Math 3560 — Prelim

10:10am–11:25am, Tuesday 2nd October 2012

"As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection." Joseph Louis Lagrange

Please answer all questions. Calculators, cell phones, music players and other electronic devices are not permitted. Notes and books may not be used.

Write your name on all exam booklets. Do not hand in any scratch paper. Unless otherwise indicated, all answers should be justified.

1. (a) Define the term *group*.

   A group is a set G with a binary operation $(a, b) \mapsto ab$ that satisfies:

   (i) $(ab)c = a(bc)$ for all $a, b, c \in G$ (associativity),

   (ii) there exists $e \in G$ such that $ea = ae = a$ for all $a \in G$ (identity),

   (iii) for all $a \in G$ there exists $b \in G$ such that $ab = ba = e$ (inverses).

   (b) Prove carefully that each element of a group has a unique inverse. Indicate how you are using the group axioms.

   Suppose $b, c \in G$ are both inverses for $a \in G$. That is, $ab = ba = e$ and $ac = ca = e$ by (iii). Then $ab = ac$. So $b(ab) = b(ac)$. And so $(ba)b = (ba)c$ by (i). So $eb = ec$ because $ab = ba = e$. And so $b = c$ by (ii).

   4 + 5 = 9 pts

2. (a) Give the definition of what it means for a group to be *cyclic*.

   A group G is cyclic when there exists $g \in G$ such that every element of G is of the form g^n for some $n \in \mathbb{Z}$.

   (b) Give an example of a non–cyclic abelian group. (*No justification is required. Just give the set and the operation.*)

   $\mathbb{Z} \times \mathbb{Z}$ with the operation $(a, b) + (c, d) = (a + c, b + d)$.

   (c) Give an example of a non–abelian group. (*No justification is required. Just give the set and the operation.*)

   S_3 with the operation composition of permutations.

   2 + 2 + 2 = 6 pts

3. What is the order $|G|$ of the group G of rotational symmetries of a cube? *Justify your answer from first principles — that is, without appealing to any knowledge you may have about groups isomorphic to G.*

Suppose F is a face of the cube. The order of G is $6 \times 4 = 24$ since each rotation of the cube is uniquely determined by which of the 6 faces it moves F to and in which of the 4 possible orientations (i.e. the four ways of rotating that face around the perpendicular axis through its mid–point).

5 pts

4. Let $G = \langle x, y \rangle$ be the subgroup of the symmetric group S_5 generated by the elements

$$x = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{bmatrix}.$$

(a) List the non–identity elements of G, writing each as a product of disjoint cycles. Explain why your list is exhaustive.

$$\begin{aligned} x &= (1\ 2), \\ y &= (3\ 4\ 5), \\ xy &= (1\ 2)(3\ 4\ 5), \\ y^2 &= (3\ 5\ 4), \\ xy^2 &= (1\ 2)(3\ 5\ 4). \end{aligned}$$

As x and y commute (as the numbers moved by x are not moved by y and vice versa), every element of G is $x^i y^j$ for some $i, j \in \mathbb{Z}$. As x and y have orders 2 and 3, respectively, $\{x^i y^j \mid i = 0, 1 \text{ and } j = 0, 1, 2\}$ exhausts the possibilities.

(b) Is G a subgroup of A_5?

No, it includes odd permutations such as x.

(c) Is G cyclic?

Yes, it is generated by xy. Indeed,

$$\begin{aligned} xy &= xy \\ (xy)^2 &= x^2 y^2 &&= y^2 \\ (xy)^3 &= x^3 y^3 &&= x \\ (xy)^4 &= x^4 y^4 &&= y \\ (xy)^5 &= x^5 y^5 &&= xy^2 \\ (xy)^6 &= x^6 y^6 &&= e. \end{aligned}$$

5 + 2 + 3 = 10 pts

5. Cayley's Theorem states that every group G is isomorphic to a subgroup of the symmetric group S_G.

(a) Define what it means for two groups to be *isomorphic*.

Two groups A and B are isomorphic when there is a bijection $\phi : A \to B$ such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$.

(b) What is meant by *the symmetric group* S_G in Cayley's Theorem?

S_G is the group of permutations of G (viewed as a set).

(c) Show that S_n is isomorphic to a subgroup of A_{n+2}.

Consider S_n as a subgroup of S_{n+2} in the natural way. Then define $\psi : S_n \to A_{n+2}$ by

$$\psi(\sigma) \;=\; \begin{cases} \sigma & \text{if } \sigma \text{ is even} \\ \sigma\,(n+1\;\, n+2) & \text{if } \sigma \text{ is odd.} \end{cases}$$

This is clearly injective. And it satisfies $\psi(\sigma\tau) = \psi(\sigma)\psi(\tau)$ for all $\sigma, \tau \in S_n$, as we shall now see. We use the fact that $(n+1\; n+2)$ commutes with everything in S_n and squares to the identity to compute:

$$\psi(\sigma)\psi(\tau) \;=\; \begin{cases} \sigma\tau & \text{if } \sigma \text{ and } \tau \text{ are both even or both odd} \\ \sigma\tau\,(n+1\;\, n+2) & \text{if one is even and the other is odd.} \end{cases}$$

If σ and τ are both even or both odd, then $\sigma\tau$ is even and so $\psi(\sigma\tau) = \sigma\tau$. If one of σ and τ is even and the other is odd, then $\sigma\tau$ is odd and so $\psi(\sigma\tau) = \sigma\tau\,(n+1\; n+2)$. In each case, we have $\psi(\sigma\tau) = \psi(\sigma)\psi(\tau)$ as required.

(d) Use part (c) and Cayley's Theorem to show that every group G of finite order n is a subgroup of A_{n+2}.

By Cayley's Theorem, if $|G| = n$, then there is an isomorphism $\phi : G \to H$ for some subgroup H of S_n. Define $\Phi : G \to A_{n+2}$ to be $\psi \circ \phi$. This is the composition of two isomorphisms and so is a an isomorphism.

2 + 2 + 4 + 2 = 10 pts

Total = 40 pts

TRR, 28 September 2012