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Math

Sigma Invariants – Strebel Notes Chapter A

1. Introduction

1.1. preliminaries.

i. Today we will talk about some invariants used to study finitely generated groups which are used to study finitely generated groups
ii. The following all appear in Strebel's “Notes on Sigma Invariants
iii. Useful especially for looking at finitely presented groups
iv. Will focus on the geometrically defined invariant Σ^1
v. Outline:
   (a) Background on Σ invariants
   (b) Definition of Σ^1.
   (c) Examples illustrating the definition
   (d) Invariance of Σ^1 up to choice of generating set and other calculation techniques
   (e) Results for Σ^1 which can be derived geometrically.
vi. Originally Σ invariants had a more algebraic flavor which Yash will explore more in the next lecture.
vii. Henceforth, assume G is always a finitely generated group.

1.2. The Character Sphere.

i. All Σ invariants depend on a character sphere.
ii. We study homomorphisms $\varphi : G \to \mathbb{R}$.
iii. Given φ of the above type, then observe that $G' \leq \ker\varphi$ because $\mathbb{R}$ is trivial:

$$\begin{array}{ccc} G & \xrightarrow[\varphi]{} & \mathbb{R} \\ \downarrow & & \| \\ G/G' & \xrightarrow[\overline{\varphi}]{} & \mathbb{R} \end{array}$$

i.e. φ is determined by its action on $G/G' \cong \mathbb{Z}^k \oplus T$orsion. Since $\mathbb{R}$ is torsionfree, φ is determined by its action on the $\mathbb{Z}^k$ part.
iv. Hence we can identify $\hom_{\mathbb{R}}(G, \mathbb{R}) \cong \mathbb{R}^k$, a vector space over $\mathbb{R}$.
v. Define an equivalence relation on $\hom_{\mathbb{R}}(G, \mathbb{R})$ by $\chi \sim \rho$ if and only if $\chi = c\rho$ for some $c > 0$.
vi.

Definition 1. *Let* $[\chi]$ *be the equivalence class of* χ *mod* $\sim$*. Define the character sphere* $S(G) = \{[\chi] : \chi \in \hom_{\mathbb{R}}(G, \mathbb{R}) \setminus 0\}$*. For a subgroup* $N \leq G$*,* $S(G, N) = \{\chi \in S(G) : \chi(N) = 0\}$*.*

vii.

Definition 2. *For each character* χ*, define* $G_\chi = \{g \in G | \chi(g) \geq 0\}$*. Observe that* G_χ *is a monoid.*

1.3. **Defining Σ invariants.**

i. Where it all started.

Definition 3. *Let* G *be finitely generated,* A *finitely generated over* $\mathbb{Z}G$

$$\Sigma^0(G; A) \equiv \{[\chi] \in S(G) : A \text{ fin gen over the monoid ring } \mathbb{Z}G_\chi\}$$

ii. If $M = G'/G''$ as a $\mathbb{Z}G$ module by conjugation. $\Sigma^0(G/G', M)$ provides a necessary condition for finite presentation in the case that G is solvable. If G is metabelian, then we obtain a sufficient condition as well.

iii. Generalized invariant

Definition 4.

$$\Sigma_{G'} \equiv \{[\chi] \in S(G) : G' \text{ fin gen}^1 \text{ over a fin gen submonoid of } G_\chi\}$$

iv. These conditions are very technical. The invariant can be redefined in terms of connectedness of subgraphs of Cayley graphs.

1.4. Σ^1**.** Blanket assumption: Let G be finitely generated by $\eta : X \to G$. $\Gamma(G, X)$ (often X will be suppressed and we will abbreviate $\Gamma(G)$) is the Cayley graph w/r/t X.

i.

Definition 5. *Let* G *be finitely generated by* $\eta : X \to G$*. Let* Γ_χ *be the subgraph of* $\Gamma(G)$ *such that* $\chi(g) \geq 0$ *for all vertices* g *in* Γ_χ*. Then* Σ^1 *is defined as:*

$$\Sigma^1(G) \equiv \{[\chi] : \Gamma_\chi \text{ is connected}\}$$

ii. We will show that Σ^1 does not depend on the generating set used.

iii. When G has no non-abelian free subgroups, Σ^1 yields a necessary condition for finite presentation analagous to that given by Σ^1.

iv. Σ^1 also yields structural information about solvable groups including information about when normal subgroups $G' \triangleleft N \triangleleft G$ are finitely generated.

v. Also Σ^1 techniques can be used to extract normal subgroups with infinite cyclic quotients such that G is an HNN extension of said normal subgroup.

2. Examples

2.1. $\mathbb{Z}^2$**.** Set $\mathbb{Z}^2 = \langle a, b | aba^{-1}b^{-1} = 1\rangle$.

i. The Cayley graph is the lattice grid.

ii. Let χ be a character. Then $\chi(a), \chi(b)$ determine χ. We see the subgraph Γ_χ is the intersection of a half plane bounded by $\chi(a) - \chi(b) = 0$ with the lattice.

iii. Show example where $\chi(a) = \chi(b) = 1$.

2.2. **BS(1,2).** Recall $BS(1,2) \equiv \mathbb{Z}[\frac{1}{2}] \rtimes \mathbb{Z} = \langle a, u | uau^{-1} = a^2\rangle$.

i. With this presentation,[2] we can see multiplication as $(a, 2^m)(b, 2^{m'}) = (a + b2^m, 2^{m+m'})$, where in the other presentation, $a = (1,0)$ and $u = (0,2)$.

ii. Draw connected component of identity of Cayley graph

iii. Let χ be a character. Then $uau^{-1} = a^2$, then $\chi(a) = 2\chi(a) \Rightarrow \chi(a) = 0$. Hence χ is determined by $\chi(u)$ and $S(G) = \{\pm[\chi_0]\}$ where $\chi_0(u) = 1$.

iv. $[\chi_0]$ is not in Σ^1, but $-[\chi_0]$ is i.e. $\Sigma^1(BS(1,2)) = \{-[\chi_0]\}$.

Proof. Look at Γ_{χ_0}. Start with $(\frac{1}{2}, 2^m)$ for $m > 0$ which lies in Γ_{χ_0}. We see that for $z \in \mathbb{Z}$:

$$(\frac{1}{2} + z, 2^k)(\pm 1, 2^0) = (\frac{1}{2} + z \pm 1, 2^k) \qquad (\frac{1}{2} + z, 2^k)(0, 2^{\pm 1}) = (\frac{1}{2} + z, 2^{k \pm 1})$$

[2] where $\mathbb{Z}$ is represented multiplicatively with $\mathbb{Z} \cong \{2^t : t \in \mathbb{Z}\}$.

Thus the first coordinate of every path leading from $(\frac{1}{2}, 2^m)$ still has a non-integer first coordinate, so no path exists to $(0, 2^0)$, the identity.

On the other hand, for $-\chi_0$, consider the arbitrary point in the graph $\Gamma_{-\chi_0}$, $(t, 2^{-m})$, for some $m \geq 0$. There exists an integer n such that $t = z \cdot 2^{-n}$ for some $z \in \mathbb{Z}$. First starting with $(z2^{-n}, 2^{-m})$, by applying $\pm u$, $|m-n|$ times as appropriate, we can obtain a path to $(z2^{-n}, 2^{-n})$ running entirely in $\Gamma_{-\chi_0}$ since the second coordinate will always have exponent ≤ 0. Then $(z2^{-n}, 2^{-n})(1, 2^0)^k = ((z-k)2^{-n}, 2^{-m})$ which runs in $\Gamma_{-\chi_0}$ so we can reach $(0, 2^{-m})$ and then applying u m times gives a path to the identity, so $\Gamma_{-\chi_0}$ is connected. □

2.3. **Free Products.**

i. F_2 has trivial invariant. More generally, the free product of two groups such that the abelianization of the free product is not finite also has trivial invariant.

Proof. Let $F_2 = \langle a, b \rangle$. Consider a character $\chi \neq 0$. WLOG $\chi(a) > 0$ by switching a for b and possibly switching inverses. Consider $a^{-1}ba$. The only path on the Cayley graph from 1 to $a^{-1}ba$ has exactly the word $a^{-1}ba$, but $\chi(a^{-1}) = -\chi(a) < 0$. □

ii. Use similar technique using canonical forms for words in general free products.

iii. Although $S(F_2)$ is a circle, it has trivial invariant and is finitely presented, so the invariant doesn't "fill" enough of the sphere. Hence requirement for no free subgroups.

3. Invariance

Theorem 6. Σ^1 *is invariant under choice of finite generating set.*

Proof. Given finite generating sets X_1, X_2, we may assume $X_2 = X_1 \cup \{z\}$ since going from one finite generating set to another involves a finite sequence of adjoining elements and deleting elements. Let $\Gamma = \Gamma(G, X_1)$ and $\Gamma' = \Gamma(G, X_2)$ and let χ be a character. We see that Γ_χ is a subgraph of Γ'_χ with the same vertices (but fewer edges), so Γ_χ connected implies that Γ'_χ is connected.

For any path p in Γ', define $v_\chi(p) = \min\{\chi(v) : v \text{ a vertex on } p\}$. To get an idea of how this works, p in Γ lies in Γ_χ if and only if $v_\chi(p) \geq 0$. Now suppose Γ' is connected and let $g, h \in G$ be vertices.

Write $z = w_1 w_2 \dots w_n$, $w_i \in X_1^{\pm}$. If $v_\chi(w_1 \dots w_n) \geq 0$ we are done, so assume not. Choose $t \in X$ with $\chi(t) > 0$. Set k so that $\chi(t^k) = k\chi(t) \geq v_\chi(w_1 w_2 \dots w_n)$.

There exists a path from $t^{-k}gt^k$ to $t^{-k}ht^k$, which still lie in Γ_χ, in Γ', let this be $p' = x_1 x_2 \dots x_\ell$, $x_i \in X_2^{\pm}$. Replace all instances of z by $w_1 \dots w_n$. Then $v_\chi(p') \geq v_\chi(w_1 \dots w_n)$ because at each endpoint of a path segment defined by $z^{\pm}1$, the path must lie in Γ_χ. Now apply the automorphism on Γ defined by left multiplication by t^k. Then $t^k p'$ gives a path from gt^k to ht^k, and $v_\chi(t^k p') = k\chi(t) + v_\chi(p') \geq 0$, so $t^k p'$ is in Γ_χ.

Then gt^k and ht^k have clear paths in Γ_χ to g, h respectively. Hence Γ_χ is connected, yielding the desired result. □

4. Calculation Tricks

4.1. The center.

Proposition 7. *Let G be finitely generated. Then $\Sigma^1(G)$ contains every point not vanishing on $Z(G)$ i.e. $S(G, Z(G))^c \subseteq \Sigma^1(G)$.*

Proof idea. Choose a generating set including the element $z \in Z(G)$ such that $\chi(z) \neq 0$. Act by $z^k w z^{-k}$ on the path until it passes through the right graph.

Example 8. *If G is abelian, $Z(G) = G$, so every nontrivial character fails to vanish on $Z(G)$, so $S(G) = \Sigma^1(G)$.*

Example 9. *If $G = \langle a, b | a^p = b^q \rangle$, the rank of G/G' is 1, so $S(G)$ has two points, $\pm\chi_0$ with $\chi_0(a) = q$, $\chi_0(b) = p$. Observe that $a^p = b^q$ commutes with a, b, so it is central, and χ_0 does not vanish at $a^p = b^q$, so $\chi_0 \in \Sigma^1(G)$.*

4.2. **Direct products.** Let $G = G_1 \times G_2$ and π_1, π_2 be the projections. If χ_1 is a character on G_1, define $\pi_1^*(\chi_1) = \chi_1 \circ \pi_1$, a character on $G_1 \times G_2$. Then we obtain maps $\pi_1^* : S(G_1) \to S(G)$ and analagously $\pi_2^* : S(G_2) \to S(G)$.

Proposition 10. *With above conditions*

$$\Sigma^1(G_1 \times G_2)^c = \pi_1^*(\Sigma_1(G_1)^c) \cup \pi_2^*(\Sigma^1(G_2))^c$$

Example 11. *Let $G = BS(1,2)$ with $(\Sigma^1(G))^c = \{+[\chi_0]\}$ so we see that $G \times G$ has two points in $\Sigma^1(G \times G)^c$.*

5. Key technical result. Skip if necessary

Suppose G is finitely generated by X. Then $\Gamma_\chi(G, X)$ is connected if and only if for all $y \in X^{\pm 1}$, there exists a path p_y from t to $y \cdot t$ with $v_\chi(p_y) > v_\chi(1, y)$.

6. Properties and Results

Normal subgroups containing the commutator also play nicely with Σ^1:

Theorem 12. *Let G be finitely generated, $N \triangleleft G$ such that $G' \triangleleft N$. Then N is finitely generated if and only if $S(G, N) \subseteq \Sigma^1(G)$.*

Ideas here involve reduction to the free abelian case.

One of the most useful results:

Theorem 13. *Suppose G is finitely presented with no non-abelian free subgroup. Then*

$$\Sigma^1(G) \cup -\Sigma^1(G) = S(G)$$

Note how $BS(1,2)$ satisfies this condition.

Useful fact: $\Sigma^1(G)$ is open in $S(G)$ with induced topology from $\mathbb{R}^k$. When the above condition holds, connectedness arguments can be made to extract rank one characters from the intersection of $\pm\Sigma^1(G)$.

Corollary 14. *IF G is finitely presented as above and rank $G/G' \geq 2$, there exists $N \triangleleft G$ such that $G/N \cong \mathbb{Z}$. (N is the kernel of a rank 1 character).*

6.1. HNN extensions.

Theorem 15. *Let χ be a rank 1 character and $t \in G$ with $\chi(t) = 1$. Then: $[\chi] \in \Sigma^1(G)$ if and only if $N = \ker\chi$ has a subgroup H such that $t^{-1}Ht \subseteq H$ and $\bigcup_{\ell\in\mathbb{N}} t^{\ell}Ht^{-\ell} = N$. In other words, G is an ascending HNN extension with finitely generated base group H and isomorphic subgroups $t^{-1}Ht \cong H$.*

6.2. Descending HNN extensions.

Proposition 16. *Let G be finitely generated by X and let χ be a rank 1 character. If C is the connected component of 1 in $\Gamma_\chi(G,X)$, then:*

i. *$C \cap \Gamma_\chi^{[0,k]}$ is connected*
ii. *$N = \ker\chi$ has a subgroup H which is finitely generated, $uHu^{-1} \subseteq H$, $\bigcup_{\ell\in\mathbb{N}} u^{-\ell}Hu^{\ell} = N$.*

In other words, G is a descending HNN extension of H with stable letter u.

Example 17. *Recall $BS(1,2) = \langle a, u | uau^{-1} = a^2\rangle$ is an HNN extension of $\langle a\rangle$ with stable letter u. Here, $\Gamma_\chi^{[0,k]} \cap C$ is just chopping off the connected component of the identity at height k which is connected.*