MATH 6310, Homework 4
Due in class 9/18

Look over §6.1 and (perhaps just in overview) §6.2.

Do §6.1: qus. 1, 14, 17, 25 (the definition of $\Phi(G)$ is at the bottom of page 198); §6.2: qu. 27.

1. Find the upper and lower central series for A_4.

2. Recall that
$$D_{2n} = \langle u, t \mid u^n = e = t^2, tut^{-1} = u^{-1} \rangle.$$
Show that D_{2n} is nilpotent if and only if n is a power of 2. In this case (where $n = 2^a$ with $a \geq 1$), determine the lower central series.

3. Suppose a group G contain elements g, h such that the commutator $[g, h]$ commutes with both g and h. Show that for all $m, n \in \mathbb{N}$,
$$[g, h^n] = [g, h]^n \qquad \text{and} \qquad [g^m, h] = [g, h]^m.$$
Suppose now that G is nilpotent and x and y are elements of coprime order m and n. Show (by induction on the class of G or otherwise) that $xy = yx$. (*Caution: do not assume G is finite!*)

4. Suppose that $G = H \times K$. Show that for each i the terms of the lower central series are related by $G^i = H^i \times K^i$. (*So if H and K are nilpotent, then so is G.*)

5. (a) Let H_1, H_2, K_1, K_2 be *normal* subgroups of G. Show that
$$[H_1H_2, K_1K_2] \;=\; [H_1, K_1][H_2, K_1][H_1, K_2][H_2, K_2].$$
   (b) Suppose that H and K are normal subgroups of G both of which are nilpotent. Show that for any given r, the subgroup
$$\underbrace{[HK, \ldots [HK, [HK, HK}_{r}]] \ldots]$$
is contained in a product of normal subgroups of type $[X_r, \ldots [X_3, [X_2, X_1]] \ldots]$ where each X_i is either H or K.
   (c) Show that in such a product $[X_r, \ldots [X_3, [X_2, X_1]] \ldots]$, that if $k+1$ of the X_i are H, then $[X_r, \ldots [X_3, [X_2, X_1]] \ldots] \leq H^k$ and if $\ell+1$ of the X_i are K, then $[X_r, \ldots [X_3, [X_2, X_1]] \ldots] \leq K^\ell$.
   (d) Deduce that if H has class c and K has class d, then HK is nilpotent of class at most $c + d$.

Read ahead in §6.3.