The geometry of groups satisfying weak almost-convexity or weak geodesic-combability conditions

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Abstract

We examine the geometry of the word problem of two different types groups: those satisfying weak almost-convexity conditions and those admitting geodesic combings whose width satisfy minimally restrictive, non-vacuous constraints. In both cases we obtain an $n!$ isoperimetric function and n^2 upper bounds on the minimal isodiametric function and the filling length function.

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1 Introduction and statement of results

Throughout this article G will denote a group, which is always assumed to admit some finite generating set $\mathcal{A}$. We begin with an introduction to the concepts we will be discussing (detailed definitions are in §2), and by giving statements of the theorems (proofs are in §3).

An *almost-convexity* condition for G (defined with respect to $\mathcal{A}$) concerns pairs of nearby points at a distance k (with $k \geq k_0$ for some constant k_0) from the identity in the Cayley graph $\Gamma(G, \mathcal{A})$. It asserts the existence of some particular function $f : \mathbb{N} \to \mathbb{R}_+$ such that between the two points there is a path α of length at most $f(k)$ that does not leave the closed ball $B(k)$ around the identity in $\Gamma(G, \mathcal{A})$.

A *combing* of G is a normal form for the elements of G, that is, a section σ of the natural map $(\mathcal{A} \cup \mathcal{A}^{-1})^\star \to G$. We say σ is a *geodesic combing* when σ chooses a geodesic word σ_g for each $g \in G$. The synchronous width $\phi(n)$ and the asynchronous width $\Phi(n)$ of a combing measure the magnitude of the divergence of the *combing lines* of nearby group elements. A *weak combability condition* asserts that a group admits a combing whose (synchronous or asynchronous) width satisfies some specified constraint.

We will show that even weak almost-convexity conditions or weak geodesic-combability conditions lead to highly restrictive constraints on the geometry of the word problem. These constraints are manifested in upper bounds on *filling functions* for G, specifically on the *Dehn function* (also known as the *minimal isoperimetric functions*), on the *minimal isodiametric function* and on the *filling length function*. When G satisfies either a (non-vacuous) weak almost-convexity condition or a (non-vacuous) weak geodesic-combability condition we will show that it is possible to *fill* null-homotopic words with other null-homotopic words of strictly shorter length. This will lead to recurrence relations on the three filling functions, from which upper bounds will then follow.

We now state the theorems. We start with bounds on filling functions of groups satisfying any almost-convexity condition that is at least as strong as I.Kapovich's $K(2)$ condition (which is essentially the minimally restrictive non-vacuous almost-convexity condition).

Theorem 1. *Suppose a group G satisfies a weak almost-convexity condition (with respect to a finite generating set $\mathcal{A}$) in which $f(k) \leq 2k - 1$ for all $k \geq k_0$. Then G is finitely presentable and* *its Dehn, minimal isodiametric and filling length functions* simultaneously *admit the bounds:*

$$\begin{aligned}\mathrm{Area}(n) &\leq \prod_{k=2k_0+2}^{n}\left(1+f\left(\frac{k-1}{2}\right)\right) \preceq n! \\ \mathrm{Diam}(n) &\preceq n^2 \\ \mathrm{FL}(n) &\preceq n^2\end{aligned}$$

(For the definitions of Area, FL, Diam, $\preceq$ and "simultaneous" see §2.3.) From this theorem we gain some understanding of the hierarchy of almost-convexity conditions. When f is constant we recover the bound

$$\mathrm{Area}(n) \preceq e^n$$

on the Dehn function that Cannon proved in [4] for groups satisfying his almost-convexity condition $AC(2)$. So the bound of Theorem 1:

$$\mathrm{Area}(n) \preceq n! \leq n^n = e^{n\log n}$$

for groups satisfying the $K(2)$ condition only represents a slight relaxation of Cannon's bound for the $AC(2)$ condition. Cannon also proved that $AC(2)$ groups have linearly bounded isodiametric functions – we will reproduce his result in Remark 7. This compares with the quadratic bound of Theorem 1 for $K(2)$ groups.

Next we give a theorem in which we bound filling functions of groups satisfying weak geodesic-combing conditions. If we weaken the conditions to allow larger asynchronous width $\Phi(n)$ then the conditions become vacuous (see Proposition 1.1 of [2]). This theorem uncovers an (apparently)[1] unrelated criteria under which groups have filling functions satisfying the same bounds as those of Theorem 1. As stated the theorem concerns asynchronously combable groups. Synchronous width $\phi(n)$ bounds asynchronous width $\Phi(n)$ (see §2.2) and so we could replace Φ by ϕ and have a weaker theorem just about synchronous combings.

Theorem 2. *Suppose a group G admits a geodesic combing σ (with respect to a finite generating set $\mathcal{A}$) such that the asynchronous width Φ satisfies $\Phi(n) \leq n-2$ for all sufficiently large n. Then G is finitely presentable and its Dehn, minimal isodiametric and filling length functions simultaneously admit the bounds:*

$$\begin{aligned}\mathrm{Area}(n) &\preceq n!\,, \\ \mathrm{Diam}(n) &\preceq n^2, \\ \mathrm{FL}(n) &\preceq n^2.\end{aligned}$$

The author is pleased to thank Martin Bridson for his suggestion that the methods of this paper might be applied in the context of general combability conditions. The finite presentability claim of Theorem 2 has been proved by Bridson in [2]. The $n!$ upper bound on the Dehn function is an improvement on that in [2], although one should note that in [2] Bridson is working in the more general context of combings in which the combing lines σ_g are not required to be geodesics.

Many groups that arise naturally in combinatorial or geometric contexts fall in the scope of the weak almost-convexity or weak geodesic combability conditions of our two theorems – see the references listed in §2.1 and §2.2. However the author is unaware of any group that realises the $n!$ Dehn function – it would be interesting to know whether such "factorial groups" can be constructed, and indeed whether there are either groups satisfying a weak almost-convexity or admitting a geodesic combing that demonstrates the bounds of the theorems to be sharp.

It is worth pointing out that not all groups have filling functions that satisfy the bounds of the theorems in this paper. For example, it follows from remarks in §2.3 that groups whose word problem is not solvable have Area(n), FL(n) and Diam(n) all growing faster than any recursive

[1] However see [12] and [13] for relationships between weak almost-convexity conditions and *tame 1-combings*.

function. The family of groups Γ_k of [11, page 79] have Dehn functions $\simeq$-equivalent to the k-times iterated exponential function $\exp\exp\ldots\exp(n)$, and so for $k \geq 2$ fail to satisfy the $n!$ upper bound on the Dehn function. Indeed for all $k \in \mathbb{N}$, Gersten's group $\langle x, y \mid x^{x^y} = x^2 \rangle$ has $\mathrm{Area}(n)$ (and hence also $\mathrm{FL}(n)$ and $\mathrm{Diam}(n)$) growing faster than a k-times iterated – see [8]. Contrasting examples are provided by the groups Γ_m of Bridson [3] that have $\mathrm{Area}(n) \simeq n^{2m+1}$ and $\mathrm{Diam}(n) \simeq n^m$ and so fail the bound $\mathrm{Diam}(n) \preceq n^2$ when $m \geq 3$.

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2 Definitions

2.1 Weak almost-convexity conditions

Definition 1. A group G with finite generating set $\mathcal{A}$ satisfies an *almost-convexity condition* if there exist $k_0 \geq 1$ and $f : \mathbb{N} \to \mathbb{R}_+$ with the following property. Suppose $k \geq k_0$ and $a, b \in G$ are elements at a distance k from the identity in the Cayley graph $\Gamma(G, \mathcal{A})$, and that $d(a, b) \leq 2$. Then there exists a path α of length at most $f(k)$ from a to b in $\Gamma(G, \mathcal{A})$ such that α is contained in $B(k)$, the closed ball of radius k around 1.

Different almost-convexity conditions arise from taking different functions f. We list some non-trivial such conditions in order from strongest to weakest:

(i). Cannon's $AC(2)$ condition: f is constant.

(ii). Poénaru's $P(2)$ condition: f is sublinear; that is,

$$\forall C > 0, \quad \lim_{k \to \infty} (\, k - Cf(k)\,) = +\infty.$$

(iii). I. Kapovich's $K(2)$ condition: $f(k) = 2k - 1$.

Of the three condition listed above, the oldest is that of Cannon [4], and it is groups satisfying $AC(2)$ that are said to be *almost-convex*. Shapiro & Stein prove in [18] that if M is a closed 3-manifold with one of Thurston's eight geometries, then $\pi_1(M)$ satisfies Cannon's $AC(2)$ condition if and only if M is not *Sol*. We refer to the other (non-vacuous) conditions as *weak almost-convexity conditions*.

Poénaru's condition $P(2)$ is introduced in [16], where he proved that if a closed irreducible 3-manifolds has infinite fundamental group satisfying $P(2)$ then its universal cover is homoeomorphic to Euclidean 3-space.

I. Kapovich's $K(2)$ condition [14] can be regarded as the "minimally restrictive" almost-convexity condition.

Note that the metric we are using on $\Gamma(G, \mathcal{A})$ is that in which each 1-cell is uniformly given length 1. It will be important that the whole 1-skeleton is metrized, not just the 0-skeleton. For technical convenience we extend f to $\mathbb{R}_+$ by setting f to be constant on the intervals $[n, n+1)$ for all $n \in \mathbb{N}$.

The conditions $AC(2)$, $P(2)$ and $K(2)$ have generalisations $AC(n)$, $P(n)$ and $K(n)$ in which the distance $d(a, b)$ is allowed to be at most n instead of at most 2. It is straight-forward to prove that a group is $AC(n)$ for some $n \geq 2$ if and only if it is $AC(2)$. (The same cannot be said of $P(2)$ and $K(2)$.) The proof of Theorem 1 can be generalised to show that the bounds also apply to $AC(n)$, $P(n)$ and $K(n)$ groups ($n \geq 2$).

2.2 Weak combability conditions

Again we take G to be a group with finite generating set $\mathcal{A}$. Let $(\mathcal{A} \cup \mathcal{A}^{-1})^\star$ denote the free monoid on $\mathcal{A} \cup \mathcal{A}^{-1}$, and let $\ell(w)$ denote the length of words w in $(\mathcal{A} \cup \mathcal{A}^{-1})^\star$. Our definitions follow those of Bridson in [2].

Definition 2. A *combing* $\sigma : G \to (\mathcal{A}\cup\mathcal{A}^{-1})^\star$ is a section of the natural map from $(\mathcal{A}\cup\mathcal{A}^{-1})^\star$ to G. Denote the image of $g \in G$ by σ_g and view this as a continuous path $[0,\infty) \to \Gamma(G,\mathcal{A})$ from the identity to g, moving at a constant speed from the identity for time $\ell(\sigma_g)$ before becoming constant. We refer to σ_g as the *combing line* of g. The combing σ is a *geodesic combing* when σ_g is a geodesic word for all $g \in G$.

The *synchronous width* $\phi : \mathbb{N} \to \mathbb{N}$ of σ is defined by:

$$\phi(n) \;:=\; \max\left\{d(\sigma_g(t), \sigma_h(t)) \mid t \in \mathbb{N},\ d(1,g) \le n,\ d(1,h) \le n,\ d(g,h) = 1\right\}.$$

We shall also define the *asynchronous width* $\Phi : \mathbb{N} \to \mathbb{N}$ of σ. First we need to define a set of *reparametrizations*:

$$R \;:=\; \{\rho : \mathbb{N} \to \mathbb{N} \mid \rho(0) = 0,\ \rho(n+1) \in \{\rho(n), \rho(n)+1\}\,\forall n,\ \rho \text{ unbounded}\}.$$

Then let

$$D_\sigma(g,h) \;:=\; \min_{\rho,\,\rho' \in R}\left\{\max_{t\in\mathbb{N}}\{d(\,\sigma_g(\rho(t)),\ \sigma_h(\rho'(t))\,)\}\right\}$$

and define

$$\Phi(n) \;:=\; \max\left\{D_\sigma(g,h) \mid d(1,g) \le n,\ d(1,h) \le n,\ d(g,h) = 1\right\}.$$

The synchronous width $\phi(n)$ corresponds to taking each $\rho \in R$ to be the identity reparametrization, and hence $\phi(n) \le \Phi(n)$ for all n.

It is convenient to extend ϕ and Φ to $\mathbb{R}_+$ by making them constant on the intervals $[n, n+1)$ for all $n \in \mathbb{N}$.

2.3 Filling functions

Filling functions are invariants of finitely presentable groups – their definition owes much to the seminal work of Gromov [11], who pursued parallels with invariants arising from filling null-homotopic loops in Riemannian manifolds.

Let $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R}\rangle$ be a finite presentation of a group G. Let w be a word in the letters $\mathcal{A}$ and their formal inverses (that is, w is an element of $(\mathcal{A}\cup\mathcal{A}^{-1})^\star$) such that $w = 1$ in G. Such a word defines an edge-circuit in the Cayley graph $\Gamma(G,\mathcal{A})$ and is said to be *null-homotopic.* Filling functions capture aspects of the "geometry of the word problem" for $\mathcal{P}$ by examining different measurements of *van Kampen diagrams* (defined below) for words that are null-homotopic in $\mathcal{P}$.

Definition 3. A *van Kampen diagram* D_w for a null-homotopic word w can be considered to be a combinatorial homotopy disc for an edge-circuit associated to w in the Cayley 2-complex for $\mathcal{P}$. More formally, a van Kampen diagram is a finite, planar, contractible, combinatorial 2-complex; its 1-cells are directed and labelled by generators, the boundary labels of each of its 2-cells are cyclic conjugates of relators or inverse relators, and one reads w (by convention anticlockwise) around the boundary circuit from a base vertex $\star$.

Definition 4. We briefly recall the definitions of the *Dehn function*, the *minimal isodiametric function* and the *filling length function* of a finitely presentation $\mathcal{P}$ of a group G. We will refer to these as Area, Diam and FL respectively; all are functions $\mathbb{N} \to \mathbb{N}$. Let w be a null-homotopic word and D_w be a van Kampen diagram for w. The *area* of D_w is its number of 2-cells; the *diameter* of D_w is the maximal distance of all vertices in D_w to the base point $\star$ with respect to the metric on the 1-skeleton skeleton of D_w in which each 1-cell is given length 1; and the *filling length* of D_w is the minimal bound on the length of the boundary loop amongst *combinatorial null-homotopies* of D_w down to $\star$. Then $\mathrm{Area}(w)$, $\mathrm{Diam}(w)$ and $\mathrm{FL}(w)$ are the minimal area, diameter and filling length respectively of van Kampen diagrams D_w for w.

For $\mathrm{M} =$ Area, Diam and FL we define $M(n)$ to be the maximum of $\mathrm{M}(w)$ amongst all null-homotopic words w of length at most n. See [8] and [9] for more details.

Any function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \ge \mathrm{Area}(n)$ for all n is referred to as an *isoperimetric function* for $\mathcal{P}$.

The filling length is the least well known of the filling functions mentioned above. An equivalent definition of $\mathrm{FL}(w)$ is that it is the minimal bound on the length of words one must encounter in the process of reducing w to the empty word by applying relators, by free reduction, and by free expansion. In terms of van Kampen diagrams these three moves correspond to *2-cell collapse*, *1-cell collapse* and *1-cell expansion* respectively – together these can be employed to collapse a diagram down to its base point down to its basepoint, and this is what we term a *"combinatorial null-homotopies"* or *"shelling"*. (See [9] for a careful treatment of this.)

The Dehn function, the minimal isodiametric function and the filling length function are all defined for a presentation but are *well-behaved* on change of presentation – up to the well-known notion of $\simeq$-equivalence given below in Definition 6, the three functions are group invariants (the proof for FL can be found in [9] and for Area and Diam in [10]). Moreover, all three functions are in fact quasi-isometry invariants up to $\simeq$-equivalence – the proof in the case of Area is in [1], and the methods there can adapted for FL and Diam.

Definition 5. For two functions $f, g : \mathbb{N} \to \mathbb{N}$ we say that $f \preceq g$ when there exists $C > 0$ such that $f(n) \leq Cg(Cn + C) + Cn + C$ for all n, and we say $f \simeq g$ if and only if $f \preceq g$ and $g \preceq f$.

Definition 6. In saying that the functions Area, Diam and FL are *simultaneously* bounded by functions f, g, h (respectively), we mean that for any given null-homotopic word w there is some van Kampen diagram D_w whose area, diameter, and filling length are respectively at most $f(\ell(w))$, $g(\ell(w))$ and $h(\ell(w))$.

One might ask whether it is worth giving upper bounds on both $\mathrm{Diam}(n)$ and $\mathrm{FL}(n)$ in this article. We remark that it is easy to prove that $\mathrm{Diam}(n) \leq \mathrm{FL}(n)$ for all n but it is an open problem ([11], page 100) whether there is a finitely presented group for which there is no $K > 0$ with $\mathrm{FL}(n) \leq K\,\mathrm{Diam}(n)$ for all n.

The Dehn function $\mathrm{Area}(n)$ can always be bounded by a double exponential of the $\mathrm{Diam}(n)$ function – see [5] and [7]. There is extensive literature concerning isoperimetric functions and isodiametric functions – see for example [2], [8], [15] and [17] and references therein.

When one (and hence all) of $\mathrm{Diam}(n)$, $\mathrm{FL}(n)$ and $\mathrm{Area}(n)$ is bounded by a recursive function, the word problem for $\mathcal{P}$ is solvable – indeed the (non-deterministic) time complexity of the word problem is $\simeq \mathrm{Area}(n)$ by Theorem 1.1 in [17]. Thus the non-deterministic time complexity of groups satisfying either the weak almost-convexity condition of Theorem 1 or the weak geodesic-combability condition of Theorem 2 is $\preceq n!$.

3 Proofs

3.1 Proof of Theorem 1

The finitely presentability of G is due to I. Kapovich (Theorem 3 of [14]) and we reproduce his result in the process of proving the bounds on the three filling functions. Let $\mathcal{R}$ be the set of null-homotopic words in $(\mathcal{A} \cup \mathcal{A}^{-1})^\star$ of length less than $2k_0 + 1$, which is a finite set as $|\mathcal{A}| < \infty$. We will show that $\mathcal{P} := \langle \mathcal{A} \mid \mathcal{R} \rangle$ is a finite presentation for G.

Suppose w is a null-homotopic word and let n be the length of w. Our strategy is to examine how the $K(2)$ condition allows us to build a van Kampen diagram for w with respect to the presentation $\mathcal{P}$. Let D be a combinatorial 2-disc, consisting of just one 2-cell with boundary loop made up of n 1-cells. Let $\Psi : D^{(1)} \to \Gamma(G, \mathcal{A})$ be a combinatorial map from the 1-skeleton of D to a loop labelled w in the Cayley graph $\Gamma(G, \mathcal{A})$ of G. Via Ψ each 1-cell of D can be considered to inherit from $\Gamma(G, \mathcal{A})$ a labelling by element an of $\mathcal{A}$ together with a direction, so that w is read anticlockwise around ∂D starting from a base vertex $\star$.

We will show how, provided that $\ell(w) \geq 2k_0 + 2$, it is possible to *fill* w with words of length at most $n - 1$. That is, we will produce from $\Psi : D^{(1)} \to \Gamma(G, \mathcal{A})$ a combinatorial map $\hat{\Psi} : \hat{D}^{(1)} \to \Gamma(G, \mathcal{A})$ in which $\hat{D}$ is a finite, planar, contractible, combinatorial 2-complex with $\partial D = \partial \hat{D}$, and such that $\hat{\Psi}$ extends Ψ. The boundary loop of each 2-cell will map to a null-homotopic word of length at most $n-1$.

Our reward will then be that we are able to iteratively apply this filling procedure to construct successive extensions $\Psi_i : D_i^{(1)} \to \Gamma(G,\mathcal{A})$ for $i = 1, 2, \ldots$, in which each D_i is a finite, planar, contractible, combinatorial 2-complex and each Ψ_i is a combinatorial map. So we will then start with $\Psi_1 := \Psi$ and $D_1 := D$, and we will construct Ψ_{i+1} from Ψ_i and D_{i+1} from D_i by using the filling procedure to refine each 2-cell. Eventually this will lead to a combinatorial map $\Psi_m : D_m^{(1)} \to \Gamma(G,\mathcal{A})$ such that the boundary loop of each 2-cell maps to a null-homotopic word of length at most $2k_0+1$. It will then be possible to extend Ψ_m to a combinatorial map to the Cayley 2-complex $\Gamma(\mathcal{P})$ of $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$. So it will follow that w is expressible in $F(\mathcal{A})$ as a product of conjugates of elements of $\mathcal{R}$, and therefore we will be able to deduce that $\mathcal{P}$ is a finite presentation for G. Further, D_m will be a van Kampen diagram for w. An inductive analysis of the area, diameter and filling length will lead to the bounds of the theorem.

So let us now focus on the null-homotopic word w of length n with its associated map $\Psi : D^{(1)} \to \Gamma(\mathcal{A})$. We treat the cases of n odd and of n even separately.

Case: n **odd.** Assume $\frac{n-1}{2} \geq k_0$. The word w defines a loop in $\Gamma(G,\mathcal{A})$. Let u and v be the two vertices of D that are a distance $\frac{n-1}{2}$ from $\star$. Suppose one of $d(\Psi(\star),\Psi(u))$ and $d(\Psi(\star),\Psi(v))$ is strictly less than $\frac{n-1}{2}$, say $d(\Psi(\star),\Psi(u))$ with out loss of generality. Produce a diagram $\hat{D}$ from D by inserting a simple p combinatorial path made up of a concatenation of $d(\Psi(\star),\Psi(u))$ 1-cells in the interior of D between $\star$ and u. Let $\hat{\Psi} : \hat{D}^{(1)} \to \Gamma(G,\mathcal{A})$ be the extension of Ψ obtained by mapping p to a geodesic between $\Psi(\star)$ and $\Psi(u)$. Then both 2-cells in $\hat{D}$ have length at most $n-1$.

Now assume $d(\Psi(\star),\Psi(u)) = d(\Psi(\star),\Psi(v)) = \frac{n-1}{2}$. By the homogeneity of $\Gamma(G,\mathcal{A})$ we may assume that Ψ maps $\star$ to the identity element $1 \in G$. Then $\Psi(u)$ and $\Psi(v)$ are a distance $\frac{n-1}{2} \geq k_0$ from 1 in $\Gamma(G,\mathcal{A})$ and are a distance at most 1 apart. So the weak almost-convexity condition allows us to find a path α in the closed ball $B(\frac{n-1}{2})$ around 1 in $\Gamma(G,\mathcal{A})$ such that the length $\ell(\alpha)$ of α is at most $f(\frac{n-1}{2})$, which is at most $n-2$.

Now join u to v by a simple combinatorial path p in the interior of D made up of a concatenation of $\ell(\alpha)$ 1-cells, and extend Ψ to a map $\Psi' : D^{(1)} \cup p \to \Gamma(G,\mathcal{A})$ so that p is mapped to α. Let $v_0, v_1, \ldots, v_{\ell(\alpha)}$ be the vertices of p so that $v_0 = u$ and $v_{\ell(\alpha)} = v$. Next join $v_1, v_2, \ldots, v_{\ell(\alpha)-1}$ to $\star$ by simple combinatorial paths $p_1, p_2, \ldots, p_{\ell(\alpha)-1}$ of length $d(1, \Psi'(v_i)) \leq \frac{n-1}{2}$ as depicted in Figure 1. Call the resulting combinatorial 2-disc $\hat{D}$. Then extend Ψ' to a map $\hat{\Psi} : \hat{D}^{(1)} \to \Gamma(G,\mathcal{A})$ by mapping these paths to geodesics in $\Gamma(G,\mathcal{A})$.

The number of 2-cells in $\hat{D}$ is $1+\ell(\alpha) \leq 1 + f(\frac{n-1}{2}) \leq n-1$. We refer to the loop in $\hat{D}^{(1)}$ made up of p together with the segment from u to v as $\bar{p}$. We claim that the length of the boundary loop of each 2-cell in $\hat{D}$ is at most $n-1$. The loop $\bar{p}$ has length at most $1 + f(\frac{n-1}{2}) \leq n-1$. Each of the remaining loops is made up of a pair p_i, p_{i+1} of adjacent paths together with one 1-cell of p (where $0 \leq i \leq \ell(\alpha)-1$, and p_0 and $p_{\ell(\alpha)}$ are the paths labelled w_0 and w_1 respectively). Both p_i and p_{i+1} have lengths at most $\frac{n-1}{2}$ but one must also be of length at most $\frac{n-1}{2} - 1$ because otherwise the midpoint of the 1-cell between v_i and v_{i+1} would be mapped by $\hat{\Psi}$ to a point outside $B(\frac{n-1}{2})$. The length of the loop is therefore at most $n-1$ as required.

Case: n **even.** The construction of $\hat{\Psi} : \hat{D} \to \Gamma(G,\mathcal{A})$ is very similar to the case when n is odd and so we provide fewer details. This time u and v are the two vertices of D that are a distance $\frac{n}{2}-1$ from $\star$, and so $d(\Psi(u),\Psi(v)) \leq 2$. Assume $\frac{n}{2}-1 \geq k_0$. Similarly to the case of n odd, we focus on the situation where $d(\Psi(\star),\Psi(u)) = d(\Psi(\star),\Psi(v)) = \frac{n}{2}-1$, for without one of these equalities the construction of $\hat{D}$ is straight-forward. The $K(2)$ condition gives a path α in $B(\frac{n}{2}-1)$ between $\Psi(u)$ and $\Psi(v)$ of length $\ell(\alpha)$ at most $f(\frac{n}{2}-1) \leq 2(\frac{n}{2}-1)-1 = n-3$. Pairs of geodesics in $\Gamma(G,\mathcal{A})$ from adjacent vertices on α to 1 both have length at most $\frac{n}{2}-1$ and one has length at most $\frac{n}{2}-2$.

The number of 2-cells in $\hat{D}$ is at most $1+\ell(\alpha) \leq n-2$. Again we refer to the loop in $\hat{D}^{(1)}$ made up of p together with the segment from u to v as $\bar{p}$. The loop $\bar{p}$ has length at most

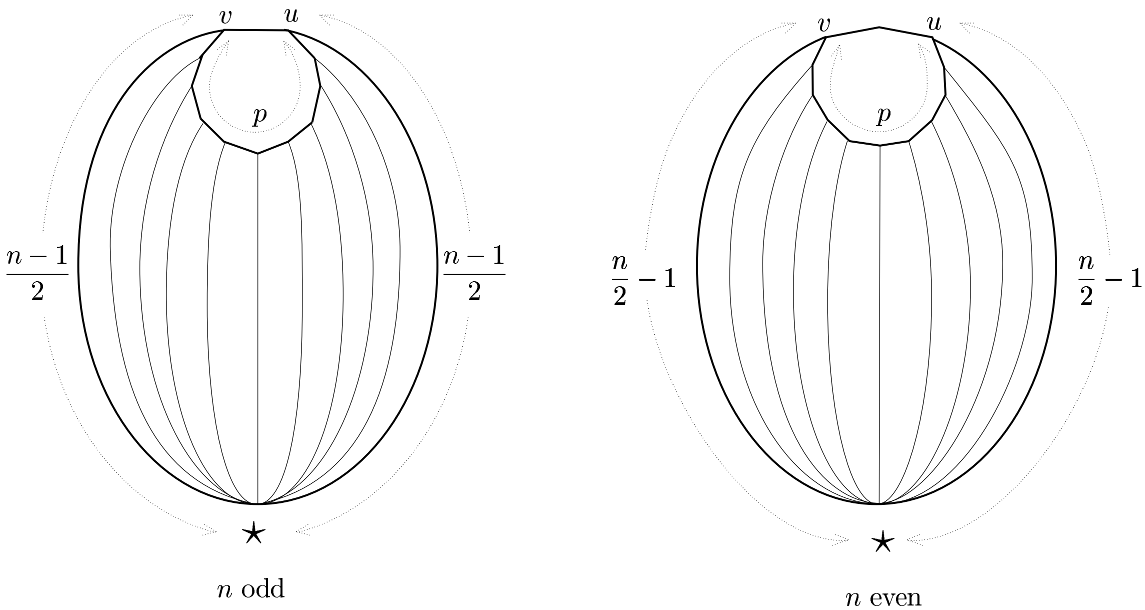

Figure 1: The diagram $\hat{D}$ for groups satisfying weak almost-convex conditions.

$\ell(\alpha)+2 \leq f(\frac{n}{2}-1)+2 \leq n-1$. The other $\ell(\alpha)$ 2-cells have boundary loops of length at most $2\frac{n}{2}-3+1=n-2$.

We can now complete our proof by using the analysis above to give recurrence relations on the Dehn, the minimal isodiametric and the filling length functions with respect to the presentation $\mathcal{P}$. For $n \geq 2k_0+2$,

$$\mathrm{Area}(n) \quad \leq \quad \left(1+f\left(\frac{n-1}{2}\right)\right)\mathrm{Area}(n-1) \tag{3.1}$$

$$\mathrm{Diam}(n) \quad \leq \quad \max\left\{\frac{n}{2}+\mathrm{Diam}\left(1+f\left(\frac{n-1}{2}\right)\right),\ \mathrm{Diam}(n-1)\right\} \tag{3.2}$$

$$\mathrm{FL}(n) \quad \leq \quad 1+f\left(\frac{n-1}{2}\right)+n+\mathrm{FL}(n-1). \tag{3.3}$$

We explain in turn how each of these inequalities follows from the construction of $\hat{D}$. We concentrate on the case where $\hat{D}$ is as in Figure 1; in the case where one of $d(\Psi(\star),\Psi(u))$ and $d(\Psi(\star),\Psi(v))$ is less than $\frac{n-1}{2}$ (case, n odd) or $\frac{n}{2}-1$ (case, n even) and $\hat{D}$ has two 2-cells, the three inequalities are easily seen to hold.

Inequality (3.1)*:* there are at most $\left(1+f\left(\frac{n-1}{2}\right)\right)$ 2-cells in $\hat{D}$. Each has boundary loop of length at most $n-1$.

Inequality (3.2)*:* the distance in $\hat{D}^{(1)}$ from any vertex of the loop $\bar{p}$ to $\star$ is at most $\frac{n}{2}$, and this loop has length at most $1+f\left(\frac{n-1}{2}\right)$. This accounts for the first entry in the set on the right-hand side of (3.1). The $\mathrm{Diam}(n-1)$ entry arises from considering the remaining loops in $\hat{D}$ (each of which we may assume also to be based at $\star$).

Inequality (3.3)*:* the diagram $\hat{D}$ can be shelled (*i.e.* combinatorially null-homotoped down to the base vertex $\star$) by first collapsing the 2-cell with boundary $\bar{p}$, and then collapsing the remaining cells working from one side of the diagram to the other. In this process the boundary curve reaches length at most $1+f\left(\frac{n-1}{2}\right)+n$. Now recall that we constructed a van Kampen diagram D_m for w by iteratively filling to give a sequence of diagrams $D_1, D_2, \ldots, D_m$ (where D_1 has one 2-cell with boundary loop labelled by the word w). We can inductively construct a shelling of D_m, by producing a shelling of D_{i+1} from a shelling of D_i: since D_{i+1} is a refinement of D_i, the image of $D_i^{(1)}$ in D_{i+1} partitions D_{i+1} into subdiagrams; shell these subdiagrams in turn in the order dictated by the shelling of D_i. Using filling length minimizing shellings of each of these subdiagrams leads to the recurrence relation of (3.3).

The hypothesis $f(k) \leq 2k-1$ (*i.e.* the $K(2)$ condition) applied to (3.1), (3.2) and (3.3) gives, for $n \geq 2k_0+2$,

$$\begin{aligned} \mathrm{Area}(n) &\leq (n-1)\mathrm{Area}(n-1) && (3.4)\\ \mathrm{Diam}(n) &\leq \frac{n}{2} + \mathrm{Diam}(n-1) && (3.5)\\ \mathrm{FL}(n) &\leq 2n-1+\mathrm{FL}(n-1). && (3.6) \end{aligned}$$

Now $\mathrm{Area}(2k_0+1) = 1$, $\mathrm{Diam}(2k_0+1) = k_0$ and $\mathrm{FL}(2k_0+1) = 2k_0+1$ since all null-homotopic words of length at most $2k_0+1$ are in $\mathcal{R}$ and hence admit van Kampen diagrams with only one 2-cell. The bounds

$$\begin{aligned} \mathrm{Area}(n) &\leq \prod_{k=2k_0+2}^{n} \left(1+f\left(\frac{k-1}{2}\right)\right) \preceq n! \\ \mathrm{Diam}(n) &\preceq n^2 \\ \mathrm{FL}(n) &\preceq n^2 \end{aligned}$$

claimed in Theorem 1 then follow from (3.1), (3.4), (3.5) and (3.6).

Remark 7. Notice that in the case where f is constant (*i.e.* the case of Cannon's $AC(2)$ condition) we can obtain Cannon's linear bound on diameter from the recurrence relation (3.2). Moreover the quadratic bound on filling length can also be improved to linear: $\mathrm{FL}(n) \preceq n$.

3.2 Proof of Theorem 2

Similarly to the proof of Theorem 1 our approach is to show that given any sufficiently long word w with associated combinatorial map $\Psi : D^{(1)} \to \Gamma(G, \mathcal{A})$ (where D is the diagram consisting of one 2-cell with boundary made up to $\ell(w)$ 1-cells) we can extend Ψ to some combinatorial map $\hat{\Psi} : \hat{D}^{(1)} \to \Gamma(G, \mathcal{A})$ where $\hat{D}$ is a refinement of D with boundary loop of length at most $\ell(w)-1$. Then we will be able to iteratively extend to produce van Kampen diagrams. It will follow that G is finitely presentable – the set of all null-homotopic words up to some sufficiently large length can be taken for the set of relators. An analysis of the inductive procedure again gives bounds on the Dehn function, the minimal isodiametric function and the filling length function.

The diagrams $\hat{D}$, whose construction we now explain, are depicted in Figure 2 in the two cases corresponding to $n := \ell(w)$ odd and even.

Case: n odd. Take u and v to be the vertices of D at a combinatorial distance $(n-1)/2$ from $\star$ in $D^{(1)}$. Let $g := \Psi(u)$ and $h := \Psi(v)$. Recall that the geodesic combing lines of g and h are denoted σ_g and σ_h respectively. Construct simple paths p_u and p_v in the interior of D between $\star$ and each of u and v, by concatenating $\ell(\sigma_g)$ and $\ell(\sigma_h)$ 1-cells respectively. Next we join vertices of these p_1 and p_2 by a *ladder* as follows. If we were working with a synchronous geodesic-combing of width $\phi(n) \leq n-2$ for all n we can take the *rungs* of this ladder to be the $(n-1)/2$ paths of length $d(\sigma_g(t), \sigma_h(t))$ for $t = 1, 2, \ldots, (n-1)/2$. In the more general context of an asynchronous geodesic-combing satisfying $\Phi(n) \leq n-2$ for all n, the *rungs* become organised more haphazardly as directed by reparametrizations; this is shown in the left-hand diagram of Figure 2.

Call the new diagram $\hat{D}$ and extend Ψ to $\hat{\Psi}$ in the natural way. There are at most $n-1$ 2-cells in the *ladder* in $\hat{D}$.

The hypothesis, $\Phi(r) \leq r-2$ for all r, on the asynchronous width of the combing means that the *rungs* of the *ladder* have length at most $(n-1)/2-2$. It follows that each of the 2-cells of $\hat{D}$ that lies between two *rungs* has boundary loop of length at most

$$2 + 2\,\Phi((n-1)/2) \;\leq\; 2\,((n-1)/2-2)+2 \;=\; n-3.$$

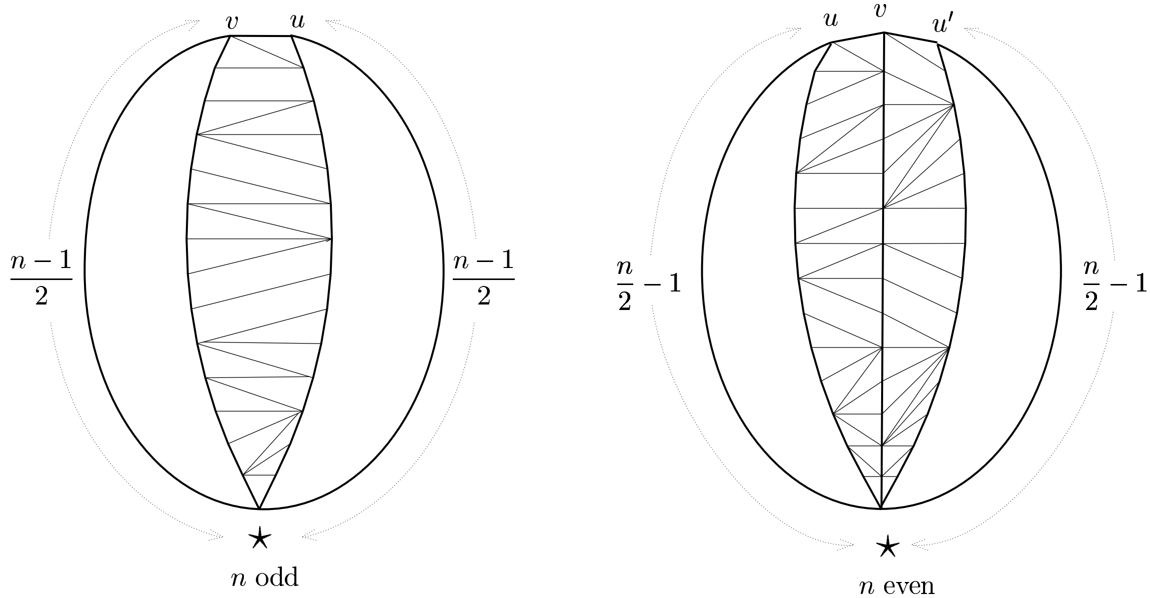

Figure 2: The diagram $\hat{D}$ for groups admitting an asynchronous geodesic-combing.

The remaining two 2-cells have length at most $n-1$.

Case: n **even.** The construction is very similar to when n is odd. We take u, and u' to be the vertices of D at a combinatorial distance $n/2-1$ from $\star$ and we take v to be the vertex at a distance $n/2$. Let $g := \Psi(u)$, $g' := \Psi(u')$ and $h := \Psi(v)$, then join u, u' and v to $\star$ by simple paths p_u, $p_{u'}$ and p_v of length $\ell(\sigma_g)$, $\ell(\sigma'_g)$ and $\ell(\sigma_h)$. Then supply the *rungs* to construct *ladders* between p_u and p_v, and between p_v and $p_{u'}$. This is depicted in the right-hand diagram of Figure 2.

The result is that each of the 2-cells of $\hat{D}$ that lies between two *rungs* has boundary loop of length at most

$$2 + 2\,\Phi(n/2) \;\leq\; 2 + 2\,(n/2-2) \;=\; n-2.$$

The remaining two 2-cells have boundary length at most $n-2$.

The following recurrence relations on filling length can be deduced from the constructions of the diagrams $\hat{D}$ above.

$$\begin{aligned} \mathrm{Area}(n) &\leq 2n\,\mathrm{Area}(2+2\,\Phi(n/2)) + 2\,\mathrm{Area}(n-1) \\ &\leq (n+2)\,\mathrm{Area}(n-1), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \mathrm{Diam}(n) &\leq \max\{\,\mathrm{Diam}(2+2\,\Phi(n/2)) + (n-1)/2,\ \mathrm{Diam}(n-1)\} \\ &\leq \mathrm{Diam}(n-1) + (n-1)/2, \end{aligned} \tag{3.8}$$

$$\mathrm{FL}(n) \;\leq\; 3n/2 + \Phi(n/2) + \mathrm{FL}(n-1). \tag{3.9}$$

We explain these inequalities in turn :

Inequality (3.7)*:* With a synchronous combing there are at most $(n-1)/2$ 2-cells in one *ladder* in $\hat{D}$ when n is odd, and there at most $n/2$ 2-cells in each of the two *ladders* when n is even. In the asynchronous case there are up to twice as many 2-cells between the combing lines on account of the *ladders* being built out of *triangles* as well as *rectangles* – see Figure 2. These 2-cells have boundaries of length at most $2+2\,\Phi(n/2)$. There are two further 2-cells in each $\hat{D}$, each of which has boundary length at most $n-1$.

Inequality (3.8)*:* recall that producing the diagram $\hat{D}$ from D is the first step in an inductive procedure for constructing a van Kampen diagram. The first entry in the set bounds the distance from a vertex in a *ladder* in $\hat{D}$ to $\star$, first along a path to a combing line in $\hat{D}$ and then down that combing line to $\star$. The second entry bounds the distance from a vertex of a 2-cell not in a *ladder* of $\hat{D}$.

Inequality (3.9)*:* one can shell $\hat{D}$ by first collapsing the left-most 2-cell (as depicted in Figure 2), then collapsing the cells in the *ladder(s)* working from the top to the bottom (shelling the left ladder followed by the right ladder in the case when n is even), and then collapsing the right-most 2-cell. In the process of this shelling of $\hat{D}$ the boundary curve has length at most $3n/2 + \Phi(n/2)$. One only has to add $\mathrm{FL}(n-1)$ to each of these to get a bound on $\mathrm{FL}(n)$ (for the same reasons as given the proof of inequality (3.3)).

The bounds of Theorem 2 follow from the recurrence relations (3.7), (3.8) and (3.9).

Remark 8. For some width functions ϕ and Φ the recurrence relations above lead to better bounds than those given in Theorem 2. For example when either ϕ or Φ is bounded (the synchronous or asynchronous "*k-fellow traveller property*" respectively) we recover the quadratic bound on the Dehn function and the linear isodiametric function of [6] and [8].

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