MATH 6510, Algebraic Topology, Spring 2017
Homework 2, Due in class 6 February

Reading.

- The details of proofs of path and homotopy lifting are pages 29–30 and 60 of Hatcher.
- Examples 1.41 and 1.42 (Hatcher, pages 73–74) of covering spaces: $M_{mn+1} \to M_{m+1}$, where M_g denotes the closed orientable surface of genus g, and $\mathbb{R}^2 \to T$ and $\mathbb{R}^2 \to K$, where T is the torus and K is the Klein bottle.
- We will soon prove that $\pi_1(S^1) = \mathbb{Z}$. (*You can use this in the exercises below.*) Read about three important consequences on pages 31–33 in Hatcher: the Fundamental Theorem of Algebra, and the dimension-2 versions of Brouwer's Fixed Point Theorem and the Borsuk–Ulam Theorem. This commentary-less video can be interpreted as illustrating what's going on in Hatcher's proof of the Fundamental Theorem of Algebra. Brouwer's Fixed Point Theorem is related to the Game of Hex—see this paper by David Gale. It and related fixed-point theorems have applications in game theory and economics. Apparently, Brouwer's dissatisfaction with the non-constructiveness of his proof led him to formulate his programme of intuitionism. There's a whole book by Jiri Matousek devoted to applications of Borsuk–Ulam. Here are biographies of Borsuk and Ulam. The latter was a major contributor to the Manhattan Project.

Exercises.

1. Show that the embedding $\mathbf{x} \mapsto (\mathbf{x}, 0)$ of S^n into S^{n+1} is homotopic to a constant map.

2. The *number of sheets* of a covering space $p : Y \to X$, where X is connected, is the cardinality of $p^{-1}(x)$ for any $x \in X$. Show this is well-defined (i.e. is independent of x).

3. (Weintraub, Ex. 2.7.4) We explained in class that a covering map $p : Y \to X$ enjoys the following three properties.
   (a) For every $x \in S$, $p^{-1}(x)$ is a discrete subset of Y.
   (b) p is a local homeomorphism.
   (c) The topology on X is the quotient topology it inherits from Y via the map.

   Exhibit a map $p : Y \to X$ which enjoys these properties but is not a covering map.

4. (Weintraub, Ex. 2.7.6) Let X be a simply connected space. Fix x_0, x_1, two arbitrary points in X. Let $f : I \to X$ and $g : I \to X$ with $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. Show that f and g are homotopic rel $\{0, 1\}$.

5. Hatcher, page 38, Qu. 8

6. Hatcher, page 38, Qu. 9

7. Recall that a map $r : X \to A$ where A is a subspace of X is a *retraction* when its restriction to A is the identity map. Show that if $a_0 \in A$, then $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.

8. Hatcher, page 39, Qu. 16 (a), (b), (c)