est-il possible que le groupe fondamental de $\mathcal{V}$ se réduise à la substitution identique, et que pourtant $\mathcal{V}$ ne soit pas simplement connexe?

En d’autres termes, peut-on tracer les cycles $K''$ et $K''$ de telle façon qu’ils ne soient pas bouclés et ne se coupent pas; que les équivalences

$$K' \equiv K_1 \equiv 0, \quad K'' \equiv K_2' \equiv 0$$

Cinquième Complément à L’Analysis Situs, 1904
$M$ a closed 3–manifold

$\Sigma = U_1 \cap U_2 = \partial U_1 = \partial U_2$

$\Gamma = \pi_1 \Sigma = \left\langle a_1, b_1, \ldots, a_g, b_g \left| \prod_{i=1}^{g} [a_i, b_i] \right. \right\rangle$

$F = \pi_1 U_1 = \pi_1 U_2$

$\Sigma \leftrightarrow U_1$ induces $\Gamma \leftrightarrow F$, i.e. $\Gamma \to F \times F$

which is surjective iff $\pi_1 M = 1$. A “splitting homomorphism.”
Equivalence of splitting homomorphisms:

\[
\Gamma \rightarrow F \times F \\
\rightarrow \cong \circ \rightarrow (\alpha, \beta) \quad \alpha, \beta \in \text{Aut}(F)
\]

\[
\Gamma \rightarrow F \times F
\]

**Stallings (1965)**

Poincaré holds iff every splitting homomorphism is equivalent to

\[
\left\langle a_1, b_1, \ldots, a_g, b_g \left| \prod_{i=1}^{g} [a_i, b_i] \right. \right\rangle \rightarrow \left\langle a_1, \ldots, a_g \right\rangle \times \left\langle b_1, \ldots, b_g \right\rangle
\]
$A = \{a_1, \ldots, a_n\}$. Every tuple of words 

$$(r_1, \ldots, r_n, q_1, \ldots, q_n)$$
on $A^{\pm 1}$ such that

$$\{ t^{-1}r_it \mid i = 1, \ldots, n; \ t \in F(q_1, \ldots, q_n) \}$$
generates $F(A)$ can be converted to

$$(a_1, \ldots, a_n, 1, \ldots, 1)$$

using

$$r_i, r_j \mapsto r_ir_j, r_j \quad q_i, q_j \mapsto q_iq_j, q_j$$

$$r_i \mapsto r_i^{-1} \quad q_i \mapsto q_i^{-1}$$

$$r_i \mapsto a_k^{\pm 1}r_ia_k^{\pm 1} \quad q_i \mapsto q_ir_j$$

G–K $\implies$ A–C
(1993) G–K holds iff every splitting homomorphism

\[ F_{2n} \rightarrow F_n \times F_n \]

is equivalent to

\[ \langle a_1, b_1, \ldots, a_n, b_n \rangle \rightarrow \langle a_1, \ldots, a_g \rangle \times \langle b_1, \ldots, b_g \rangle \]
THE GENERALIZED POINCARÉ CONJECTURE

If a closed $n$–manifold $M$ is homotopic to an $n$–sphere, is it an $n$–sphere?
THE GENERALIZED POINCARE CONJECTURE

If a closed $n$–manifold $M$ is homotopic to an $n$–sphere, is it an $n$–sphere?
\[ \Gamma = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle \]

If \( m = n \) and \( \Gamma \) is trivial, then \( \chi(K) = 1 \) and \( K \) is contractible.
$K \hookrightarrow \mathbb{R}^5$

$N = \text{regular neighbourhood of } K \text{ in } \mathbb{R}^5$

$$= B^5 \cup (1\text{-handles}) \cup (2\text{-handles})$$

$B^1 \times B^4$s
$B^2 \times B^3$s

$M = \partial N$ is a closed 4–manifold.

If the presentation is balanced, then $\chi(M) = \chi(S^4) = 2$.

$\pi_1 M = \pi_1 N = \pi_1 K$

So if $\pi_1 K = 1$, then $M$ is homotopy equivalent to $S^4$. 
A–C–moves correspond to handle–slides.

So potential counterexamples to A–C yield potential counterexamples to the Smooth 4–Dimensional Poincaré Conjecture.

Gompf (1991). \( \langle a, b \mid aba = bab, a^5 = b^4 \rangle \) yields a standard 4–sphere.
$K, \ L$ CW–complexes

Elementary collapses: $\xrightarrow{\text{Elementary expansions}}$

$X, \ Y$ “simple homotopic” when homeomorphic to complexes related by a sequence of elementary collapses and expansions.

$X$ “collapsible” when homeomorphic to a complex reducible to a point through a sequence of elementary collapses.

Whitehead: a PL–manifold is collapsible iff it is a ball.
1964. If $K$ is a finite contractible 2–complex then $K \times I$ is collapsible.

E.g. the dunce hat $\langle a \mid a^2a^{-1} \rangle \qquad K = \bigtriangleup$
Folklore, P. Wright, J.R. Stallings
A–C with stabilization holds iff every finite contractible 2–complex is simple homotopic to a point via dimension at most 3.

So Zeeman $\implies$ A–C with stabilization.
Also, Zeeman $\iff$ Poincaré!

Suppose $M$ is a closed simply connected 3-manifold. Assume $M$ is simplicially triangulated. Let $N$ be $M$ with the interior of a 3-simplex removed. Let $K$ be a spine of $N$.

\[ \chi(K) = \chi(N) = \chi(M) + 1 = 1 \]
\[ \pi_1 K = \pi_1 N = \pi_1 M = 1 \]
\[ \implies K \text{ contractible} \]

\[ \begin{align*}
N \times I &\searrow K \times I \searrow \text{pt} \\
N \times I &\cong B^4 \\
\partial N &\cong S^2 \\
N \times \{0\} &\subseteq \partial(N \times I) \cong S^3
\end{align*} \]

A PL-$S^2$ in an $S^3$ bounds a $B^3$ by Schönflies.

So $N \cong B^3$ and $M \cong S^3$. 
R. Lickorish (1973)
\[ \langle a, b \mid a^2b^3, a^3b^4 \rangle \] does not give a counterexample to Zeeman.

M. Cohen (1975)
If \( K \) is a finite contractible 2–complex then \( K \times I^6 \) is collapsible.

M. Cohen (1977)
For all \( n \geq 3 \), there is a finite contractible \( n \)–complex \( K \) such that \( K \times I \) is not collapsible.

Zeeman is true for 2–complexes arising from Poincaré.
Is every connected subcomplex of an aspherical 2-complex itself aspherical?
Why are these questions hard?
There is no algorithm to decide which finite 2–complexes have trivial $\pi_1$.

Open Question

Is there an algorithm to decide which finite simplicial 2–complexes are contractible?

...equivalently, which finite balanced presentations give the trivial group?
There is no algorithm to decide which infinite recursively described acyclic aspherical 2-complexes are contractible.

Leary (2019)

There is an infinite recursively described acyclic aspherical 2-complex which is contractible iff the Collatz Conjecture is true.
THANK YOU
Further references / acknowledgements

D. Calegari, The 4–dimensional Poincaré Conjecture, unpublished notes available on his web page

A. Kupers, Zeeman’s Conjecture, unpublished notes available on his web page


D. Rolfsen, Cousins of the Poincaré Conjecture, unpublished notes on his web site