MATH 6310, Homework 5
Due in class 9/25

Continue to look over §6.1 and, in overview, §6.2. Additionally look over §6.3 .

Do §6.3, Questions 1 and 3.

1. Show that for a group G, the following are equivalent.

   (i) There exists a normal subgroup $A \trianglelefteq G$ such that both A and G/A are abelian.

   (ii) $G'' = 1$. (*Notation:* G' *denotes the derived subgroup* $[H,H]$ *and* H'' *denotes* $(H')'$.)

   Such a group is called *metabelian*.

2. Which of the following are nilpotent and which are solvable (or neither, or both)?

   (a) S_4

   (b) $\langle a, b \mid b^{-1}ab = a^2 \rangle$

   (c) $\mathbb{Z} \wr \mathbb{Z}$ which is defined to be $S \rtimes_\varphi \mathbb{Z}$ where $S = \bigoplus_{\mathbb{Z}} \mathbb{Z}$ denotes the set of finitely supported functions $f : \mathbb{Z} \to \mathbb{Z}$ and $\mathbb{Z}$ acts on S by
   $$\left(\varphi(r)(f)\right)(n) = f(n+r).$$
   (*Cf. the definition of the wreath products we saw in Qu. 23 from §5.5. Another way to describe* S *is as the set of* $\mathbb{Z}$*–indexed integer sequences with only finitely many non–zero entries, and then the action is to shift the indexing of these sequences.*)

   (d) The group of orientation–preserving isometries of the plane.

   (e) The unitriangular matrix group $\mathrm{UT}_n(\mathbb{Z})$, the multiplicative group of upper triangular $n \times n$ integer matrices with all 1s on the diagonal.

   (f) The Borel subgroup $B_n(\mathbb{R})$, consisting of invertible upper triangular real $n \times n$ matrices.

   For those that are nilpotent, what is their class? For those that are solvable what is their solvability length (a.k.a. derived length)?

3. Prove that $\langle\, a, b, c \mid a^{-1}b^{-1}ab = c,\ ac = ca,\ bc = cb \,\rangle$ presents $\mathcal{H}_3$, the three–dimensional integral Heisenberg group, which is the multiplicative group of three–by–three matrices of the form
   $$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$
   where $x, y, z \in \mathbb{Z}$. (*In other words,* $\mathcal{H}_3 = \mathrm{UT}_3(\mathbb{Z})$ *of the previous question.*)

   Show that every element of $\mathcal{H}_3$ can be expressed uniquely as $a^r b^s c^t$ for some $r, s, t \in \mathbb{Z}$.

4. Suppose $\langle A \mid R \rangle$, where the generating set is $A = \{a_1, \ldots, a_s\}$ and the defining relators are $R = \{r_1, \ldots, r_m\}$, presents a group $G \cong F(A)/\langle\langle R \rangle\rangle$.

   The defining relations tell us when words w and w' represent the same group element: specifically, when w' can be obtained from w by a finite sequence of the moves –

- *free reduction*: remove a substring $a_i {a_i}^{-1}$ or ${a_i}^{-1} a_i$ from within a word;
- *free expansion*: insert a substring $a_i {a_i}^{-1}$ or ${a_i}^{-1} a_i$ into a word;
- *apply a defining relation*: replace a substring u in a word with a new substring v such that uv^{-1} or vu^{-1} is a cyclic permutation of one of the words in R.

A *null–sequence* for a word w is such a sequence that transforms w to the empty word. The words that admit null–sequences are those that represent the identity in the group.

The *Dehn function* $f : \mathbb{N} \to \mathbb{N}$ maps n to the minimal number N such that if w is a word of length at most n that represents the identity, then there is a null–sequence for w involving at most N applications–of–defining–relations moves. (*There are only finitely many words of length at most n since the alphabet A is finite, and so N is well–defined.*)

(a) Find a null–sequence for $a^{-2}b^{-2}a^2b^2$ with respect to the presentation $\langle a, b \mid a^{-1}b^{-1}ab \rangle$ of $\mathbb{Z} \times \mathbb{Z}$.

(b) Find a null–sequence for $a^{-2}b^{-1}ab^2ab^{-1}$ with respect to the presentation of $\mathcal{H}_3$ in Question **??**.

(c) Show that there is a constant $C > 0$ such that the Dehn function $f(n)$ of the presentation $\langle a, b \mid a^{-1}b^{-1}ab \rangle$ of $\mathbb{Z} \times \mathbb{Z}$ satisfies $f(n) \leq Cn^2$ for all n. *Bonus: find the minimal C.*

(d) Show that there is a constant $C > 0$ such that the Dehn function $f(n)$ of the presentation of $\mathcal{H}_3$ from Question **??** satisfies $f(n) \leq Cn^3$ for all n. *Hint: use the last part of that question.*

(e) Show that, we can equivalently define the Dehn function by declaring $f(n)$ to be the minimal N such that if w is a word of length at most n that represents the identity, then w is equal in $F(A)$ to a product

$$\left({u_1}^{-1}{r_{j_1}}^{\epsilon_1}u_1\right)\cdots\left({u_M}^{-1}{r_{j_M}}^{\epsilon_M}u_M\right)$$

of some $M \leq N$ conjugates of defining relations or their inverses — that is, each u_i is a word on $A^{\pm 1}$, each r_{j_i} is in R, and each ϵ_i is ± 1.

(f) The words $w_k := a^{-k}b^{-k}a^kb^k$ have length $4k$ and represent the identity in $\langle a, b \mid a^{-1}b^{-1}ab \rangle$.

Suppose, as per part (a), that u_i are words on $\{a, b\}^{\pm 1}$ and $\epsilon_i = \pm 1$ so that

$$W_k \;=\; \left({u_1}^{-1}(a^{-1}b^{-1}ab)^{\epsilon_1}u_1\right)\cdots\left({u_M}^{-1}(a^{-1}b^{-1}ab)^{\epsilon_M}u_M\right)$$

equals w_k in $F(a, b)$.

i. Show that W_k represents the same element as $c^{\epsilon_1}\cdots c^{\epsilon_M}$ in $\mathcal{H}_3$.

ii. Show that w_k and c^{k^2} represent the same element in $\mathcal{H}_3$ and that c has infinite order in $\mathcal{H}_3$.

iii. Show that $M \geq k^2$.

Together parts (e) *and* (f) *establish a quadratic lower bound on the Dehn function of* $\langle a, b \mid a^{-1}b^{-1}ab \rangle$. *A similar argument can be used to establish a cubic lower bound on the Dehn function of* $\mathcal{H}_3$.

Read ahead in §7.