Math 3560 — Prelim

11:40am–12:55pm, Tuesday 8th October 2013

"Mathematics is the art of giving the same name to different things."
Henri Poincaré, in response to the statement that poetry is the art of giving different names to the same thing.

Please answer all questions. Calculators, cell phones, music players and other electronic devices are not permitted. Notes and books may not be used.

USE ONE ANSWER BOOKLET FOR QUESTIONS 1 & 2 AND ANOTHER FOR QUESTIONS 3 & 4.

Write your name on all exam booklets. Do not hand in any scratch paper. Unless otherwise indicated, all answers should be justified. You may invoke without proof results proved in class or in the textbook, provided you state them clearly.

1. Which of the following form a *group*. (*In each case justify your answer fully.*)

   (a) The set $\{1, 2, 4, 5, 6, 8, 9\}$ under multiplication mod 10.

   (b) The rotational symmetries of a dodecahedron that induce an even permutation of the vertices.

   (c) The set $S = \left\{ a + b\sqrt{2} \,\middle|\, a, b \in \mathbb{Q} \right\} \smallsetminus \{0\}$, under multiplication.

   (d) The permutations of $\{1, 2, 3, 4, 5\}$ which have order at most 3.

   (e) The bijections $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \geq x$ for all $x \in \mathbb{R}$, under composition.

   2 + 2 + 2 + 2 + 2 = 10 pts

2. (a) Show that the group of complex numbers $\{1, -1, i, -i\}$ under multiplication is cyclic.

   (b) Prove that if G is a finite group, every row and every column of its multiplication table is a permutation of the group elements. (*Remark: it is not really essential that G be finite here, except that we tend to discuss multiplication tables only for finite groups.*)

   (c) Rewrite and complete the following multiplication table for the Quaternion group (which contains $\{1, -1, i, -i\}$ as a subgroup) in your answer booklet:

	1	-1	i	-i	j	-j	k	-k
1					j			
-1					-j	j	-k	k
i					k	-k	-j	j
-i					-k	k	j	-j
j		-j	-k	k	-1	1	i	-1
-j		j			1	-1	-i	i
k		-k	j	-j	-i	i	-1	
-k			-j	j	i	-i	1	-1

Please circle the entries that you put in the blank spaces.

(d) Draw the Cayley diagram (Cayley graph) of the Quaternion group with respect to the generators i and j.

(e) Is the Quaternion group cyclic?

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3. (a) The dihedral group D_n of order $2n$ (where $n \geq 3$) can be defined as the group of rotational symmetries of what object?

(b) Describe (without proof) symmetries r and s of orders n and 2 which satisfy $sr = r^{-1}s$.

(c) Explain, by means of appropriate diagrams, why $sr = r^{-1}s$ in D_3.

(d) Show how it follows from $r^n = 1$, $s^2 = 1$ and $sr = r^{-1}s$ that $sr^2 = r^{n-2}s$ in D_n.

2 + 2 + 3 + 3 = 10 pts

4. Explain why the group G of rotational symmetries of the tetrahedron is isomorphic to the alternating group A_4.

10 pts

Total = 40 pts

TRR, 28 September 2013