Math 3560 — Prelim

10:10am–11:25am, Tuesday 2nd October 2012

"As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection." Joseph Louis Lagrange

Please answer all questions. Calculators, cell phones, music players and other electronic devices are not permitted. Notes and books may not be used.

Write your name on all exam booklets. Do not hand in any scratch paper. Unless otherwise indicated, all answers should be justified.

1. (a) Define the term *group*.

   (b) Prove carefully that each element of a group has a unique inverse. Indicate how you are using the group axioms.

   4 + 5 = 9 pts

2. (a) Give the definition of what it means for a group to be *cyclic*.

   (b) Give an example of a non–cyclic abelian group. (*No justification is required. Just give the set and the operation.*)

   (c) Give an example of a non–abelian group. (*No justification is required. Just give the set and the operation.*)

   2 + 2 + 2 = 6 pts

3. What is the order $|G|$ of the group G of rotational symmetries of a cube? *Justify your answer from first principles — that is, without appealing to any knowledge you may have about groups isomorphic to G.*

   5 pts

4. Let $G = \langle x, y \rangle$ be the subgroup of the symmetric group S_5 generated by the elements

$$x = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{bmatrix} \qquad \text{and} \qquad y = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{bmatrix}.$$

(a) List the non–identity elements of G, writing each as a product of disjoint cycles. Explain why your list is exhaustive.

(b) Is G a subgroup of A_5?

(c) Is G cyclic?

5 + 2 + 3 = 10 pts

5. Cayley's Theorem states that every group G is isomorphic to a subgroup of the symmetric group S_G.

(a) Define what it means for two groups to be *isomorphic*.

(b) What is meant by *the symmetric group* S_G in Cayley's Theorem?

(c) Show that S_n is isomorphic to a subgroup of A_{n+2}.

(d) Use part (c) and Cayley's Theorem to show that every group G of finite order n is a subgroup of A_{n+2}.

2 + 2 + 4 + 2 = 10 pts

Total = 40 pts

TRR, 28 September 2012