Math 4530 — Final Exam

2:00pm–4:30pm, Friday 11th December 2015

"Point set topology is a disease from which the human race will soon recover." Henri Poincaré

Calculators, cell phones, music players and other electronic devices are not permitted. Notes and books may not be used.

Please answer Parts 1 and 2 in separate booklets. Write your name and the part number on all booklets. Do not hand in any scratch paper.

$\mathbb{R}^n$ and its subspaces have their usual topology, unless otherwise stated.

Unless otherwise indicated, answers should be justified.

PART 1

Answer question 1 and one other question in this part.

1. TRUE or FALSE? *Do NOT justify your answers.*

   (a) If a subspace of a topological space is connected, then its interior is also connected.

   (b) If a subspace of a topological space is connected, then its closure is also connected.

   (c) Suppose a knot K_1 is obtained from another knot K_2 by reversing one crossing (that is when one strand went over another, it is instead made to go under), then $\pi_1(\mathbb{R}^3 \setminus K_1)$ is not isomorphic to $\pi_1(\mathbb{R}^3 \setminus K_2)$.

   (d) Subspaces of simply connected topological spaces are simply connected.

   (e) Quotients of simply connected spaces are simply connected.

   (f) If $E \to B$ is a covering map with $p(e) = b$ and $\pi_1(B, b)$ is abelian (that is, every pair of group elements x and y satisfies $xy = yx$), then $\pi_1(E, e)$ is abelian.

   (g) Every continuous map $f : S^2 \to S^2$ has a fixed point.

   (h) Every continuous map $f : [0, 1]^2 \to [0, 1]^2$ has a fixed point.

   (i) For points x and y in a topological space X, the fundamental groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic.

   (j) Simply connected subspaces of the plane are contractible.

   (*2 pts each = 20 pts.*
   Incorrect answers will score 0.)

2. Suppose $\{X_\alpha\}_{\alpha\in\Lambda}$ is a collection of nonempty sets.

   The Axiom of Choice is that there exists a function $f : \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha$ with $f(\alpha) \in X_\alpha$ for all $\alpha \in \Lambda$. The purpose of this question is to show that the Axiom of Choice follows from Tychonoff's Theorem. (*We used Zorn's Lemma, which is equivalent to the Axiom of Choice, to prove Tychonoff's Theorem, so it will follow that Tychonoff's Theorem and the Axiom of Choice are equivalent.*)

   Given a set X_α add a new element ∞_α to define a set $Y_\alpha := X_\alpha \cup \{\infty_\alpha\}$.

   (a) Show that $\{\emptyset, \{\infty_\alpha\}, X_\alpha, Y_\alpha\}$ defines a topology on Y_α and that with this topology, Y_α is compact.

   (b) Let $Y = \prod_{\alpha\in\Lambda} Y_\alpha$. What is meant by the *product topology* on Y? What does Tychonoff's Theorem tell us about Y with this topology?

   (c) For $\beta \in \Lambda$, let $U_\beta := \prod_{\alpha\in\Lambda} V_\alpha$ where $V_\alpha := Y_\alpha$ for all $\alpha \neq \beta$ and $V_\beta := \{\infty_\beta\}$. Explain why $\{U_\beta\}_{\beta\in\Lambda}$ is a family of open subsets Y. Suppose $\beta_1, \ldots, \beta_n \in \Lambda$. Explain why it is possible to construct an element of $Y \setminus (U_{\beta_1} \cup \cdots \cup U_{\beta_n})$ by making only finitely many choices.

   (d) Explain why it follows that $\{U_\beta\}_{\beta\in\Lambda}$ is not a cover for Y, and then deduce the Axiom of Choice (as stated at the start of this question).

   (*5 + 5 + 5 + 5 = 20 pts*)

3. Prove the Heine–Borel Theorem: a subset $X \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded. (*You may take as given the facts that the unit interval* $[0, 1]$ *is compact and products of compact sets are compact. Please prove everything else you need.*)

   (*20 pts*)

PART 2

Answer question 1 and one other question in this part.

1. What are the fundamental groups of the following spaces? *Do NOT justify your answers.*

   (a) A circle and 2-sphere glued together at one point.

   (b) The quotient $S^2/\sim$ of the 2-sphere in which $\mathbf{x} \sim -\mathbf{x}$ for all $\mathbf{x} \in S^2$.

   (c) The subspace of $\mathbb{R}^2$ that is the union of the circles of radius 1 centered at $(0,0)$ and $(3/4,0)$.

   (d) The quotient $S^2/\sim$ of the 2-sphere in which $(0,0,1) \sim (0,0,-1)$ and all other equivalence classes are singleton sets.

   (e) The quotient S^1/C_3 of the circle by the cyclic group of order three acting by rotation: $C_3 = \{e, g, g^2\}$ and $x^g = xe^{2\pi i/3}$ for $x \in S^1$.

   (f) $\mathbb{R}^2$ with the metric $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$.

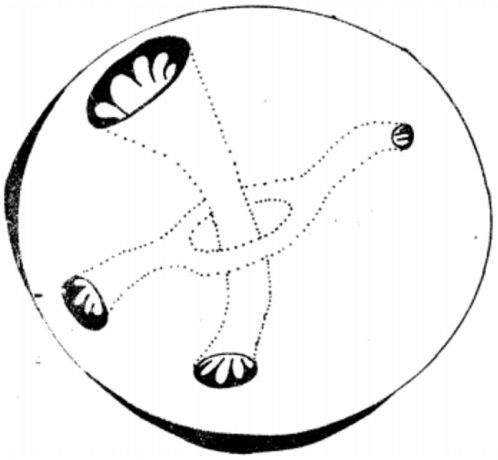
   (g) A Möbius band.

   (h) A torus with one point removed.

   (i) The circle S^1 with the discrete topology.

   (j) The surface shown below (a 2-sphere with two holes running through it and one of the holes threaded through a hole in the other hole!).

2 pts each = 20 pts. Incorrect answers will score 0.



2. Suppose $p : E \to B$ is a covering map with $p(e_0) = b_0$ for some $e_0 \in E$ and $b_0 \in B$.

   (a) What does it mean to say p is a *covering map*?

   (b) Suppose $f : I \to B$ is a path with $f(0) = b_0$. Recall that a lift of f is a path $\tilde{f} : I \to E$ such that $p \circ \tilde{f} = f$. Prove that f has a unique lift $\tilde{f}$ such that $\tilde{f}(0) = e_0$.

   (c) What is meant by the induced map $p^* : \pi_1(E, e_0) \to \pi_1(B, b_0)$? Why is it well-defined?

   (d) Show that the induced map $p^* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective. (*You may use results about homotopy lifting without proof, provided you state them correctly.*)

(3 + 8 + 4 + 5 = 20 pts)

3. (a) What does it mean to say that maps $f, g : X \to Y$ between topological spaces are homotopic?

   (b) Recall that a topological space X is *contractible* when it is homotopic to a point. Explain why an equivalent definition is that the identity map $X \to X$ is homotopic to a constant map $X \to X$.

   (c) Prove that if a space X is contractible, then it is path connected.

   (d) Show that if X is contractible and Y is path connected, then any two maps $f, g : X \to Y$ are homotopic.

(5+ 5 + 5 + 5 = 20 pts)

TRR, 8 December 2015