MATH 6510, Algebraic Topology, Spring 2017
Homework 3, Due in class 13 February

Reading.

- The details of the proof of the Lifting Criterion are on pages 61–62 of Hatcher.
- The construction of a simply connected covering space on pages 64ff.

The wedge product $X \vee Y$ is the quotient of the disjoint union of X and Y in which one point in X is identified with one point in Y. Real projective space $\mathbb{R}P^n$ is the quotient of S^n in which pairs of antipodal points are identified. You may take it as given that S^n is simply connected for all $n > 1$.

Exercises.

1. (*Unique lifting.*) Suppose E is a connected space and $p : Y \to X$ is a covering map. Show that if two lifts $\widetilde{f}, \widetilde{f}' : E \to Y$ of a map $f : E \to X$ agree at one point, then they are equal.

2. (a) What is $\pi_1(\mathbb{R}P^2)$?
   (b) Find all the connected covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^2$ and exhibit how they correspond to all the subgroups of $\pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^2)$.

3. (Hatcher) Construct a simply-connected covering space for each of the following spaces. (*Example 1.21 on page 43 of Hatcher in the special case where the X_α are simply connected, is a technical result you can use to justify formally that many of the spaces you construct are simply connected.*)
   (a) $S^1 \vee S^2$.
   (b) The union of S^2 with an arc joining two distinct points $x, y \in S^2$.
   (c) S^2 with two points identified.
   (d) $\mathbb{R}P^2 \vee \mathbb{R}P^2$.
   (e) S^2 with an arc joining two points x, y and another arc joining z, w, where all four points x, y, z, w are distinct.
   (f) $S^1 \vee \mathbb{R}P^2$.
   (g) $\mathbb{R}P^2$ with an arc joining two distinct points.
   (h) $S^1 \vee T^2$ where T^2 is the torus $S^1 \times S^1$.

4. Hatcher, page 81, Qu 24.

5. A group is a *topological group* when it has a topology that makes the group's binary operation and the inverse operation continuous. Show that the fundamental group (based at the identity) of a topological group is always abelian.

6. Working directly from the construction of the universal cover $\widetilde{X}$ of X as per pages 64ff of Hatcher, explain why it is equivalent to $\mathbb{R}$ in the case $X = S^1$.