MATH 6310, Homework 2
Due in class 9/4

Review group actions in §4.

Do §4.5, questions 16, 30, 43.

1. Let P be a Sylow p–subgroup of a finite group G and N a normal subgroup of G. Show that
   (a) $P \cap N$ is a p–Sylow subgroup of N, and
   (b) PN/N is a Sylow p–subgroup of G/N.

2. Show that a group of order 6 has exactly one Sylow 3–subgroup. Using this result, show that A_4 has no subgroup of order 6.

3. (a) Describe the conjugacy classes of elements of A_5.
   (b) Prove that A_5 is simple.

4. (a) Suppose $h \in S_n$ maps α to β where $\alpha \neq \beta$. Let γ and δ be distinct elements of $\{1, 2, \ldots, n\}$ and x be a permutation that maps α to γ and β to δ. What does $x^{-1}hx$ map γ to?
   (b) Let N be a non–trivial normal subgroup of A_n. Show that N must act transitively on $\{1, 2, \ldots, n\}$.
   (c) Show that if $n \geq 5$, then N must contain non–trivial permutations with fixed points. Deduce that $N \cap A_{n-1} \neq \{1\}$ (where A_{n-1} is identified with a subgroup of A_n in the natural way).
   (d) Use induction and Question 3 to prove that A_n is simple for $n \geq 5$.

5. Let G be a finite group and N a normal subgroup. Let P be a Sylow p-subgroup of N. Show that for $g \in G$, gPg^{-1} is also a Sylow p–subgroup of N. Deduce that $gPg^{-1} = nPn^{-1}$ for some $n \in N$ and that $G = N_G(P)N$. [Frattini]

6. (a) Show that $\mathrm{PSL}(2, p)$ has exactly $p + 1$ Sylow p–subgroups.
   (b) Exhibit a Sylow 5–subgroup of $\mathrm{SL}(2, 5)$.
   (c) Exhibit a Sylow 3–subgroup of $\mathrm{SL}(2, 5)$.
   (d) Exhibit a Sylow 2–subgroup of $\mathrm{SL}(2, 5)$.

7. (a) Show that $\mathrm{PSL}(2, 2) \cong S_3$.
   (b) Show that $\mathrm{PSL}(2, 3) \cong A_4$.
   (c) Show that $\mathrm{PSL}(2, 4) \cong A_5 \cong \mathrm{PSL}(2, 5)$.

   Hint: for (a) *and* (b) *find appropriate* 3– *and* 4–*element sets* $\mathrm{PSL}(2, 2)$ *and* $\mathrm{PSL}(2, 3)$ *act on. An elegant solution to* (c) *along the same lines would be welcome.*

Read ahead in §5.