Math 3560 — Final Examination

7:00pm–9:30pm, Wednesday 5th December 2012

“Groups, as men, will be known by their actions” — Guillermo Moreno.

Please answer TWO QUESTIONS FROM SECTON I and TWO QUESTIONS FROM SECTION II. Please write your answers to Section I questions in DIFFERENT BOOKLETS to those in Section II. Label the booklets according.

Calculators, cell phones, music players and other electronic devices are not permitted. Notes and books may not be used.

Write your name on ALL exam booklets. Do not hand in any scratch paper. Unless otherwise indicated, all answers should be justified.

SECTION I

1. (a) Define the direct product $H \times K$ of groups H and K. (*What is it as a set? What is the group operation?*)

   (b) Suppose H and K are subgroups of a group G and $H \cap K = \{e\}$, $hk = kh$ for all $h \in H$ and $k \in K$, and $G = HK$. Show that $G \cong H \times K$.

   (c) Recall that O_3 denotes the group of 3×3 matrices A with entries in $\mathbb{R}$ and the property that $A^t A = I$ and SO_3 is the subgroup consisting of those matrices in O_3 that have determinant 1. Let K be the subgroup $\{I, -I\}$ of O_3. Show that $O_3 \cong \mathrm{SO}_3 \times K$.

   8 + 9 + 8 = 25 pts

2. (a) Assume N is a normal subgroup of a group G.

      i. What does it mean to say N is *normal*?

      ii. Show that $(aN)(bN) = abN$ for all $a, b \in G$.

      iii. Explain what is meant by the *quotient group* G/N and verify that it is a group.

   (b) Let N be the subgroup generated by r^2 in the dihedral group $D_4 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$.

      i. Show that N is normal in D_4.

      ii. Show that D_4/N is isomorphic to the Klein 4–group V.

   (5 + 5 + 5) + (5 + 5) = 25 pts

3. Suppose H is a subgroup of a group G.

   (a) Write $a \sim b$ when $a^{-1}b \in H$. Show that $\sim$ defines an equivalence relation on G and that the equivalence classes are the left cosets of H in G.

   (b) Show that there is a bijection between any two left cosets of H in G.

   (c) State Lagrange’s Theorem and deduce it from parts (a) and (b).

   (d) Show that if p is prime then any non-zero element generates $\mathbb{Z}/p\mathbb{Z}$.

   6 + 5 + 8 + 6 = 25 pts

SECTION II

4. (a) What does it mean to say that a group G acts on a set X? For $x \in X$, what is meant by the *stabilizer* G_x and the *orbit* $G(x)$?
   (b) State the Orbit–Stabilizer Theorem, relating G_x, $G(x)$ and G when a finite group G acts on a finite set X and $x \in X$.
   (c) Consider the group G of rotational symmetries of a tetrahedron acting on the set X of vertices of a tetrahedron. Show that $|G| = 12$ and deduce that $G \cong A_4$.
   (d) Give a geometric description of an element of the full symmetry group of the tetrahedron which cycles the four vertices.

8 + 4 + 7 + 6 = 25 pts

5. (a) How many different ways are there to build a fixed $2 \times 2 \times 2$ cube using red and green $1 \times 1 \times 1$ cubes?
   (b) State The Counting Theorem for a finite group G acting on a finite set X.
   (c) Use the Counting Theorem to calculate how many ways there are to build a $2 \times 2 \times 2$ cube from red and green $1 \times 1 \times 1$ cubes up to rotational symmetry. (*Please give an explicit numerical answer.*)

2 + 7 + 16 = 25 pts

6. (a) State the First Isomorphism Theorem.
   (b) Show that $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) \cong \mathbb{R} \setminus \{0\}$. ($\mathrm{GL}_n(\mathbb{R})$ *is the group of invertible* $n \times n$ *matrices with real entries and* $\mathrm{SL}_n(\mathbb{R})$ *is the subgroup consisting of those matrices with determinant* 1.)
   (c) The Second Isomorphism Theorem says that if H and J are subgroups of G with J normal in G, then
      i. HJ is a subgroup of G,
      ii. $H \cap J$ is a normal subgroup of H, and
      iii. $\dfrac{H}{H \cap J} \cong \dfrac{HJ}{J}$.

   Prove these three statements. (*Hint: for iii, define a suitable homomorphism $H \to HJ/J$ and apply the First Isomorphism Theorem.*)

5 + 5 + (5 + 5 + 5) = 25 pts

TRR, 29 November 2012