What is a Dehn function?

Tim Riley

“What is...?” Seminar

Cornell, September 9, 2009
Euclidean space “enjoys a quadratic isoperimetric function.”
\( \rho \) a loop in a simply connected space \( X \)

\( \text{Area}(\rho) \) is the infimum of the areas of discs spanning \( \rho \).

\( \text{Area}_X : [0, \infty) \rightarrow [0, \infty] \) is defined by

\[
\text{Area}_X(l) = \sup\{\text{Area}(\rho) \mid \ell(\rho) \leq l\}.
\]
$\mathbb{R}^3$

$\Delta$

Area($\Delta$) = $\#$ 2-cells
\( \mathcal{P} = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle \) a finite presentation of a group \( \Gamma \)

The presentation 2-complex of \( \mathcal{P} \):

\[
K = \begin{array}{cccc}
\hspace{1cm} & r_1 & \hspace{1cm} & r_2 & \hspace{1cm} & r_3 & \hspace{1cm} \\
\hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
r_4 & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
\hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
r_n & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
\hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
a_1 & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
a_2 & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
a_3 & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
a_4 & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
a_m & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} & \hspace{1cm} \\
\end{array}
\]

\( \pi_1(K) = \Gamma \)

The universal cover \( \tilde{K} \) is the Cayley 2-complex of \( \mathcal{P} \).

Its 1-skeleton \( \tilde{K}^{(1)} \) is the Cayley graph of \( \mathcal{P} \).
\[ \langle a, b, c, d \mid a^{-1}b^{-1}ab\ c^{-1}d^{-1}cd \rangle \]

\[ \langle a, b, c, d \mid a^{-1}b^{-1}ab\ c^{-1}d^{-1}cd \rangle \]

\[ \langle a, b \mid a^{-1}b^{-1}ab \rangle \]

\[ \mathbb{Z}^2 \]

\[ \langle a, b \mid b^{-1}aba^{-2} \rangle \]

\[ (a, b, c, d) \]

\[ (a, b, c, d) \]
A van Kampen diagram for $\mathbb{Z}^3 = \langle a, b, c \mid b\ a^{-1} c a^{-1} b c b^{-1} c a^{-1} b^{-1} c b^{-1} b^{-2} a c a c^{-1} a^{-1} c^{-1} a b a \rangle$.

A van Kampen diagram for a word $w$ is a finite planar contractible 2-complex $\Delta$ with edges directed and labelled so that around each 2-cell one reads a defining relation and around $\partial \Delta$ one reads $w$. 
The Filling Theorem. If $\mathcal{P}$ is a finite presentation of the fundamental group of a closed Riemannian manifold $\tilde{M}$ then

$$\text{Area}_\mathcal{P} \simeq \text{Area}_{\tilde{M}}.$$
\[ \langle a, b, c, d \mid a^{-1}b^{-1}ab c^{-1}d^{-1}cd \rangle \]

Area\( (n) \simeq n \)

\[ \langle a, b \mid a^{-1}b^{-1}ab \rangle \]

Area\( (n) \simeq n^2 \)

\[ \langle a, b \mid b^{-1}aba^{-2} \rangle \]

Area\( (n) \simeq 2^n \)
**van Kampen’s Lemma.** For a group $\Gamma$ with finite presentation $\langle A \mid R \rangle$, and a word $w$, the following are equivalent.

(i) $w = 1$ in $\Gamma$,

(ii) $w = \prod_{i=1}^{N} u_i^{-1} r_i^{\varepsilon_i} u_i$ in $F(A)$ for some $r_i \in R$, $\varepsilon_i = \pm 1$, words $u_i$,

(iii) $w$ admits a van Kampen diagram.

Moreover, $\text{Area}(w)$ is the minimal $N$ as occurring in (ii).
Examples of Dehn functions

- finite groups: \[ \leq Cn \]

- free groups: \[ \leq Cn \]

- hyperbolic groups := groups with Dehn function \[ \leq Cn \]

- finitely generated abelian groups: \[ \leq Cn^2 \]

- 3-dimensional integral Heisenberg group: \[ \simeq n^3 \]

- class \( c \) free nilpotent group on 2 letters: \[ \simeq n^{c+1} \]
The closure of $\text{IP}$ is

$$\text{IP} = \{ \alpha > 0 \mid n \mapsto n^\alpha \text{ is } \simeq \text{ a Dehn function} \}$$

Gromov, Bowditch, N. Brady, Bridson, Olshanskii, Sapir...
Theorem. (Sapir–Birget–Rips.) If $\alpha > 4$ is such that $\exists c > 0$ and a Turing Machine calculating the first $m$ digits of the decimal expansion of $\alpha$ in time $\leq c2^{2cm}$ then $\alpha \in 1P$. Conversely, if $\alpha \in 1P$ then there exists a Turing Machine that calculates the first $m$ digits in time $\leq c2^{2cm}$ for some constant $c > 0$. 
Dehn functions and the Word Problem

For a finitely presented group, t.f.a.e.:

• the word problem is solvable,
• the Dehn function is recursive,
• the Dehn function is bounded from above by a recursive function.

But groups with large Dehn function may have efficient solutions to their word problem.

Example:

\[ \langle a, b \mid b^{-1}aba^{-2} \rangle \leq GL_2(\mathbb{Q}) \]

via \[ a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \]

Dehn function “is” a geometric time–complexity measure
Big Dehn functions

\[ \langle a, b \mid b^{-1}ab = a^2 \rangle \quad \exp(n) \]

\[ \langle a, b, c \mid b^{-1}ab = a^2, c^{-1}bc = b^2 \rangle \quad \exp(\exp(n)) \]

\[ \langle a, b, c, d \mid b^{-1}ab = a^2, c^{-1}bc = b^2, d^{-1}cd = c^2 \rangle \quad \exp(\exp(\exp(n))) \]

\[ \vdots \]

\[ \langle a, b \mid (b^{-1}ab)^{-1}a(b^{-1}ab) = a^2 \rangle \quad \exp(\ldots(\exp(n))\ldots) \]

\[ \log n \]
Hydra Groups

The group generated by

\[ a_1, \ldots, a_k, p, t \]

subject to

\[
\begin{align*}
  t^{-1}a_it &= a_ia_{i-1} \quad \text{for all } i > 1 \\
  t^{-1}a_1t &= a_1 \\
  [p, a_it] &= 1 \quad \text{for all } i
\end{align*}
\]

has Dehn function like the \( k \)-th of Ackermann’s functions

\[ n \mapsto A_k(n). \]
There are finitely presented groups with unsolvable Word Problem (Novikov–Boone).

These have Dehn function which cannot be bounded from above by a recursive function.