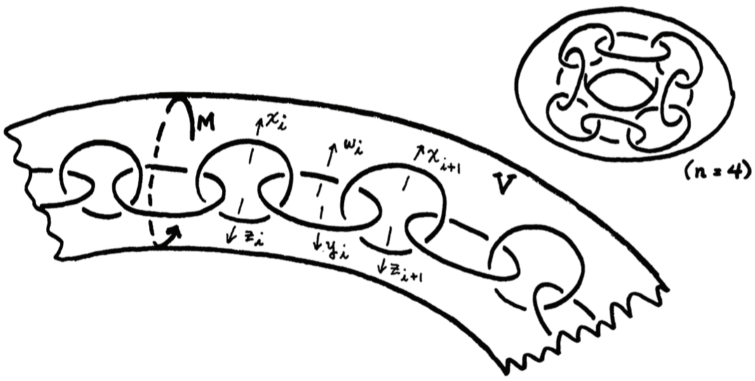
MATH 6510, Algebraic Topology, Spring 2017
Midterm 1, Due at the start of class 6 March, 8:40am

This midterm has two pages and four questions. You may typeset your solutions or write them by hand, as you prefer. You are reminded to follow Cornell's Academic Integrity Code and that papers may be subject to submission for textual similarity review to Turnitin.com. Please abide by the following statement, which you should reproduce and sign on your exam ('our textbooks' refers to 'Algebraic Topology' by Hatcher and 'Fundamentals of Algebraic Topology' by Weintraub):

> *"This is all my own work. I have not consulted any books (apart from our textbooks), web sites, on-line forums, people, or any other sources."*

1. *(Wirtinger presentations for knot groups.)* Hatcher page 55, Qu. 22.

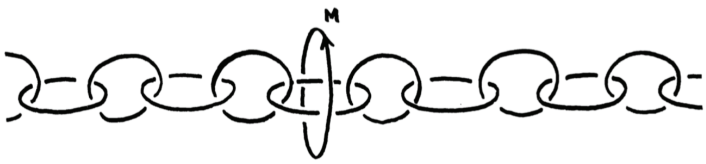
2. *(Chains are as good as ropes.)* Consider the link C below of $2n$ unknotted circles in a chain running around a solid torus $V = D^2 \times S^1$, standardly embedded in $\mathbb{R}^3$:



   (a) The Wirtinger presentation for $\pi_1(\mathbb{R}^3 \setminus C)$ where C is as shown has $4n$ generators x_i, y_i, z_i, w_i (as shown). What are its defining relations?

   (b) Show that the meridian M of V is not homotopically trivial in $\mathbb{R}^3 \setminus C$ or in $V \setminus C$. Show that M represents an element of infinite order in $\pi_1(\mathbb{R}^3 \setminus C)$. *(Hint: use a suitable homomorphism from $\pi_1(\mathbb{R}^3 \setminus C)$ onto the free group $F(a, b)$.)*

   (c) Show that the loop M is not contractible in the complement of the infinite chain:



3. *(Residual finiteness of free groups.)*

   (a) Hatcher page 87, Qu. 9

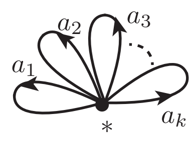
   (b) Hatcher page 87, Qu. 10

   (c) Hatcher page 87, Qu. 11

4. Recall that the free group $F = F(a_1, \dots, a_k)$ of rank k is the fundamental group of R_k, where $R_k = \bigvee_{i=1}^{k} S^1$ is k circles wedged at $*$ and directed and labelled as shown below.

The aim of this question is to prove this theorem: *If H is a finitely generated subgroup of F, then there is a finite index subgroup $G \leq F$ such that $G = H * K$ for some group K.*

R_k



(a) Suppose H is generated by elements of F represented by words $w_1, w_2, \dots, w_m$ on $a_1, a_2, \dots, a_k$. Take m copies of S^1 all wedged together at a vertex $*$. Let Γ' be the graph obtained by then subdividing the i-th S^1 into directed edges labelled by $a_1, \dots, a_k$ in such a way that we read w_i around it.

Form a new graph Γ by quotienting Γ' in such a way that when two adjacent edges emanate from the same vertex in Γ' and have the same label, they become identified:

For example, here are Γ' and Γ when $F = F(a, b)$ and $H = \langle ab^{-2}, ba^{-2} \rangle$:

Explain why (in general) $\pi_1(\Gamma, *) = H$.

(b) Explain why Γ is not necessarily a cover of R_k.

(c) Let $\widehat{\Gamma}$ be Γ with the orientations of all the edges reversed. Explain how, in our example, to connect Γ to $\widehat{\Gamma}$ with finitely many additional edges to form a connected cover of R_k (the covering map sending each edge labelled a_i to the edge in R_k labelled a_i while respecting the edge-orientations). Explain why this works in general.

(d) Deduce the theorem stated at the start of this question.