Math 3560 — Prelim

11:40am–12:55pm, Tuesday 8th October 2013

"Mathematics is the art of giving the same name to different things."
Henri Poincaré, in response to the statement that poetry is the art of giving different names to the same thing.

Please answer all questions. Calculators, cell phones, music players and other electronic devices are not permitted. Notes and books may not be used.

USE ONE ANSWER BOOKLET FOR QUESTIONS 1 & 2 AND ANOTHER FOR QUESTIONS 3 & 4.

Write your name on all exam booklets. Do not hand in any scratch paper. Unless otherwise indicated, all answers should be justified. You may invoke without proof results proved in class or in the textbook, provided you state them clearly.

1. Which of the following form a *group*. (*In each case justify your answer fully.*)

   (a) The set $\{1, 2, 4, 5, 6, 8, 9\}$ under multiplication mod 10.

   This is not a group. For example, $2 \cdot 5 = 10$, which is 0 modulo 10, and so multiplication is not a binary operation on this set.

   (b) The rotational symmetries of a dodecahedron that induce an even permutation of the vertices.

   This is a group. If two rotational symmetries induce an even permutation of the vertices, then their composition induces a permutation which is a composition of two even permutations and so is also even. The identity rotation induces the trivial permutation, which is even. The inverse of a rotation σ induces the inverse of the permutation induced by σ. Composition of these rotations is associative, since composition (when defined) is always associative.

   (c) The set $S = \left\{ a + b\sqrt{2} \,\middle|\, a, b \in \mathbb{Q} \right\} \smallsetminus \{0\}$, under multiplication.

   This is a group. Suppose $a, b, c, d \in \mathbb{Q}$ and $a+b\sqrt{2}$ and $c+d\sqrt{2}$ are non-zero. Then their product is non-zero, since the product of two non-zero real numbers is always non-zero. And $1 \in S$ and is the identity: $(a + b\sqrt{2})1 = 1(a + b\sqrt{2}) = (a + b\sqrt{2})$. We can invert $a + b\sqrt{2}$ since $a^2 - 2b^2 \neq 0$ as $\sqrt{2} \notin \mathbb{Q}$:

   $$(a + b\sqrt{2}) \left(\frac{a - b\sqrt{2}}{a^2 - 2b^2} \right) \;=\; \left(\frac{a - b\sqrt{2}}{a^2 - 2b^2} \right) (a + b\sqrt{2}) \;=\; 1$$

   and $\dfrac{a - b\sqrt{2}}{a^2 - 2b^2} \in S$. Associativity holds since multiplication of real numbers is always associative.

(d) The permutations of $\{1, 2, 3, 4, 5\}$ which have order at most 3.

This is not a group. For example, (123) and (45) have orders 3 and 2, respectively, but their product (123) and (45) has order 6: the lowest common multiple of the sizes of the cycles in its cycle decomposition (123)(45). So this set of permutations is not closed under composition.

(e) The bijections $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \geq x$ for all $x \in \mathbb{R}$, under composition.

This is not a group. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$ for all $x \in \mathbb{R}$. Then f is a bijection and satisfies $f(x) \geq x$ for all $x \in \mathbb{R}$, but its inverse $g(x) = x - 1$ fails to satisfy $g(x) \geq x$ for all $x \in \mathbb{R}$.

2 + 2 + 2 + 2 + 2 = 10 pts

2. (a) Show that the group of complex numbers $\{1, -1, i, -i\}$ under multiplication is cyclic.

The element i is a generator: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. So the group is cyclic.

(b) Prove that if G is a finite group, every row and every column of its multiplication table is a permutation of the group elements. (*Remark: it is not really essential that G be finite here, except that we tend to discuss multiplication tables only for finite groups.*)

For $g \in G$, the elements in row g of the multiplication table are all gx as x ranges over G. They are a permutation of G since the map $G \to G$ given by $x \mapsto gx$ is invertible: it has inverse $G \to G$ given by $y \mapsto g^{-1}y$. Similarly, for all $g \in G$, column g is a permutation of G since its elements are all xg for $x \in H$ and $x \mapsto xg$ is invertible, having inverse $y \mapsto yg^{-1}$.

(c) Complete the multiplication table for the Quaternion group (which contains $\{1, -1, i, -i\}$ as a subgroup):

	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

(d) Draw the Cayley diagram (Cayley graph) of the Quaternion group with respect to the generators i and j.

See page 53 of *Visual Group Theory* by Nathan Carter.

(e) Is the Quaternion group cyclic?

No. It is not abelian (for example, $ij \neq ji$) and so is not cyclic.

2 + 2 + 2 + 2 + 2 = 10 pts

3. (a) The dihedral group D_n of order $2n$ (where $n \geq 3$) can be defined as the group of rotational symmetries of what object?

   It is the group of rotational symmetries of a regular n-gon plate.

   (b) Describe (without proof) symmetries r and s of orders n and 2 which satisfy $sr = r^{-1}s$.

   The element r is rotation through $2\pi/n$ about the axis perpendicular to the plate, and s is rotation through π about an axis of symmetry parallel to the plate.

   (c) Explain, by means of appropriate diagrams, why $sr = r^{-1}s$ in D_3.

   This is done by Figure 4.2 on page 16, expect it uses r^2 instead of r^{-1}.

   (d) Show how it follows from $r^n = 1$, $s^2 = 1$ and $sr = r^{-1}s$ that $sr^2 = r^{n-2}s$ in D_n.

$$sr^2 \;=\; (sr)r \;=\; (r^{-1}s)r \;=\; r^{-1}(sr) \;=\; r^{-1}(r^{-1}s) \;=\; r^n r^{-2}s \;=\; r^{n-2}s.$$

2 + 2 + 3 + 3 = 10 pts

4. Explain why the group G of rotational symmetries of the tetrahedron is isomorphic to the alternating group A_4.

   If we label the vertices of the tetrahedron $1, 2, 3, 4$, then rotations induce permutations of $\{1, 2, 3, 4\}$, giving us a map $\phi : G \to S_4$. This map satisfies $\phi(\sigma\tau) = \phi(\sigma)\phi(\tau)$ since the permutation induced by the product of two rotations is the composition of the permutations they each induce.

   The order of G is 12. This is because the elements of G can be partitioned into four sets depending on whether they map vertex 1 to vertex 1, 2 , 3, or 4. Each of these sets contains exactly three elements: once the image of 1 is determined, there are three choices for the image of 2 and each of these choices can be realized by a unique rotational symmetry.

   Every 3-cycle is in the image of ϕ: given a 3-cycle c either the rotation $2\pi/3$ or $4\pi/3$ about the axis through the vertex fixed by c and through the mid-point of the opposite face maps to c. So, as the 3-cycles generate A_4, we have that $A_4 \subseteq \mathrm{Im}\phi$. But $|A_4| = 4!/2 = 12 = |G|$, so ϕ is injective and gives an isomorphism between G and its image A_4.

10 pts

Total = 40 pts

TRR, 28 September 2013