Math 6310, Homework 1
Due in class 8/28

Review your undergraduate group theory as needed. To begin with, you need a good grasp of normal subgroups, quotients, and the isomorphism laws. (See Section 3.3 for the latter.)

Do §3.4, Qu. 7 and 8.

In the following questions you can use the fact that A_n is simple for $n \geq 5$. (An exercise on Homework 2 will guide you through proof.)

1. (a) Exhibit a composition series for S_4.
   (b) Show S_n is solvable for $n \leq 4$.
   (c) Use the Jordan–Hölder Theorem to prove that S_n is not solvable for $n \geq 5$.

2. Give an example of an infinite group which has a composition series.

3. The elements of a group G are the $120^2 \times 2$ ordered triples of the form (a, b, c) where $a, b \in S_5$ and $c = \pm 1$. Multiplication in G is defined by

$$(a_1, b_1, c_1)(a_2, b_2, c_2) \ = \ \begin{cases} (a_1 a_2, b_1 b_2, c_1 c_2) & \text{if } \ c_1 = 1, \\ (a_1 b_2, b_1 a_2, c_1 c_2) & \text{if } \ c_1 = -1. \end{cases}$$

   Show that $K := \{(a, b, c) \mid c = 1\}$ is a normal subgroup of G. Hence or otherwise find the composition factors of G.

4. Let G be an Ω–group with an Ω–composition series. If H is a normal Ω–subgroup of G, show that G has an Ω–composition series in which H is one of the terms. Deduce that H and G/H have Ω–composition series.

5. If G is an Ω–group with a composition series, the length of some (every) composition series is called the length of G and is denoted $l(G)$. With the notation of the previous exercise, show that $l(G) = l(H) + l(G/H)$. If H_1 and H_2 are two normal Ω–subgroups of G, show that $l(H_1 H_2) = l(H_1) + l(H_2) - l(H_1 \cap H_2)$. Deduce from this a familiar result from linear algebra about dimensions of subspaces.

Read ahead in §4.