MATH 6310, Homework 3
Due in class 9/11

Review direct and semidirect products and abelian groups in §5.

Do §5.1, question 17; §5.2, questions 2(c), 3(c), 4(c), 9; §5.4, question 20; and §5.5, questions 12, 16, 23

1. (a) Show that the dihedral group D_8 can be expressed as a split extension of Z_4 by Z_2 and also as a split extension of V_4 by Z_2 (where $V_4 \cong Z_2 \times Z_2$ is the Klein 4–group).

   (b) Show that S_4 is a split extension of V_4 by S_3 and also of A_4 by Z_2.

2. Say a group G is *p–nilpotent* if there is a normal subgroup K (called a *normal p–complement*) such that for any Sylow p–subgroup P we have $G = PK$ and $K \cap P = \{e\}$.

   (a) Show that if G is p–nilpotent, then $K = \{g \in G \mid p \nmid o(g)\}$. (Here $o(g)$ denotes the order of g.)

   (b) Deduce that G is p–nilpotent if and only if $\{g \in G \mid p \nmid o(g)\}$ is a subgroup; and also that if G is p–nilpotent, then its normal p-complement is unique.

   (c) Prove that subgroups and factor groups of p–nilpotent groups are p–nilpotent.

   (d) Prove that if G is p-nilpotent, then for any p–subroup Q of G, we have $N_G(Q)/C_G(Q)$ is a p-group.

   [*The converse of* (d) *is a theorem of Frobenius.*]

3. Suppose that M and N are normal subgroups of G. By considering the map $g \mapsto (gM, gN)$ ($g \in G$), show that $G/(M \cap N)$ is isomorphic to a subgroup of $G/M \times G/N$. Show further that if $[G : M]$ and $[G : N]$ are coprime, then

$$G/(M \cap N) \;\cong\; G/M \times G/N.$$

   Deduce that G is p–nilpotent for all prime divisors p of $|G|$ if and only if G is the direct product of its Sylow subgroups.

Read ahead in §6.