What is a Dehn function?

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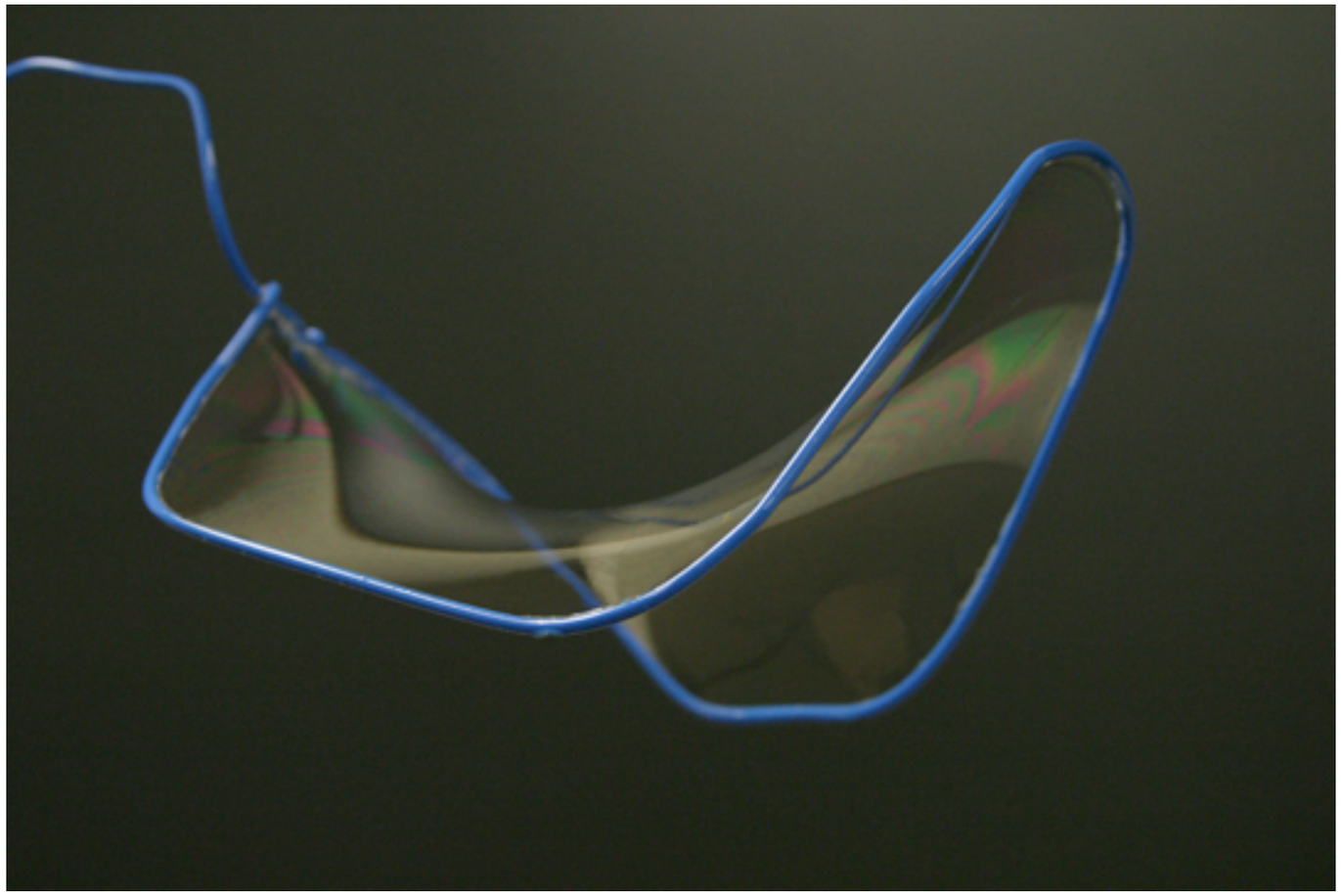
"What is...?" Seminar

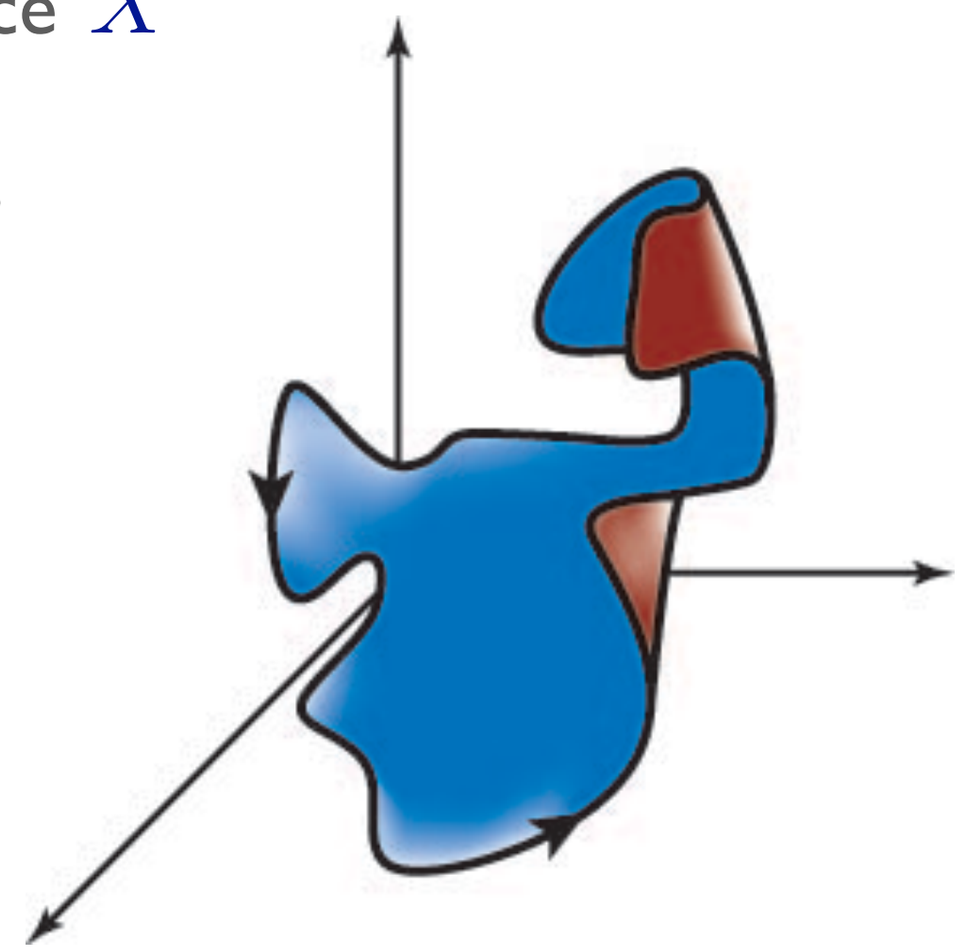
Cornell, September 9, 2009



Euclidean space “enjoys a quadratic isoperimetric function.”

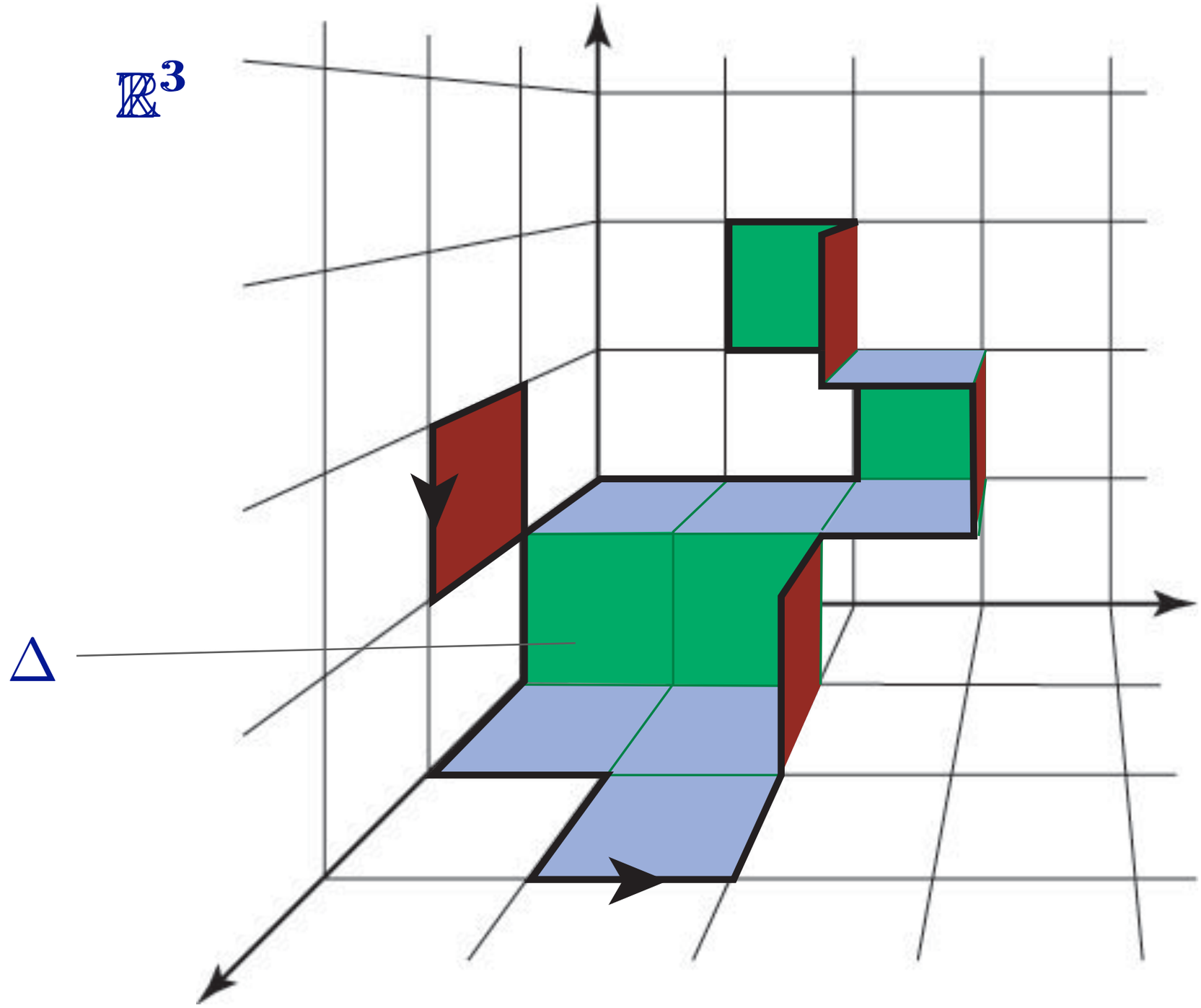
ρ a loop in a simply connected space X

$\mathrm{Area}(\rho)$ is the infimum of the areas of discs spanning ρ.



$\mathrm{Area}_X : [0, \infty) \to [0, \infty]$ is defined by

$$\mathrm{Area}_X(l) \;=\; \sup\{\mathrm{Area}(\rho) \mid \ell(\rho) \leq l\}.$$

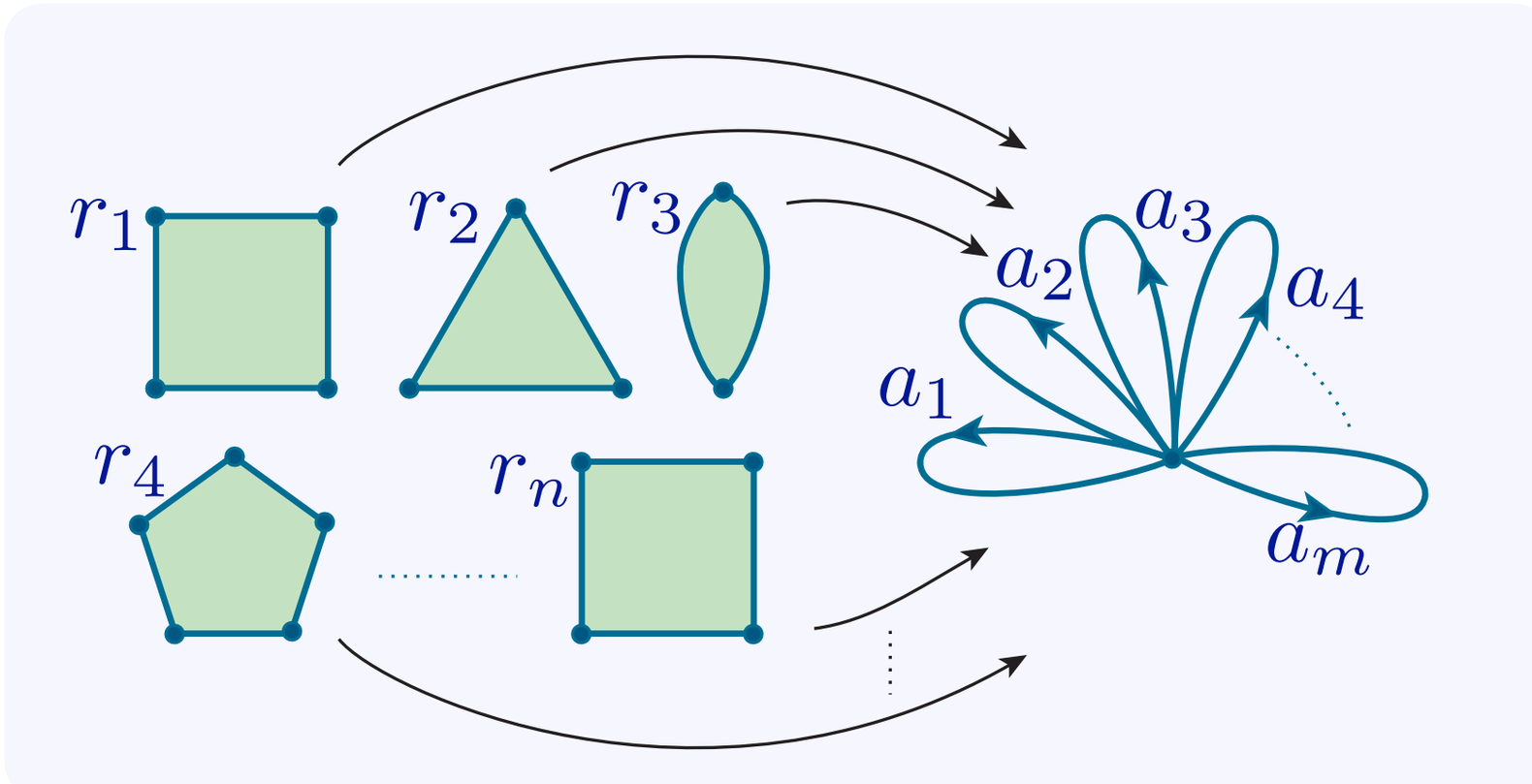


$$\mathrm{Area}(\Delta) = \#\ \text{2-cells}$$

$\mathcal{P} = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$ a finite presentation of a group Γ

The presentation 2-complex of $\mathcal{P}$:

$K =$

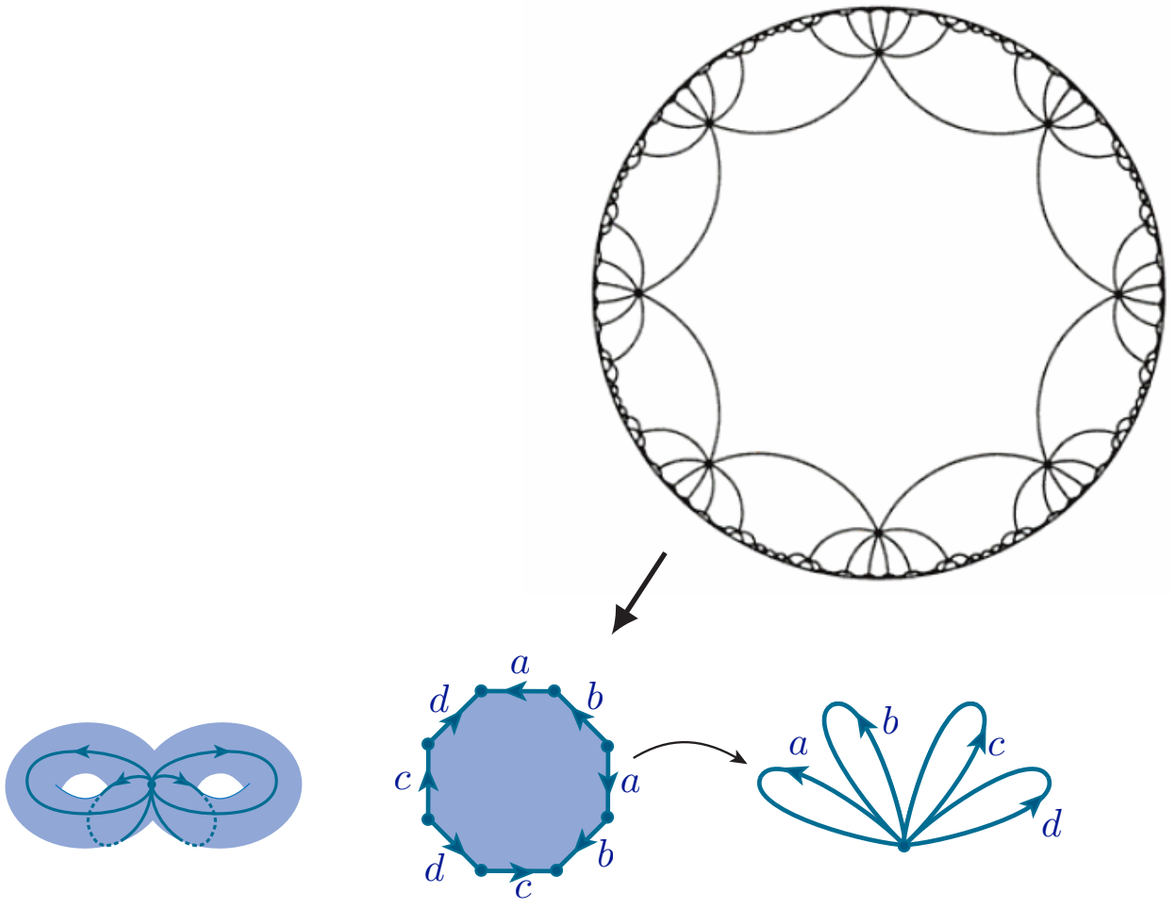


$\pi_1(K) = \Gamma$

The universal cover $\widetilde{K}$ is the *Cayley 2-complex* of $\mathcal{P}$.

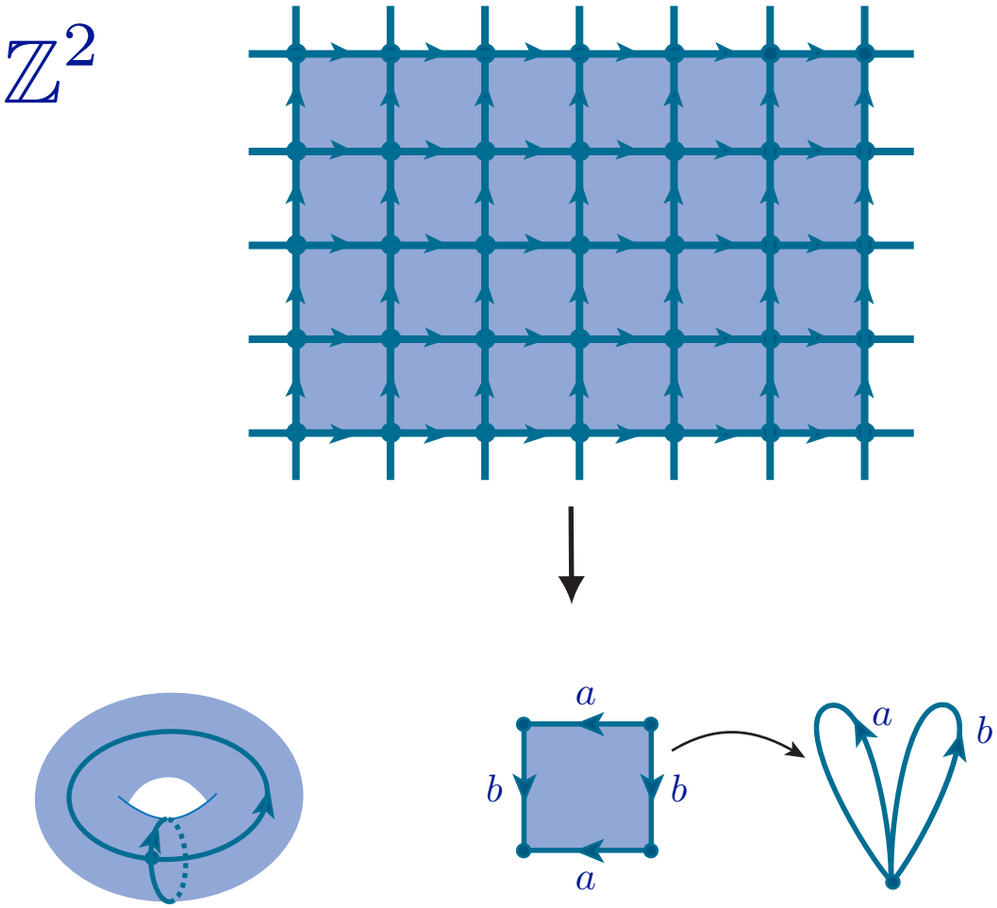
Its 1-skeleton $\widetilde{K}^{(1)}$ is the *Cayley graph* of $\mathcal{P}$.

$\langle\, a, b, c, d \;\mid\; a^{-1}b^{-1}ab\,c^{-1}d^{-1}cd \,\rangle$

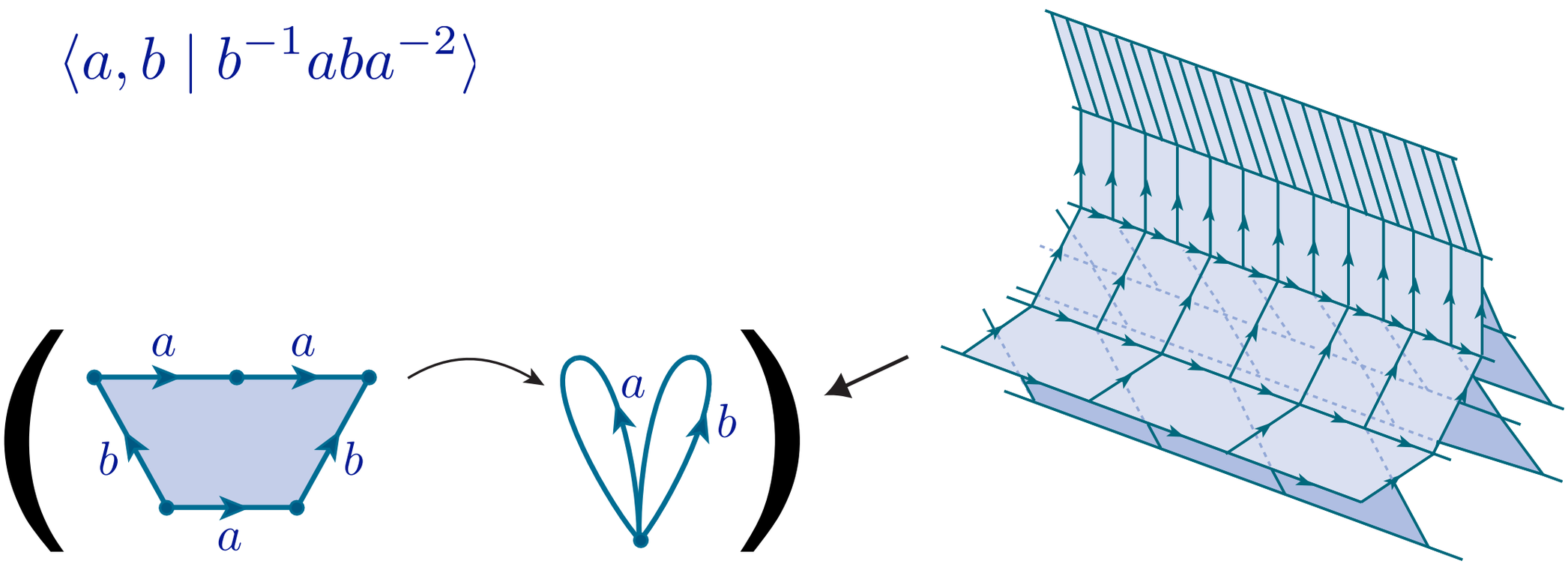


$\langle a, b \mid a^{-1}b^{-1}ab \rangle$

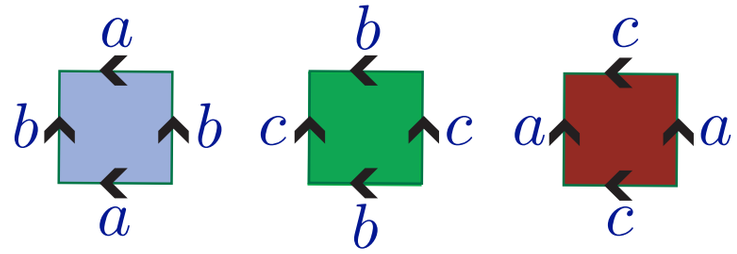
$\mathbb{Z}^2$



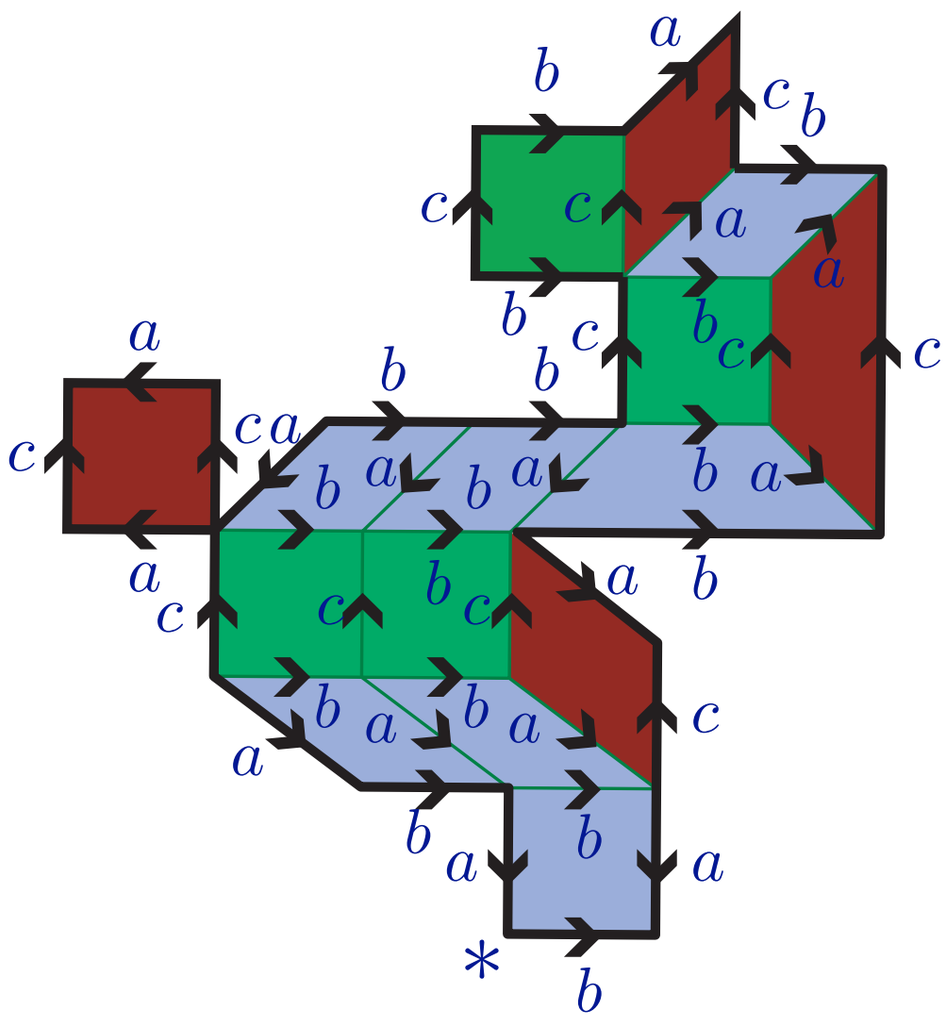
$\langle a, b \mid b^{-1}aba^{-2} \rangle$

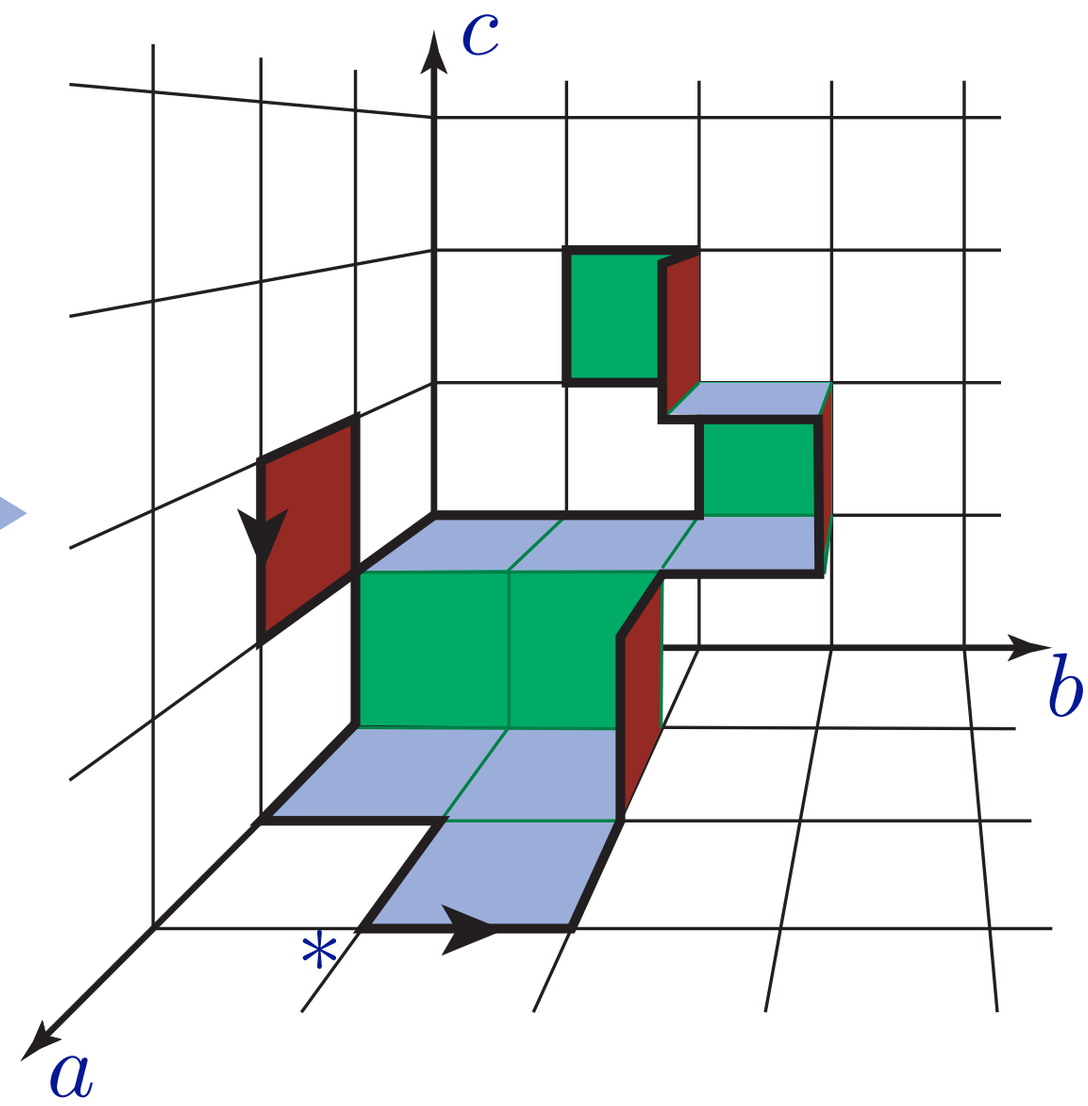


$$\mathbb{Z}^3 = \langle\ a, b, c \ \mid$$



$$\ \rangle$$





A *van Kampen diagram for* $ba^{-1}ca^{-1}bcb^{-1}ca^{-1}b^{-1}c^{-1}bc^{-1}b^{-2}acac^{-1}a^{-1}c^{-1}aba$.

A *van Kampen diagram* for a word w is a finite planar contractible 2-complex Δ with edges directed and labelled so that around each 2-cell one reads a defining relation and around $\partial\Delta$ one reads w .

For an edge-loop ρ in the Cayley 2-complex of a finite presentation $\mathcal{P}$, $\mathrm{Area}(\rho)$ is the minimum of $\mathrm{Area}(\Delta)$ over all van Kampen diagrams spanning ρ.

The *Dehn function* $\mathrm{Area}_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ of a finite presentation $\mathcal{P}$ with Cayley 2-complex $\widetilde{K}$ is

$$\mathrm{Area}_{\mathcal{P}}(n) \;=\; \max\{\mathrm{Area}(\rho) \,|\text{edge-loops } \rho \text{ in } \widetilde{K} \text{ with } \ell(\rho) \leq n\}.$$

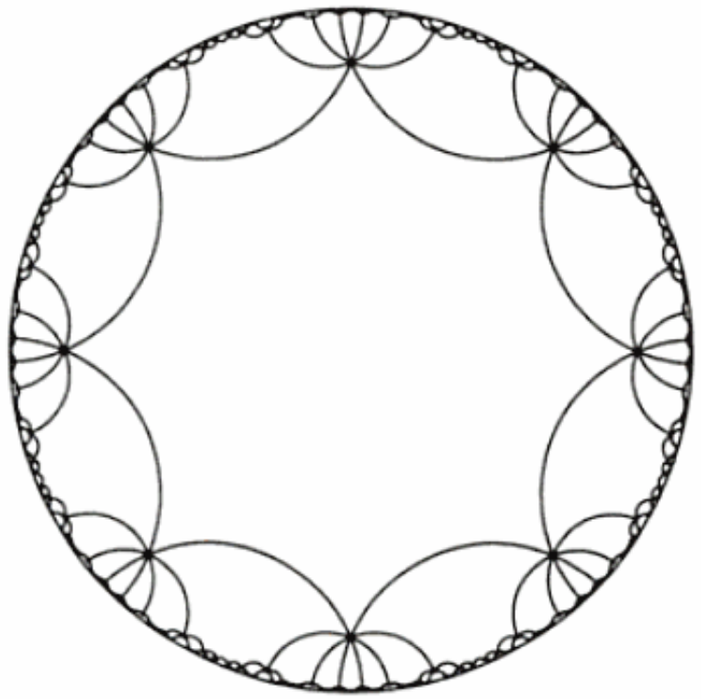
The Filling Theorem. If $\mathcal{P}$ is a finite presentation of the fundamental group of a closed Riemannian manifold M then

$$\mathrm{Area}_{\mathcal{P}} \;\simeq\; \mathrm{Area}_{\widetilde{M}}.$$

$f \preceq g$ when $\exists C > 0$ such that $\forall n > 0,\ f(n) \leq Cg(Cn + C) + Cn + C$.

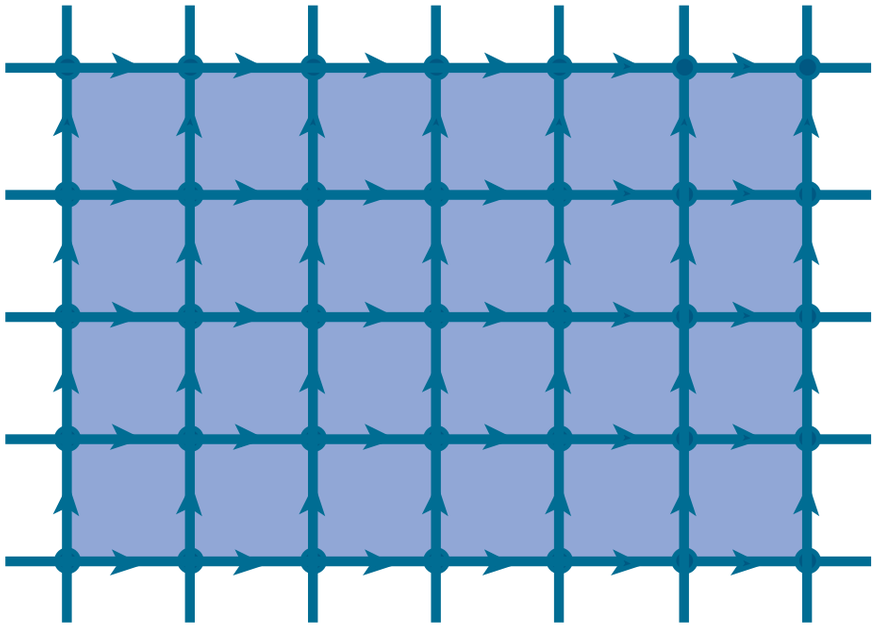
$f \simeq g$ when $f \preceq g$ and $g \simeq f$.

$\langle\, a, b, c, d \;\mid\; a^{-1}b^{-1}ab\, c^{-1}d^{-1}cd \,\rangle$



$\mathrm{Area}(n) \simeq n$

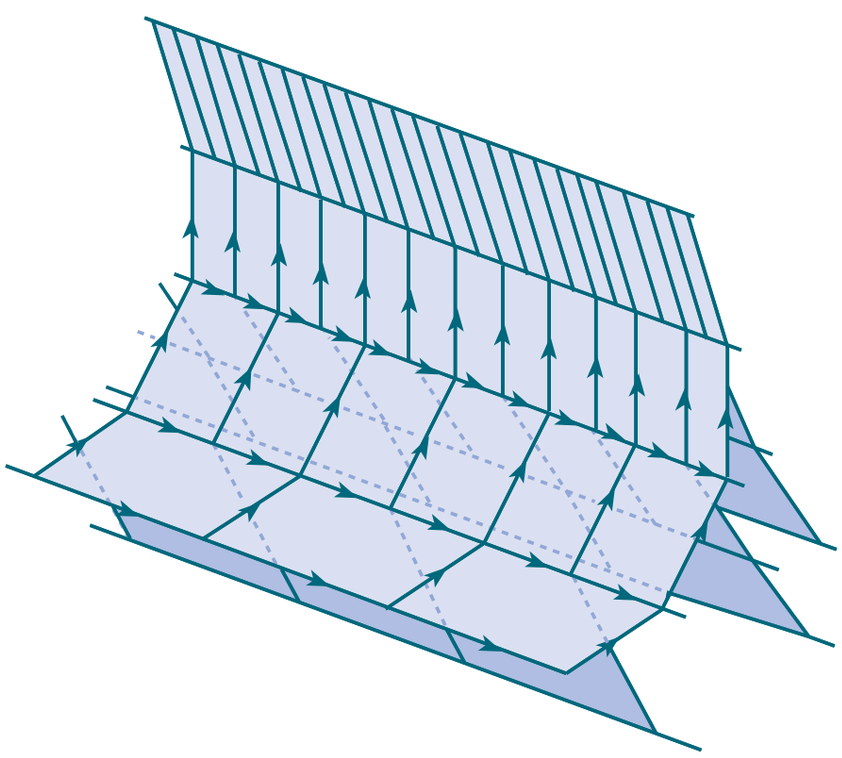
$\langle a, b \mid a^{-1}b^{-1}ab \rangle$

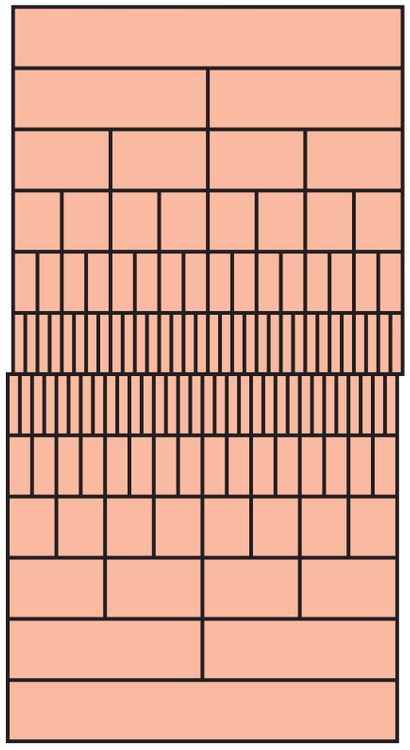


$\mathrm{Area}(n) \simeq n^2$

$\langle a, b \mid b^{-1}aba^{-2} \rangle$

$\mathrm{Area}(n) \simeq 2^n$



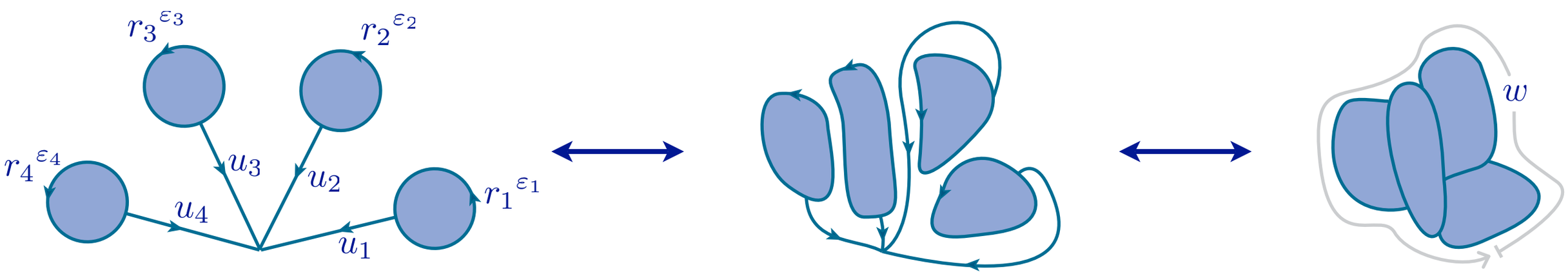


van Kampen's Lemma. For a group Γ with finite presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$, and a word w, the following are equivalent.

(i) $w = 1$ in Γ,

(ii) $w = \prod_{i=1}^{N} {u_i}^{-1} {r_i}^{\varepsilon_i} u_i$ in $F(\mathcal{A})$ for some $r_i \in \mathcal{R}$, $\varepsilon_i = \pm 1$, words u_i,

(iii) w admits a van Kampen diagram.



Technicalities hidden!

Moreover, $\mathrm{Area}(w)$ is the minimal N as occurring in (ii).

Examples of Dehn functions

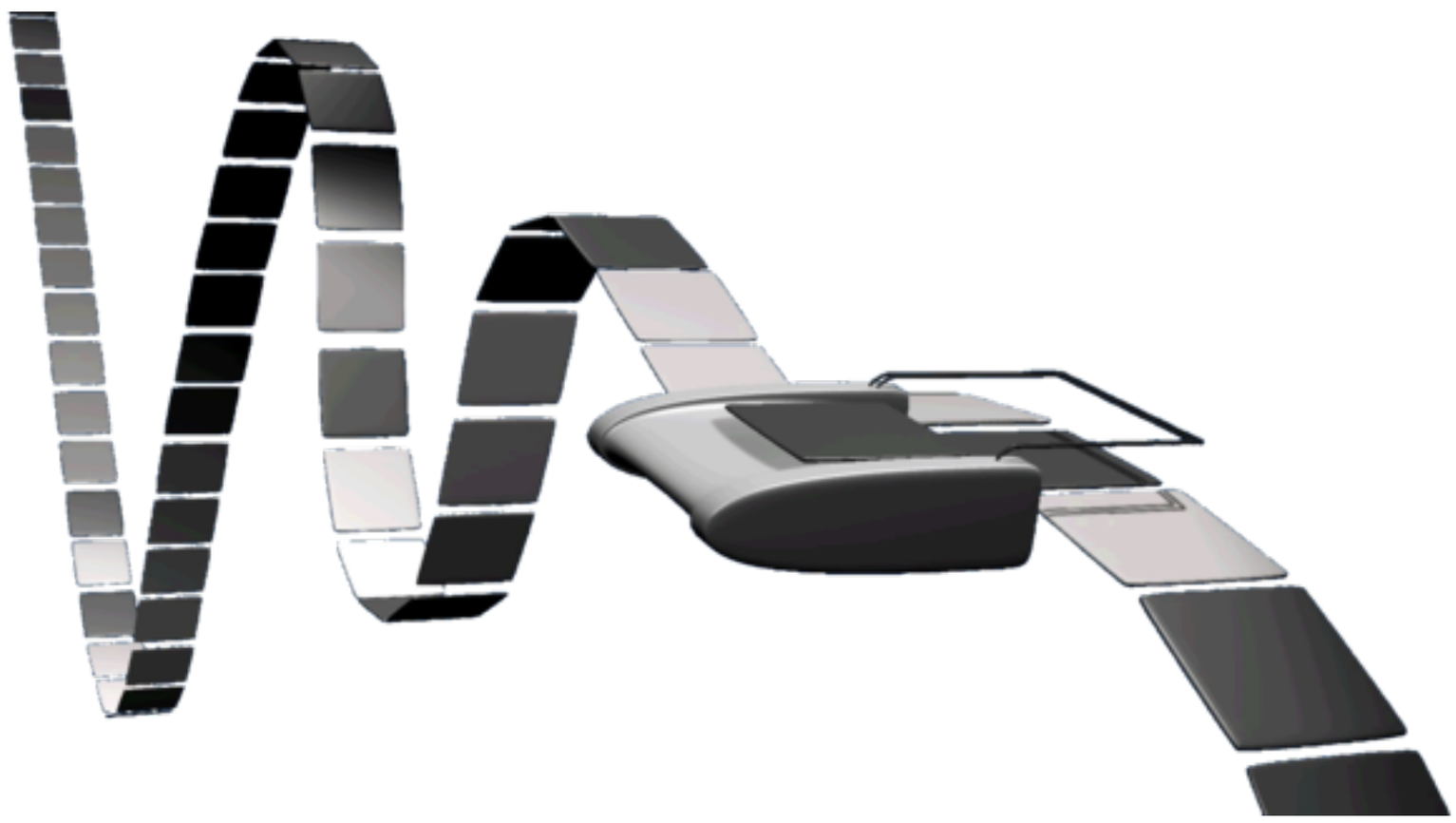
- finite groups: $\leq Cn$
- free groups: $\leq Cn$
- hyperbolic groups $:=$ groups with Dehn function $\leq Cn$
- finitely generated abelian groups: $\leq Cn^2$
- 3-dimensional integral Heisenberg group: $\simeq n^3$
- class c free nilpotent group on 2 letters: $\simeq n^{c+1}$

$$\mathrm{IP} \;=\; \{\, \alpha > 0 \mid n \mapsto n^{\alpha} \text{ is } \simeq \text{ a Dehn function} \,\}$$

The closure of IP is

Gromov, Bowditch, N. Brady, Bridson, Olshanskii, Sapir...

Theorem. (*Sapir–Birget–Rips.*) If $\alpha > 4$ is such that $\exists c > 0$ and a Turing Machine calculating the first m digits of the decimal expansion of α in time $\leq c2^{2^{cm}}$ then $\alpha \in \mathrm{IP}$. Conversely, if $\alpha \in \mathrm{IP}$ then there exists a Turing Machine that calculates the first m digits in time $\leq c2^{2^{2^{cm}}}$ for some constant $c > 0$.



Dehn functions and the Word Problem

For a finitely presented group, t.f.a.e.:

- the word problem is solvable,
- the Dehn function is recursive,
- the Dehn function is bounded from above by a recursive function.

Dehn function "is" a geometric time–complexity measure

But groups with large Dehn function may have efficient solutions to their word problem.

Example:

$$\langle a, b \mid b^{-1}aba^{-2} \rangle \leq GL_2(\mathbb{Q})$$

via $a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $b \mapsto \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$

Big Dehn functions

$$\langle a, b \mid b^{-1}ab = a^2 \rangle \qquad \exp(n)$$

$$\langle a, b, c \mid b^{-1}ab = a^2,\ c^{-1}bc = b^2 \rangle \qquad \exp(\exp(n))$$

$$\langle a, b, c, d \mid b^{-1}ab = a^2,\ c^{-1}bc = b^2,\ d^{-1}cd = c^2 \rangle \qquad \exp(\exp(\exp(n)))$$

$$\vdots \qquad\qquad \vdots$$

$$\langle a, b \mid (b^{-1}ab)^{-1}a(b^{-1}ab) = a^2 \rangle \qquad \underbrace{\exp(\ldots(\exp}_{\log n}(n))\ldots)$$

Hydra Groups

The group generated by

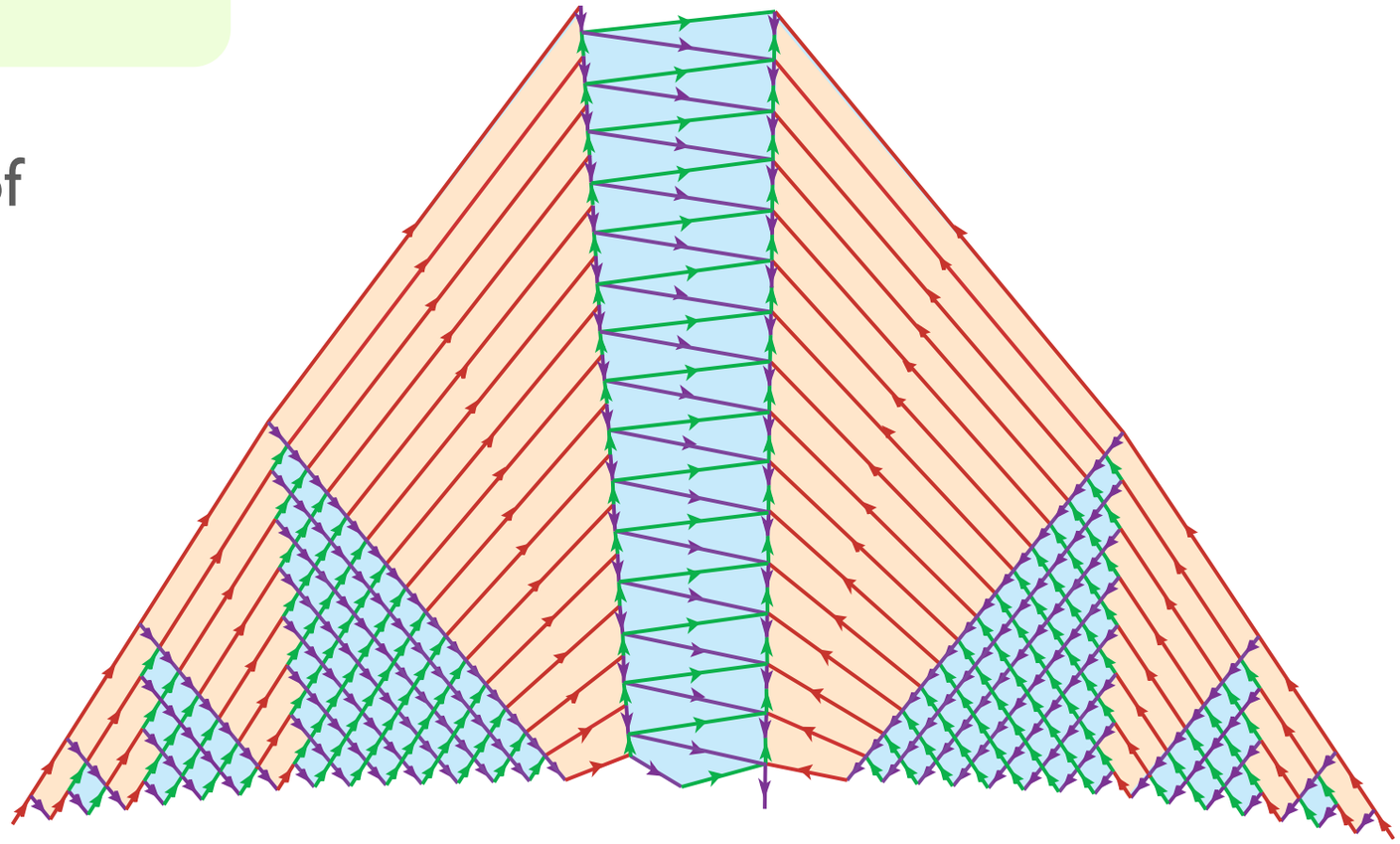
$$a_1, \ldots, a_k, p, t$$

subject to

$$t^{-1} a_i t = a_i a_{i-1} \text{ for all } i > 1$$
$$t^{-1} a_1 t = a_1$$
$$[p, a_i t] = 1 \text{ for all } i$$

has Dehn function like the k-th of Ackermann's functions

$$n \mapsto A_k(n)\,.$$



There are finitely presented groups with unsolvable Word Problem (*Novikov–Boone)*.

These have Dehn function which cannot be bounded from above by a recursive function.