Higher connectedness of asymptotic cones

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Abstract

We give coarse geometric conditions for a metric space X to have N-connected asymptotic cones. These conditions are expressed in terms of certain filling functions concerning filling N-spheres in an appropriate coarse sense.

We interpret the criteria in the case where X is a finitely generated group Γ with a word metric. This leads to upper bounds on filling functions for groups with simply connected cones – in particular they have linearly bounded filling length functions. We prove that if all the asymptotic cones of Γ are N-connected then Γ is of type $\mathcal{F}_{N+1}$ and we provide N-th order isoperimetric and isodiametric functions. Also we show that the asymptotic cones of a virtually polycyclic group Γ are all contractible if and only if Γ is virtually nilpotent.

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1 Introduction

In the book *Asymptotic Invariants of Infinite Groups* [25] Gromov says of a finitely generated group Γ with a word metric d:

> "This space may at first appear boring and uneventful to a geometer's eye since it is discrete and the traditional local (*e.g.* topological and infinitesimal) machinery does not run in Γ."

The asymptotic cone presents a different perspective in which to look at Γ. Imagine viewing Γ from increasingly distant vantage points, *i.e.* scaling the metric by a sequence $\mathbf{s} = (s_n)$ with $s_n \to \infty$. Via some nonstandard analysis, an asymptotic cone provides a limit of the sequence $(\Gamma, \frac{1}{s_n}d)$. This limit represents a coalescing of Γ to a more continuous object that is amenable to attack by topological and infinitesimal machinery, and which "fills our geometer's heart with joy" (to quote Gromov [25] again).

In particular one can study the homotopy groups of the asymptotic cones of Γ, which, as we will see, impart information about the coarse geometry of Γ. The essential technique is as follows. Maps of spheres or discs into asymptotic cones of Γ can be *pulled back* to sequences of maps into Γ. Information about Γ can then be gleaned from *filling functions*. (This idea goes back to Gromov [25, Ch. 5]; it has been pursued further in relation to 1-connectedness of asymptotic cones of finitely generated groups in [5], [10], [27], [37] – see §5.)

It turns out that an asymptotic cone is a rather general construction, not just applying to groups but in fact to all metric spaces. Hence in §3 and §4 we work to understand what it means for *any given metric space* to have highly connected asymptotic cones, before interpreting what the results mean in the subsequent sections in the context of groups. Whilst the definitions of §3 and the results of §4 are given in the full generality of *any* metric space X, the reader may find it helpful to keep in mind the examples where X is a (quasi-) homogeneous, non-compact metric space, *e.g.* X is quasi-isometric to a finitely generated group (with its word metric) or is a non-compact Lie group.

The vanishing of particular homotopy groups is a bi-Lipschitz invariant property of asymptotic cones of a space X, and any bi-Lipschitz invariant of the cones provides a quasi-isometry invariant for the underlying metric spaces X – see Proposition 2.5. Thus one motivation for examining what the N-connectedness of the asymptotic cones of X means for X, is to find quasi-isometry invariants. In particular the two conditions in Theorem A below are both quasi-isometry invariants for metric spaces.

Important applications of asymptotic cones have exploited this fact that bi-Lipschitz invariants of cones give quasi-isometry: this fact is used by Kleiner and Leeb [30] in establishing quasi-isometric rigidity results for symmetric spaces of non-compact type, and by M. Kapovich and Leeb [29] in distinguishing quasi-isometry classes of fundamental groups of closed Haken 3-manifolds. Uses of asymptotic cones have also been found by Druţu who gives a new proof in [12] of a result about quasi-isometries of irreducible non-uniform lattices in certain semi-simple Lie groups, and in [11] finds bounds on the Dehn function of certain non-cocompact, irreducible $\mathbb{Q}$-rank 1 lattices.

This article is structured as follows. We define asymptotic cones, quasi-isometries, combinatorial complexes, $\simeq$-equivalence of functions $\mathbb{R} \to \mathbb{R}$, van Kampen diagrams and some related notions from Geometric Group Theory in §2. Then in §3 we recursively define the filling functions $\text{Fill}^k_{\mathbf{R},\boldsymbol{\mu}} : [0,\infty) \to \mathbb{N} \cup \{\infty\}$ for metric spaces that we will use in characterising spaces with highly connected asymptotic cones.

The definition of $\text{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$ makes use of finite *combinatorial structures* (defined in §2.3) for discs and spheres. The sequence $\mathbf{R} = (R_i)$ of positive integers constrains the combinatorial complexity of the complexes used. Given a combinatorial structure C and a map $\gamma : C^{(0)} \to X$, with domain the 0-skeleton of C, define

$$\text{mesh}(C, \gamma) \ := \ \sup \{ d(\gamma(a), \gamma(b)) \mid a \text{ and } b \text{ are the end points of a 1-cell in } C \}.$$

Suppose C is, in fact, a combinatorial structure for the N-sphere. The function $\text{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}} : [0,\infty) \to \mathbb{N} \cup \{\infty\}$ tells us about extending any such $\gamma : C^{(0)} \to X$ in a coarse sense across the $(N+1)$-disc $\mathbb{D}^{N+1}$. This extension is built up through the dimensions; that is, by first extending (in a coarse sense) across 1-cells, then across 2-cells, and so on, until finally across $\mathbb{D}^{N+1}$. The sequence of positive reals $\boldsymbol{\mu} = (\mu_i)$ introduce *error terms* into the definition, and consequently a *coarseness* into the filling functions; this will be appropriate because asymptotic cones ignore local geometry.

In §4 we prove the characterisation of metric spaces with highly connected asymptotic cones that will be at the heart of all the subsequent results in the chapter. Recall that a topological space is said to be N-connected when its homotopy groups $\pi_0, \pi_1, \ldots, \pi_N$ are all trivial. In the statement of this theorem $\mathbf{e} = (e_n)$ is a sequence of base points in X and $\mathbf{s} = (s_n)$ is a sequence of scaling factors with $s_n \to \infty$; both are part of the definition of an asymptotic cone.

Theorem A. *Let X be a metric space, let ω be a non-principal ultrafilter, and $N \geq 0$. The following are equivalent.*

- *The asymptotic cones* $\mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$ *are* N*-connected for all* $\mathbf{e}$ *and* $\mathbf{s}$.
- *There exist* $\mathbf{R}$, $\boldsymbol{\mu}$ *such that the filling functions* $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$ *are bounded.*

In §5 we show how the 2-dimensional filling function $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ can be reinterpreted to give the following algebraic characterisation of finitely generated groups Γ with 1-connected cones:

Theorem B. *Let* Γ *be a group with finite generating set* $\mathcal{A}$*. Fix any non-principal ultrafilter* ω*. The following are equivalent.*

- *The asymptotic cones* $\mathrm{Cone}_\omega(\Gamma, \mathbf{1}, \boldsymbol{s})$ *of* Γ *are simply connected for all* $\boldsymbol{s}$.
- *There exist* $K, L \in \mathbb{N}$ *such that for all null-homotopic words* w *of length* $\ell(w) \geq L$ *there is an equality*

$$w \;=\; \prod_{i=1}^{K} u_i w_i u_i^{-1}$$

in the free group $F(\mathcal{A})$ *for some words* u_i *and* w_i *such that the* w_i *are null-homotopic and have length* $\ell(w_i) \leq \ell(w)/2$ *for all* i.

This theorem is then used to obtain information about filling invariants of Γ. We recover results of Bridson [5], Druţu [10], Gromov [25], and Papasoglu [37] that say Γ has a polynomially bounded Dehn function and a linearly bounded isodiametric function, and we are able to add that the *filling length function* also has a linear bound. The constants K and L in the statement of Theorem C are those arising in Theorem B.

Theorem C. *Suppose that the asymptotic cones* $\mathrm{Cone}_\omega(\Gamma, \mathbf{1}, \boldsymbol{s})$ *of a finitely generated group* Γ *are simply connected for all sequences of scalars* $\boldsymbol{s} = (s_n)$ *with* $s_n \to \infty$*. Then there exists a finite presentation* $\langle \mathcal{A} \mid \mathcal{R} \rangle$ *for* Γ *with respect to which, for all* $n \in \mathbb{N}$ *the Dehn function, the minimal isodiametric function, and the filling length function satisfy*

$$\begin{aligned} \mathrm{Area}(n) &\leq Kn^{\log_2(K/L)}, \\ \mathrm{Diam}(n) &\leq (K+1)n, \\ \mathrm{FL}(n) &\leq 2(K+1)n, \end{aligned}$$

for some constants $K, L > 0$*. Further, given a null-homotopic word* w *with* $\ell(w) = n$*, there is a van Kampen diagram* D_w *for* w *on which these three bounds are realised simultaneously.*

The filling length function FL is a filling invariant discussed extensively in [18]; it measures the length of the contracting boundary loop of a van Kampen diagram for null-homotopic words w in the course of a combinatorial analogue of a null-homotopy. Equivalently filling length bounds the maximum length of words one must encounter when reducing w down to the empty word using the relators. In particular, in the light of Pansu [36], we learn from the above theorem that finitely generated nilpotent groups have linear filling length. Using the result of Papasoglu [37] we learn that this is also the case for groups with quadratically bounded Dehn functions (this includes Thompson's Group F – see Guba [26]).

In §6 we discuss finiteness properties and higher order isoperimetric and isodiametric functions for finitely generated groups. This is in preparation for §7 in which we prove bounds on the 2-variable N-th order isoperimetric function $\delta^{(N)}(n,\ell)$ and isodiametric function $\eta^{(N)}(n,\ell)$ for groups with N-connected cones. These functions concern the combinatorial filling volume and filling diameter, respectively, of singular combinatorial N-spheres in terms of the combinatorial N-volume n of the N-spheres and the diameter of their images ℓ. The theorem is:

Theorem D. *Let Γ be a finitely generated group with a word metric. Suppose that the asymptotic cones of Γ are all N-connected ($N \geq 1$). Then Γ is of type $\mathcal{F}_{N+1}$.*

Further, fix any finite $(N+1)$-presentation for Γ. There exist $a_N, b_N \in \mathbb{N}$ and $\alpha_N > 0$ such that for all $n \in \mathbb{N}$ and $\ell \geq 0$,

$$\begin{aligned} \delta^{(N)}(n,\ell) &\leq a_N \, n \, \ell^{\alpha_N}, \\ \eta^{(N)}(n,\ell) &\leq b_N \, \ell. \end{aligned}$$

Moreover these bounds are always realisable simultaneously.

The theorem of §2.8 is:

Theorem E. *Let Γ be a virtually polycyclic group and let ω be any non-principal ultrafilter. The following are equivalent.*

- Γ *is virtually nilpotent.*
- $\mathrm{Cone}_\omega(\Gamma, \mathbf{1}, \boldsymbol{s})$ *is contractible for all sequences of scalars* $\mathbf{s}$.

In the proof we appeal to results of Harkins [28] and Pansu [36]. It is immediate from Pansu's work that the asymptotic cones of nilpotent groups are contractible. Harkins shows that if Γ is virtually polycyclic but not virtually nilpotent then one of its higher order Dehn functions is exponential. We use Harkins' techniques to show that the higher order isoperimetric and isodiametric inequalities of §2.7 must fail for such a Γ in some dimension. It follows that the higher homotopy groups of the asymptotic cones of Γ cannot all be trivial.

Related literature. Detailed references to work related to Theorems A to E can be found in the text of this article. Here is a brief summary. Theorem A has origins in work of Papasoglu [37], which in turn is based on ideas of Gromov [25]. The half of Theorem B that begins with the assumption that the asymptotic cones are simply connected was essentially proved by R. Handel [27] and subsequently by Gromov [25]. The reverse implication is used by Papasoglu in [37] and our proof develops the arguments he gives. Gromov went on to deduce the polynomial bound on the Dehn function in Theorem C. An exegesis of Gromov's proof was given by Druţu [10]. The linear bound on diameter in Theorem C was observed by Papasoglu [37] and Bridson [5]. The inequalities in Theorem D are in a similar vein to those obtained in different contexts in Epstein *et al.* [14, Theorem 10.2.1] (*"mass times diameter estimate"*) and Gromov [23] (the "*cone inequality*"). Theorem E was suggested to me by Martin Bridson, although there were many details to pursue; the proof includes appeals to [28] and [36] as discussed above.

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2 Preliminaries

2.1 Asymptotic cones

Asymptotic cones were introduced by van den Dries and Wilkie in [44], who saw nonstandard analysis as the natural context for the constructions used by Gromov in his proof (in [22]) that groups of polynomial growth are virtually[1] nilpotent.

The construction is very general, applying not just to groups with word metrics but to any metric space (X, d). Asymptotic cones encode large scale information about X whilst ignoring local geometry. The idea is to view X from increasingly distant vantage points. That is, we scale the metric by a sequence of strictly positive reals $\mathbf{s} = (s_n)$ with $s_n \to \infty$ and we seek a limit of the sequence $(X, \frac{1}{s_n}d)$.

So what limit of $(X, \frac{1}{s_n}d)$ should we take? In restricted circumstances a *Gromov–Hausdorff* limit can be used (see [9, pages 70ff]) – in fact the groups for which the asymptotic cone construction agrees with the taking of the Gromov–Hausdorff limit are precisely the virtually nilpotent groups. But, in general, we need a device from non-standard analysis to force convergence. This is a non-principal ultrafilter ω, which has the crucial property of selecting a (possibly infinite) limit point, the *ultralimit*, $\lim_\omega a_n \in \mathbb{R} \cup \{\pm\infty\}$ of any given sequence of reals (a_n). Non-principal ultrafilters and ultralimits are defined and discussed in Appendix A. One concise way of defining non-principal ultrafilters is to say that they are finitely additive probability measures on $\mathbb{N}$, taking values in $\{0, 1\}$. Non-principal ultrafilters cannot be constructed explicitly; their existence is ensured by Zorn's Lemma.

In addition to a metric space (X, d) the ingredients of the definition of an asymptotic cone are:

- a non-principal ultrafilter ω,
- a ***sequence of basepoints*** $\mathbf{e} = (e_n)_{n\in\mathbb{N}}$ in X,
- a ***sequence of scalars***[2] $\mathbf{s} = (s_n)_{n\in\mathbb{N}}$ of strictly positive reals with $s_n \to \infty$.

Definition 2.1. Define the ***asymptotic cone*** of (X, d) with respect to $\mathbf{e}, \mathbf{s}$ and ω, to be

$$\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}) := \left\{ \mathbf{a} \ = \ (a_n)_{n\in\mathbb{N}} \ \middle| \ \lim_\omega \frac{1}{s_n} d(e_n, a_n) < \infty \right\} \Big/ \sim$$

[1]A group is said *virtually* to admit some property if it has a subgroup of finite index with that property.

[2]Throughout this article we insist that the sequences of scalars $\mathbf{s} = (s_n)$ used in defining an asymptotic cone tends to infinity. It is possible to relax this requirement and still get a well defined $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ but this is not useful for our viewpoint of discarding local information, focusing only on large-scale behaviour.

where the equivalence relation is

$$\mathbf{a} \sim \mathbf{b} \quad \Leftrightarrow \quad \lim_{\omega} \frac{d(a_n, b_n)}{s_n} = 0.$$

The cone is given the metric

$$d([\mathbf{a}], [\mathbf{b}]) \;:=\; \lim_{\omega} \frac{d(a_n, b_n)}{s_n}.$$

Here $[\mathbf{a}]$ denotes the equivalence class of the sequence $\mathbf{a} = (a_n)$, but henceforth we will regularly abuse notation and refer to $\mathbf{a}$ as an element of $\mathrm{Cone}_{\omega}(X, \mathbf{e}, \mathbf{s})$.

The language of nonstandard analysis provides another way of looking at this construction – via the ultraproduct X^* (with respect to ω) of X. The ultraproduct has a natural distance function $d^*(\mathbf{a}, \mathbf{b}) = (d(a_n, b_n))$ taking values in the non-negative hyperreals. The sequence of scalars $\mathbf{s}$ defines an infinite hyperreal, which is used to scale d^* to $\frac{1}{\mathbf{s}} d^*$. Then $\mathrm{Cone}_{\omega}(X, \mathbf{e}, \mathbf{s})$ consists of the set of $\mathbf{a} \in X$ for which $\frac{1}{\mathbf{s}} d^*(\mathbf{a}, \mathbf{e})$ is a finite hyperreal, quotiented by the equivalence relation $\mathbf{a} \sim \mathbf{b}$ when $\frac{1}{\mathbf{s}} d^*(\mathbf{a}, \mathbf{b})$ is infinitesimal.

It is important to note that the definition of the cone involves choices. In the first place there is a dependence on the sequence of basepoints $\mathbf{e}$. In many common contexts this is not critical because when X is homogeneous (for example when X is a finitely generated group with a word metric) or more generally is *quasi-homogeneous*[3] (for example when X is a Cayley Graph – see §2.5), we can appeal to the following well known lemma.

Lemma 2.2. *Let X be a quasi-homogenous metric space, and suppose $\mathbf{e} = (e_n)$ and $\mathbf{e}' = (e'_n)$ are two sequences of base points in X. Let $\mathbf{s} = (s_n)$ be a sequence of scalars with $s_n \to \infty$. Then the asymptotic cones $\mathrm{Cone}_{\omega}(X, \boldsymbol{e}, \boldsymbol{s})$ and $\mathrm{Cone}_{\omega}(X, \boldsymbol{e}', \boldsymbol{s})$ are isometric.*

Proof. The quasi-homogeneity hypothesis allows us to find isometries $\Phi_n : X \to X$ such that

$$d(\Phi_n(e_n), e'_n) \leq \mathrm{diam}(X/\mathrm{Isom}X) < \infty.$$

Define $\mathbf{\Phi} := (\Phi_n)$ to be the induced map $\mathrm{Cone}_{\omega}(X, \mathbf{e}, \mathbf{s})$ to $\mathrm{Cone}_{\omega}(X, \mathbf{e}', \mathbf{s})$.

Then for $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ in $\mathrm{Cone}_{\omega}(X, \mathbf{e}, \mathbf{s})$,

$$d(\mathbf{\Phi}(\mathbf{a}), \mathbf{\Phi}(\mathbf{b})) = \lim_{\omega} \frac{d(\Phi_n(a_n), \Phi_n(b_n))}{s_n} = \lim_{\omega} \frac{d(a_n, b_n)}{s_n} = d(\mathbf{a}, \mathbf{b}).$$

Also

$$d(\mathbf{\Phi}(\mathbf{e}), \mathbf{e}') = \lim_{\omega} \frac{d(\Phi_n(e_n), e'_n)}{s_n} \leq \lim_{\omega} \frac{\mathrm{diam}(X/\mathrm{Isom}X)}{s_n} = 0.$$

It follows that $\mathbf{\Phi}$ is well defined, is an isometry, and maps $\mathbf{e}$ to $\mathbf{e}'$. ∎

More critical to applications is the dependence of the definition of an asymptotic cone on the sequence of scalars and the non-principal ultrafilter. These are interrelated – changes in the sequence of scalars can alternatively be achieved by altering the ultrafilter. In Appendix B we prove a result which makes the relationship more precise. Essentially we show that given an asymptotic cone $\mathrm{Cone}_{\omega}(X, \mathbf{e}, \mathbf{s})$ with $\mathbf{s} = (s_n)$ not tending to infinity too slowly, there is an ultrafilter ω' and a sequence of base points $\mathbf{e}'$ such that $\mathrm{Cone}_{\omega}(X, \mathbf{e}, \mathbf{s})$ is isometric to

[3] A metric space X is *quasi-homogeneous* when $\mathrm{diam}(X/\mathrm{Isom}X) < \infty$.

$\mathrm{Cone}_{\omega'}(X, \mathbf{e}', \mathbb{N})$. (In this context we use $\mathbb{N}$ to denote the sequence (n) rather than the set of natural numbers.)

Often authors wish to use $\mathbb{N}$ for the sequence of scalars, and take an obvious sequence of base points – typically the constant sequence $\mathbf{1} = (1)$ at the identity in a finitely generated group Γ with a word metric. In this circumstance the cone may be more concisely denoted $\mathrm{Cone}_\omega \Gamma$.

In our applications we will find it most natural to fix the ultrafilter and then state results whose hypothesis is that some conditions hold in all $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ as $\mathbf{e}$ and $\mathbf{s}$ vary[4]. In this way we capture characteristics of the large scale geometry of X, but avoid the loss of information that can occur when we just focus on one $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. The following, which is Proposition 3.1.1 of Druţu [10], is an example.

Proposition 2.3. *Let Γ be a finitely generated group and ω a non-principal ultrafilter. Then Γ is hyperbolic if and only if $\mathrm{Cone}_\omega(\Gamma, \mathbf{1}, \boldsymbol{s})$ is an $\mathbb{R}$-tree for all $\boldsymbol{s}$.*

The insistence that a condition in the cone holds *for all* $\mathbf{s}$ is often necessary, as the following theorem testifies.

Theorem 2.4 (Thomas & Velickovic [43]). *There exists a finitely generated group which has two non-homeomorphic asymptotic cones.*

The examples given by Thomas & Velickovic are not finitely presentable. (The question of whether a *finitely presentable* group can have two non-homeomorphic cones remains open.) Their examples Γ are defined using an infinite sequence of relators (r_i) satisfying a small cancellation property. Such groups Γ are not finitely presentable and so must have at least one cone that is not 1-connected (see Theorem C). Thomas & Velickovic show that one can choose the r_i and an ultrafilter ω in such a way that in any neighbourhood of the base point, $\mathrm{Cone}_\omega \Gamma$ resembles the cone of a finitely presented small cancellation group Γ_R. Such a group Γ_R is hyperbolic and $\mathrm{Cone}_\omega \Gamma_R$ is an $\mathbb{R}$-tree. It follows that $\mathrm{Cone}_\omega \Gamma$ is itself an $\mathbb{R}$-tree (and so is 1-connected). Thomas & Velickovic achieve these different cones by using two different ultrafilters, whilst fixing the sequence of scalars as $\mathbb{N}$. However, their methods can easily be adapted so that the difference is realised by two different sequences of scalars but a fixed ultrafilter.

If X is a quasi-homogenous metric space then its asymptotic cones are homogeneous. When Γ is a non-elementary (that is, not virtually cyclic) hyperbolic group, $\mathrm{Cone}_\omega(\Gamma, \mathbf{1}, \mathbf{s})$ is an everywhere branching $\mathbb{R}$-tree. In fact this $\mathbb{R}$-tree turns out to be the (uniquely) *everywhere $2^{\aleph_0}$-branching universal $\mathbb{R}$-tree* (see [13]).

An important property of asymptotic cones is that they are *complete* metric spaces. Proofs can be found in [9, page 79] and [44].

2.2 Quasi-isometries

We now recall an important notion of large scale equivalence of metric spaces that is designed to respect only global properties of the space, ignoring local geometry.

Let X and Y be metric spaces. Let $\lambda \geq 1$ and $\mu \geq 0$. A (not-necessary continuous) map $\Phi : X \to Y$ is a (λ, μ)-***quasi-isometry*** if

[4] Fixing $\mathbf{s}$ and varying $\mathbf{e}$ and ω would work similarly. In either case the point is not to lose information on some subsequence of $(X, \frac{1}{s_n} d)$.

1. $\forall x_1, x_2 \in X, \;\; \frac{1}{\lambda} d(x_1, x_2) - \mu \;\leq\; d(\Phi(x_1), \Phi(x_2)) \;\leq\; \lambda d(x_1, x_2) + \mu,$ and

2. $\forall\, y \in Y, \;\; \exists\, x \in X, \;\; d(\Phi(x), y) \;\leq\; \mu.$

When such Φ, λ, μ exist we say X and Y are ***quasi-isometric***. This defines an equivalence relation on any given set of metric spaces.

The first condition says Φ combines the stretching λ of a bi-Lipschitz map with an amount of tearing bounded by μ. The second condition tells us that $\mathrm{Im}\Phi$ is ***quasi-dense*** in Y. If we discard the second condition then Φ is called a ***quasi-isometric embedding***.

The following proposition is well known.

Proposition 2.5. *A (λ, μ)-quasi-isometry $\Phi : X \to Y$ induces a λ-bi-Lipschitz homeomorphism*

$$\boldsymbol{\Phi} : \; \mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s}) \;\to\; \mathrm{Cone}_\omega(Y, \boldsymbol{\Phi}(\boldsymbol{e}), \boldsymbol{s}).$$

Proof. Suppose that $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ are elements of $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. Then

$$d\left(\boldsymbol{\Phi}(\mathbf{a}), \boldsymbol{\Phi}(\mathbf{b})\right) \;=\; \lim_\omega \frac{d(\Phi(a_n), \Phi(b_n))}{s_n} \leq \lim_\omega \frac{\lambda\, d(a_n, b_n) + \mu}{s_n} \;=\; \lambda\, d(\mathbf{a}, \mathbf{b}).$$

The other half of the bi-Lipschitz condition is proved similarly. It remains to show that $\boldsymbol{\Phi}$ is surjective. Well suppose $\mathbf{y} = (y_n)$ is a point in $\mathrm{Cone}_\omega(Y, \boldsymbol{\Phi}(\mathbf{e}), \mathbf{s})$. Then there is a sequence $\mathbf{x} = (x_n)$ in X such that $d(\Phi(x_n), y_n) \leq \mu$. One readily checks that $d(\boldsymbol{\Phi}(\mathbf{x}), \mathbf{y}) = 0$ and that

$$d(\mathbf{e}, \mathbf{x}) \;\leq\; \lambda\, d\left(\boldsymbol{\Phi}(\mathbf{e}), \boldsymbol{\Phi}(\mathbf{x})\right) \;=\; \lambda\, d\left(\boldsymbol{\Phi}(\mathbf{e}), \boldsymbol{\Phi}(\mathbf{y})\right) \;<\; \infty.$$

So $\mathbf{x}$ is a well defined point in $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ and is mapped to $\mathbf{y}$ by $\boldsymbol{\Phi}$. ∎

Here is the reward of this proposition.

Corollary 2.6. *Any bi-Lipschitz invariant of the asymptotic cones of a metric space is a quasi-isometry invariant of metric spaces.*

The identity map $(\Gamma, d_{\mathcal{A}}) \to (\Gamma, d_{\mathcal{B}})$ associated to two finite generating sets (and hence two word metrics) of a finitely generated group $\Gamma = \langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle$ is a $(\lambda, 0)$-quasi-isometry for some $\lambda \geq 1$. Hence we have the following further corollary.

Corollary 2.7. *Any topological (bi-Lipschitz) invariant (*e.g.* N-connectedness) of the cones of a finitely generated group Γ is a group invariant, in other words, is independent of the particular choice of generating set.*

Let Γ be a group with finite generating set $\mathcal{A}$ with respect to which it has word metric denoted $d_{\mathcal{A}}$. The Cayley graph $C(\Gamma, \mathcal{A})$ of Γ, whose definition we will recall in §2.5 is given the metric in which each edge has length 1. Then $(\Gamma, d_{\mathcal{A}})$ can be identified with the 0-skeleton of $C(\Gamma, \mathcal{A})$, and the inclusion is a $(1, \frac{1}{2})$-quasi-isometry. So here is a further corollary.

Corollary 2.8. *The asymptotic cones of a group Γ with word metric associated to some finite generating set $\mathcal{A}$ are the same as those of $C(\Gamma, \mathcal{A})$.*

2.3 Combinatorial complexes

In order to study homotopy groups of asymptotic cones we would like to relate maps of spheres into cones to sequences of maps of spheres into the original space X. For example, a sequence of λ_n-Lipschitz N-spheres $\gamma_n : (\mathbb{S}^N, \star) \to (X, e_n)$ with $\lambda_n \to \infty$ yields a 1-Lipschitz map

$$\gamma \ := \ (\gamma_n) : \ \mathbb{S}^N \to \mathrm{Cone}_\omega(X, \mathbf{e}, \boldsymbol{\lambda}) \,,$$

where $\boldsymbol{\lambda} := (\lambda_n)$. This is because for $a, b \in \mathbb{S}^N$

$$d(\gamma(a), \gamma(b)) \ = \ \lim_\omega \frac{d(\gamma_n(a), \gamma_n(b))}{\lambda_n} \ \leq \ \lim_\omega \frac{\lambda_n d(a,b)}{\lambda_n} \ = \ d(a,b).$$

However we lack control when we *pull back* a continuous map $\gamma : \mathbb{S}^N \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ to a sequence (γ_n) of maps $\gamma_n : \mathbb{S}^N \to X$ such that $\gamma(a) = (\gamma_n(a))$ for all $a \in \mathbb{S}^N$. We can only at best hope for coarse information, and given a set $J \subseteq \mathbb{N}$ of ω-measure 0, we can deduce no constraints on γ_n for $n \in J$. What is particularly troublesome is that we cannot get information uniformly constraining the behaviour of any γ_n over the whole of $\mathbb{S}^N$. It is much easier to *pull back* finite sets of points: let $\mathcal{C}$ be a finite set of points and let τ be a map $\mathcal{C} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$; express τ as (τ_n) for some maps $\tau_n : \mathcal{C} \to X$ such that each τ_n is a map $\mathcal{C} \to X$; then for a given error term $\varepsilon > 0$ we can find a set $J \subseteq \mathbb{N}$ of ω-measure 1 such that for all $n \in J$, the distances between pairs of points of $\tau_n(\mathcal{C})$ in $(X, \frac{1}{s_n} d)$ differ by at most ε from the distances between respective pairs of points of $\tau(\mathcal{C})$.

This discussion leads to work with *combinatorial configurations of points* – specifically the 0-skeleta of *combinatorial structures* for N-spheres and $(N+1)$-discs. Thus we give the following review of definitions.

Combinatorial complexes are defined by recursion on dimension. We follow the definition of Bridson & Haefliger [9, page 153]. Define a 0-dimensional combinatorial complex just to be a set with the discrete topology, each point being termed both an ***open cell*** and a ***closed cell***.

Next we define a continuous map $C_1 \to C_2$ between combinatorial complexes to be ***combinatorial*** if its restriction to each open cell of C_1 is a homeomorphism onto an open cell of C_2.

To complete the definition we explain how we use combinatorial maps to provide the attaching maps necessary to obtain N-dimensional combinatorial complexes from those of dimension $N-1$. An N-dimensional combinatorial complex is a topological space C that can be obtained in the following way. Take the disjoint union U of an $(N-1)$-dimensional combinatorial complex $C^{(N-1)}$ and a family $(e_\lambda)_{\lambda \in \Lambda}$ of closed N-discs. Suppose the boundaries ∂e_λ of the e_λ have combinatorial structures: that is, for each e_λ there is an $(N-1)$-dimensional combinatorial complex S_λ for which there is a homeomorphism $\partial e_\lambda \to S_\lambda$. Further suppose there are combinatorial maps $S_\lambda \to C^{(N-1)}$. The attaching maps are the compositions $\partial e_\lambda \to S_\lambda \to C^{(N-1)}$. Then C is obtained from U by quotienting via the attaching maps in the usual way (and is given the quotient topology). The ***open cells*** of C are defined to be the (images of) open cells in $C^{(N-1)}$ and the interiors of the e_λ. The ***closed cells*** of C are defined to be the closed cells of $C^{(N-1)}$ together with the N-discs e_λ, equipped with their boundary combinatorial structures $\partial e_\lambda \to S_\lambda$. So a closed N-cell is a combinatorial complex in its own right, having one open N-cell e_λ together with the combinatorial structure S_λ for an $(N-1)$-sphere on its boundary. (However a closed N-cell of C need not embed as a subcomplex in C on account of the identifications that may occur under the attaching map $S_\lambda \to C^{(N-1)}$.)

It is often only the *combinatorial type* of a complex that we are interested in. Therefore define two combinatorial complexes to be ***combinatorially equivalent*** (or ***of the same combinatorial type***) when there exists a ***combinatorial isomorphism*** between them – that is, a homeomorphism which is combinatorial and has combinatorial inverse.

A ***combinatorial structure for a topological space*** V is a combinatorial complex C together with a homeomorphism $V \xrightarrow{\cong} C$. It is common to suppress the homeomorphism and regard the cells of C as subsets of V. Two combinatorial structures $\phi_1 : V \xrightarrow{\cong} C_1$ and $\phi_2 : V \xrightarrow{\cong} C_2$ are said to be equivalent when $\phi_2 \circ \phi_1^{-1} : C_1 \to C_2$ is a combinatorial isomorphism.

We use the notation $\#_N(C)$ to denote the number of open N-cells in a combinatorial complex.

A combinatorial complex is ***triangular***[5] when the combinatorial structure on each attaching sphere is always that of the boundary of a simplex (of the appropriate dimension). A ***triangulation*** of a topological space is a combinatorial structure for the space in which the combinatorial complex used is triangular.

More generally, given a sequence $\mathbf{R} = (R_N)$ such that each $R_N \in \mathbb{N} \cup \{\infty\}$ and $R_N \geq N+2$, define an $\mathbf{R}$***-combinatorial complex*** to be a combinatorial complex in which, for all N, all the combinatorial structures $\mathbb{S}^N \cong S_\lambda$ for N-spheres used to attach $(N+1)$-cells e_λ (via combinatorial maps $\partial e_\lambda \to S_\lambda$) have $\#_N(S_\lambda) \leq R_N$. (Note that this implicitly forces the combinatorial structures S_λ to be $\mathbf{R}$-combinatorial also.) The reason we insist that $R_N \geq N+2$ for each N is to ensure that the $\mathbf{R}$-combinatorial complexes include the triangular combinatorial complexes.

We will find $\mathbf{R}$-combinatorial complexes particularly useful (in the proofs in §2.3) because the combinatorial type of an $\mathbf{R}$-combinatorial complex is restricted: if the entries $R_0, R_1, \dots, R_{N-1}$ are all finite then in an N-dimensional $\mathbf{R}$-combinatorial complex there are only finitely many possible combinatorial structures for the $(N-1)$-spheres used to attach N-cells (up to combinatorial equivalence); so given an integer $M > 0$ there are only finitely many N-dimensional non-equivalent combinatorial complexes C such that $\#_N(C) \leq M$.

A ***refinement*** of an N-dimensional combinatorial structure $V \xrightarrow{\cong} C$ on a topological space V is, roughly speaking, another combinatorial structure $V \xrightarrow{\cong} \bar{C}$ for V which can be obtained from C by subdividing the cells in C in a way that is matches up across shared i-cells in the boundaries of two $(i+1)$-cells. We produce $\bar{C}$ by *refining* first the 1-cells in C then the 2-cells and so on, until finally the N-cells.

For example consider the combinatorial structure $C \cong \mathbb{D}^2$ where C is made up four 2-cells, twelve 1-cells and nine 1-cells assembled to make a 2-by-2 chessboard complex. If we subdivided each of the four 2-cells into 2-by-2 chessboard complexes (say) we would have a 4-by-4 chessboard and this would be a *refinement* of C. However if we subdivided some 2-cell in C into a 2-by-2 chessboard complex and an adjacent 2-cell into a 3-by-3 chessboard complexes then the result would fail to be a refinement of C because the subdivision would not agree across the common edge.

Formally, refining a combinatorial structure C for a space V is an inductive process. First we define $\bar{C}^0 := C^{(0)}$. Then for $k = 1, 2, \dots, N$ we shall explain how to *refine the k-skeleton $C^{(k)}$ of C subject to $\bar{C}^{k-1}$* to produce $\bar{C}^k$. Then a refinement $\bar{C}$ of C is defined to be any

[5]This definition is not the same as that of a *simplicial triangulation* because two N-cells can meet across multiple $(N-1)$-cells.

$\bar{C}^N$ that can be obtained from a sequence $\bar{C}^0, \bar{C}^1, \ldots, \bar{C}^N$ in which each $\bar{C}^k$ is the result of refining $C^{(k)}$ subject to $\bar{C}^{k-1}$.

Recall from the definition of a combinatorial complex C that each closed k-cell e^k of C has a combinatorial structure $\partial e^k \xrightarrow{\cong} S$ on its boundary and a combinatorial map $S \to C^{(k-1)}$ such that composing gives the attaching map $f_{e^k} : \partial e^k \to C^{(k-1)}$. The refinement $\bar{C}^{k-1}$ induces a *refined attaching map* $\bar{f}_{e^k} : \partial e^k \xrightarrow{\cong} \bar{S} \to \bar{C}^{k-1}$. Now suppose we have any combinatorial structure $e^k \xrightarrow{\cong} D$ on e^k (so D is a combinatorial complex which is topologically a k-disc) such that $\partial D = \bar{S}$ as combinatorial complexes and $e^k \to D$ restricts to $\partial e^k \to \bar{S}$ on the boundary. Then D can be attached to $\bar{C}^{k-1}$ via $\bar{f}_{e^k}$. So our k-complex $\bar{C}^k$ is obtained by attaching any such combinatorial k-discs to $\bar{C}^{k-1}$ in place of the k-cells of C. Any $\bar{C}^k$ that can be obtained in this way is referred to as a refinement of the k-skeleton $C^{(k)}$ of C subject to $\bar{C}^{k-1}$.

We will also need ***singular combinatorial maps*** and ***singular combinatorial complexes***. Their definition, which is due to Bridson in [8], is similar to that of combinatorial complexes – using recursion on dimension. A continuous map $C_1 \to C_2$ between singular combinatorial complexes is a singular combinatorial map when, for all N, each open N-cell of C_1 is either mapped homeomorphically onto an N-cell of C_2, or ***collapses***. In saying an N-cell "collapses" we mean that it maps into the image of its boundary (and hence into the $(N-1)$-skeleton of C_2). Then singular combinatorial complexes are built up through the dimensions similarly to combinatorial complexes: the boundary N-spheres of $(N+1)$-discs are given (non-singular) combinatorial structures and are glued to an N-dimensional complex via singular combinatorial attaching maps.

2.4 Geodesic metric spaces

We say that a metric space X is a ***geodesic metric space*** when, given $a, b \in X$, there is an isometrically embedded continuous path $\gamma : [0, d(a,b)] \to X$ with $\gamma(0) = a$ and $\gamma(d(a,b)) = b$. Such an isometrically embedded continuous path is referred to as a ***geodesic*** from a to b.

Any Cayley graph (defined in §2.5) is an example of a geodesic metric space. More generally the same can be said of the 1-skeleton of a combinatorial complex that has been equipped with the *combinatorial metric* (that is, each 1-cell has uniformly been given length 1).

2.5 The geometry of the word problem

The *word problem* for a finite presentation $\mathcal{P}$ (defined below) of a group Γ asks for an algorithm which, on input of a word w in the generators, decides whether or not $w = 1$ in Γ. (It is straight-forward to show that the existence of the algorithm does not depend on the finite presentation.) In the 1950s celebrated examples of groups with undecidable word problem were constructed by Novikov [33] and Boone [3]. However this is by no means the end of the story as far as Geometric Group Theory in concerned.

It turns out that complexity measures associated to naive approaches to solving the word problem provide such invariants. The most well known is called the *minimal isoperimetric function* (*a.k.a.* the *Dehn function*). Moreover, through insights of Gromov in [24] and [25], these invariants can really be seen to capture information about the geometry of Γ.

Here are some of the standard basic definitions.

Let $\mathcal{A}$ be an ***alphabet***, that is, a set of symbols (***letters***). A ***word*** w ***in*** $\mathcal{A}$ is a finite string of letters from $\mathcal{A}$ and their formal inverses – that is, w is an element of the free monoid $(\mathcal{A} \cup \mathcal{A}^{-1})^\star$. Denote the length of w by $\ell(w)$. For a word $w = {a_1}^{\varepsilon_1}{a_2}^{\varepsilon_2} \dots {a_s}^{\varepsilon_s}$, where each $a_i \in \mathcal{A}$ and each $\varepsilon_i = \pm 1$, the ***inverse word*** w^{-1} is ${a_s}^{-\varepsilon_s} \dots {a_2}^{-\varepsilon_2}{a_1}^{-\varepsilon_1}$.

We say that a group Γ is ***generated by a subset*** $\mathcal{A}$ when the natural map $(\mathcal{A} \cup \mathcal{A}^{-1})^\star \to \Gamma$ is surjective. (So Γ is said to be ***finitely generatable*** when it admits some finite generating set.) We say that a word w in $\mathcal{A}$ is ***null-homotopic*** when $w = 1$ in Γ.

A ***presentation*** $\mathcal{P}$ consists of a set $\mathcal{A}$ (the ***alphabet***) and a set of words $\mathcal{R}$ (***relators***) and is denoted by writing $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$. The group presented by $\mathcal{P}$ is $F(\mathcal{A})/\langle\langle \mathcal{R} \rangle\rangle$, the quotient of the free group on $\mathcal{A}$ by the normal closure $\langle\langle \mathcal{R} \rangle\rangle$ in $F(\mathcal{A})$ of the elements represented by words in $\mathcal{R}$. A presentation is ***finite*** when both $\mathcal{A}$ and $\mathcal{R}$ are finite sets. We say a group is ***finitely presentable*** when it is isomorphic to the group presented by some finite $\langle \mathcal{A} \mid \mathcal{R} \rangle$.

The ***Cayley graph*** $C(\Gamma, \mathcal{A})$ associated to a group Γ finitely generated by a set $\mathcal{A}$ is the graph defined as follows. The vertex set of $C(\Gamma, \mathcal{A})$ is Γ, and, for each $a \in \mathcal{A}$ and $u \in \Gamma$, there is an oriented edge labelled by a from u to ua. The Cayley graph in equipped with the combinatorial metric d, in which each edge is uniformly given length 1. The restriction of this metric to the 0-skeleton agrees with the ***word metric*** on Γ, which is the left-invariant metric d such that $d(1, u)$ is the minimal length of words that evaluate to u in Γ.

The ***Cayley 2-complex*** $C(\mathcal{P})$ associated to a finite presentation $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ of a group Γ is the universal cover $\widetilde{\mathcal{K}^2}$ of the finite 2-complex $\mathcal{K}^2$ constructed as follows. Start with a *rose*: this is a 1-complex with one vertex $\star$ and one edge-loop for each element a of $\mathcal{A}$, oriented and labelled by a. Then for each relator $r \in \mathcal{R}$ attach a $\ell(r)$-sided 2-cell to the rose using r to describe the attaching map. The fundamental group of this finite 2-complex is Γ (by the Seifert-van Kampen theorem – see [41] for example). The Cayley graph $C(\Gamma, \mathcal{A})$ is the 1-skeleton of $C(\mathcal{P})$.

2.5.1 Van Kampen diagrams

The terminology used above of a *"null-homotopic"* word w in a finitely presented group is, with good reason, borrowed from algebraic topology. If one starts at some vertex v in the Cayley 2-complex $C(\mathcal{P})$ (the homogeneity of $C(\mathcal{P})$ renders the particular choice of v unimportant) and follows successive edges in such a way as to reads a null-homotopic word w, then one will finish at v. In this way null-homotopic words define *edge-circuits* in $C(\mathcal{P})$. A *van Kampen diagram* D_w for a null-homotopic word w can be considered to be a combinatorial homotopy disc for an edge-circuit associated to w in the Cayley 2-complex for $\mathcal{P}$.

More formally, a ***van Kampen diagram*** D_w for w is a finite, planar, contractible, combinatorial 2-complex; its 1-cells are directed and labelled by generators, the boundary labels of each of its 2-cells are cyclic conjugates of relators or inverse relators, and one reads w (by convention anticlockwise) around the boundary circuit from a base vertex $\star$.

Here is an equivalent definition of a van Kampen diagram that is intuitively closer to the notion of a homotopy disc for a loop in $C(\mathcal{P})$ defined by a null-homotopic word w. Say that $D_w := S^2 \smallsetminus e_\infty$ is a van Kampen diagram for w whenever S^2 is a combinatorial cell structure on the 2-sphere with a distinguished 2-cell e_∞ and a combinatorial map f from $S^2 \smallsetminus e_\infty$ to $C(\mathcal{P})$ such that the attaching map of e_∞ is then mapped by f to w. (The orientation and labelling of the edges of D_w is then inherited from $C(\mathcal{P})$ via f.)

Note that a van Kampen diagram can, in general, be a *singular* disc – it is convenient to think of a van Kampen diagram as a planar tree-like arrangement of topological discs and topological arcs as displayed in Figure 1.

It is an immediate corollary of the forthcoming Lemma 2.5.4 (*"van Kampen's Lemma"*) that a word w in $\mathcal{P}$ is null-homotopic if and only if it admits a van Kampen diagram.

2.5.2 $\simeq$-equivalence of functions

We recall a well known equivalence relation on functions $[0,\infty) \to [0,\infty)$. Given two functions $f_1, f_2 : [0,\infty) \to [0,\infty)$ we say $f_1 \preceq f_2$ when there exists $M > 0$ such that $f_1(\ell) \leq M f_2(M\ell + M) + M\ell + M$, for all $\ell \geq 0$. Then $f_1 \simeq f_2$ if and only if $f_1 \preceq f_2$ and $f_2 \preceq f_1$.

Similarly we can define $f_1 \preceq f_2$ and $f_1 \simeq f_2$ for functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$.

2.5.3 Filling functions for finite presentations of groups

We give the 1-skeleton of a van Kampen diagram the combinatorial metric: each 1-cell has length 1. An insight of Gromov [25] is to pursue parallels with filling null-homotopic loops in Riemannian manifolds and define group invariants (***"filling functions"***) that concern different measurements one can make of the geometry of van Kampen diagrams. In particular we will be concerned with:

- the ***area***, $\mathrm{Area}(D_w)$, which is the number of 2-cells,
- the ***diameter***[6], $\mathrm{Diam}(D_w)$, which is the maximal distance (in the combinatorial metric on the 1-skeleton) of vertices in D_w from the basepoint $\star$,
- and the ***filling length***, $\mathrm{FL}(D_w)$, which is the minimal bound on the length of the *contracting boundary curve* amongst *shellings* of D_w.

The third of these, the *filling length*, requires further explanation. A ***shelling*** of D_w is the combinatorial analogue of a null-homotopy: the boundary circuit of a singular combinatorial 2-disc D is *homotoped* to the basepoint $\star$. More precisely, we have a sequence of van Kampen diagrams:

$$D_w \;=\; D_0, D_1, \ldots, D_m \;=\; \star$$

in which D_{i+1} is obtained from D_i by one of the following three types of moves:

- A ***1-cell collapse.*** Remove a pair (e^1, e^0) such that $e^0 \in \partial e^1$ is a 0-cell in D_i which is not the base point $\star \in D_i$, and e^1 is a 1-cell only attached to the rest of the diagram at one 0-cell which is not e^0.
- A ***1-cell expansion.*** Suppose (e^1, e^0) is a pair such that e^1 is a 1-cell in the interior of D_i and $e^0 \in \partial e^1 \cap \partial D_i$. Make a cut along e^1 starting from e^0, so two copies of e^0 and e^1 are found in D_{i+1}. This has the effect of introducing two new 1-cells into the boundary of the diagram.
- A ***2-cell collapse.*** Remove a pair (e^2, e^1) where e^2 is a 2-cell of D_i with e^1 a 1-cell of $\partial e^2 \cap \partial D_i$ (note that the 0-skeleton of D_i is the same as that of D_{i+1}).

[6] It is conventional in this context to define diameter to be the maximum distance to the basepoint, rather than the maximum distance between two vertices. Obvious inequalities relate the two alternatives.

The filling length of the shelling $D_0, D_1, \ldots, D_m$ of D_w is defined to be

$$\max \{\ell(\partial D_i) \mid 0 \leq i \leq m\}.$$

The filling length $\mathrm{FL}(w)$ of D_w is defined to be the minimal filling length amongst all shellings of D_w.

The ***area (resp. diameter, filling length) of a null-homotopic word*** w is the minimal area (resp. diameter, filling length) of all van Kampen diagrams filling w.

For $\mathrm{M} =$ Area, Diam and FL we define $\mathrm{M}(n)$ to be the maximum of $\mathrm{M}(w)$ amongst all null-homotopic words w of length at most n. Thus we have defined three functions:

- the ***Dehn function*** $\mathrm{Area} : \mathbb{N} \to \mathbb{N}$,
- the ***minimal isodiametric function*** $\mathrm{Diam} : \mathbb{N} \to \mathbb{N}$,
- and the ***filling length function*** $\mathrm{FL} : \mathbb{N} \to \mathbb{N}$.

Any function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \geq \mathrm{Area}(n)$ for all n is referred to as an ***isoperimetric function*** for $\mathcal{P}$. Thus the Dehn function can alternatively be referred to as the ***minimal isoperimetric function***. Similarly, any function that is an upper bound for Diam is referred to as an ***isodiametric function***.

We say that functions $f(n)$, $g(n)$ and $h(n)$ that are upper bounds for $\mathrm{Area}(n)$, $\mathrm{Diam}(n)$ and $\mathrm{FL}(n)$ are ***realisable simultaneously*** when, for any given null-homotopic word w of length n there is a van Kampen diagram D_w with

$$\begin{aligned} \mathrm{Area}(D_w) &\leq f(n) \\ \mathrm{Diam}(D_w) &\leq g(n) \\ \mathrm{FL}(D_w) &\leq h(n). \end{aligned}$$

It is important to note that Area, Diam and FL are defined with respect to a fixed finite presentation for Γ. However they are group invariants in the sense that if $\mathcal{P}$ and $\mathcal{Q}$ are two finite presentations for Γ then the respective functions are $\simeq$-equivalent (see Definition 2.5.2). This is proved in [20] for Area and Diam and in [18] for FL.

There is an extensive literature concerning isoperimetric functions and isodiametric functions – see, for example, [4], [7], [16], [35] and [42] and references therein. Filling length is less well known; some of its properties are discussed in [18]. There is increasing evidence of its importance – in particular it plays a pivotal role in a recent proof of a long-standing open question about isoperimetric functions for nilpotent groups (see [17] and §5.7 of this article).

Another approach to the definition of Area and FL is via *null-sequences*. Suppose, as before, that w is a null-homotopic word in the presentation $\mathcal{P}$. Then a ***null-sequence*** for w is a sequence

$$w \ = \ w_0, w_1, \ldots, w_m \ = \ 1$$

of words (where 1 denotes the empty word) in which w_{i+1} is obtained from w_i by applying a relator, inserting an inverse pair, or removing an inverse pair. The filling length of a null-sequence is the length of the longest of the words w_i. An equivalent definition of the filling length $\mathrm{FL}(w)$ of w is as the minimal filling length amongst all null-sequences for w. Similarly the Area of a null-sequence is the number of *application of a relator* moves it involves, and

Area(w) is the minimal area amongst all null-sequences for w. For a carefully treatment of this please refer to [18, Proposition 1].

One sees that Area and FL are non-deterministic space and time complexity measure, respectively, for the word problem working within the limited context of null-sequences (sometimes called the *Dehn proof system*) – more details can be found in [17], [18] and [40].

2.5.4 Van Kampen's Lemma

An alternative definition of the area Area(w) a null-homotopic word w in a finite presentation $\mathcal{P} = \langle \mathcal{A} \mid \mathcal{R} \rangle$ is as the minimal N such that there is an equality

$$w = \prod_{i=1}^{N} u_i^{-1} r_i u_i$$

in the free group $F(\mathcal{A})$ for some $r_i \in \mathcal{R}^{\pm 1}$ and words u_i. That this formulation agrees with the definition given in §2.5.3 follows from van Kampen's Lemma with $\mathcal{R}_w = \mathcal{R}$ below. (In fact, van Kampen's Lemma is often given with $\mathcal{R}_w = \mathcal{R}$ for all w, however we will need a slightly more general version.) First we need the following definition of a *diagram* (compare Definition 2.5.1).

Definition 2.9. A ***diagram*** is a finite, planar, contractible, combinatorial 2-complex. One can think of a *diagram* as a planar tree-like arrangement of topological 2-discs and 1-dimensional arcs.

Lemma 2.10 (van Kampen's Lemma). *Let w be a null-homotopic word in a generating set $\mathcal{A}$ of a group Γ with Cayley graph $C(\Gamma, \mathcal{A})$. Let $\mathcal{R}_w$ be a set of null-homotopic words and let $K \in \mathbb{N}$. The following are equivalent:*

(i). There is an equality

$$w = \prod_{i=1}^{K'} u_i w_i u_i^{-1} \tag{1}$$

in the free group $F(\mathcal{A})$, for some $K' \leq K$, and some words u_i and w_i such that the w_i are in ${\mathcal{R}_w}^{\pm 1}$.

(ii). There exists a diagram D_w (depicted in Figure 1) together with a combinatorial map $\Phi : D_w^{(1)} \to C(\Gamma, \mathcal{A})$ satisfying:

- *the number of 2-cells of D_w is at most K;*
- *Φ maps ∂D_w (reading from a basepoint $\star$) to the edge circuit defined by w;*
- *the boundary circuit of each 2-cell of D_w is mapped by Φ to an edge circuit around which one reads a word from $\mathcal{R}_w$ (reading in one direction or the other from some starting vertex).*

Proofs of van Kampen's Lemma can be found in Bridson [7], Lyndon and Schupp [31], or Ol'shanskii [34].

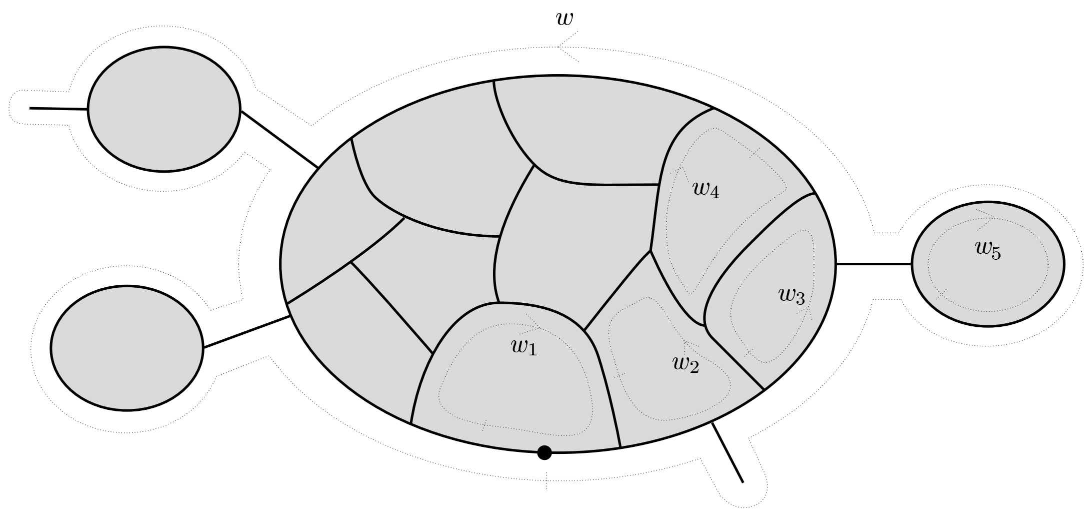

Figure 1: The diagram D_w of van Kampen's Lemma.

3 Filling functions in a coarse geometric setting

Let X be a metric space.

In the forthcoming definitions of filling functions for X, the appropriate notion of a coarse N-sphere in a space X will be a map $\gamma : C^{(0)} \to X$ of the 0-skeleton of certain combinatorial structures C for $\mathbb{S}^N$ into X. Roughly speaking, a filling (a "*partition*") will be an extension $\bar{\gamma} : \bar{C}^{(0)} \to X$ of γ, where $\bar{C}$ is a combinatorial structure for the $(N+1)$-disc and $\partial\bar{C}$ is a refinement of C. What will make such a filling effective is that we restrict the combinatorial type of $\bar{C}$ and the *mesh*, which we now define, decreases (approximately halves, in fact).

Definition 3.1. Suppose X is a metric space, C is a (possibly singular) combinatorial complex and $\gamma : C^{(0)} \to X$ is a map from the 0-skeleton of C to X. Then we define the ***mesh*** of the pair (C, γ) by

$$\mathrm{mesh}(C,\gamma) \;:=\; \max\left\{d(\gamma(a),\gamma(b)) \mid a \text{ and } b \text{ are the end vertices of a 1-cell in } C\right\}.$$

3.1 The definition of the 1-dimensional filling function $\mathrm{Fill}^1_{\mu_1}$

Fix any $\mu_1 \geq 0$. We shall define the 1-dimensional function

$$\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}} \;=\; \mathrm{Fill}^1_{\mu_1} : \; [0,\infty) \to \mathbb{N} \cup \{\infty\}$$

for our metric space X.

In general the k-filling function $\mathrm{Fill}^k_{\mathbf{R},\boldsymbol{\mu}}$ will be defined with reference to the entries $R_1, R_2, \ldots, R_{k-1}$ of the sequence $\mathbf{R} = (R_i)$ and the entries $\mu_1, \mu_2, \ldots, \mu_k$ of the sequence $\boldsymbol{\mu} = (\mu_i)$. But this means that $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}$ only depends on μ_1 and hence we adopt the more concise notation $\mathrm{Fill}^1_{\mu_1}$.

Definition 3.2. We define $\mathrm{Fill}^1_{\mu_1}(\ell)$ to be the least integer K such that given any $a, b \in X$ with $d(a,b) \leq \ell$, there are $a^0, a^1, \ldots, a^K$ with $a^0 = a$ and $a^K = b$, such that for $i = 0, 1, \ldots, K-1$

$$d(a^i, a^{i+1}) \;\leq\; \frac{\ell}{2} + \mu_1.$$

But if no such least K exists or if for some $a, b \in X$ no such sequence exists then $\mathrm{Fill}^1_{\mu_1}(\ell) := \infty$.

Examples 3.3.

1. A metric space X is bounded if and only if there exists $\mu_1 \geq 0$ such that $\mathrm{Fill}^1_{\mu_1} \equiv 1$. If X is bounded then, in fact, its asymptotic cones each consist of just one point.

2. In any finitely generated group Γ with a word metric, each pair of points a^0, a^2 has a mid-point a^1 modulo a possible error term of $\frac{1}{2}$: that is,

$$\max\left\{d(a^0, a^1), d(a^1, a^2)\right\} \ \leq \ \frac{1}{2} d(a^0, a^2) + \frac{1}{2}.$$

So Γ satisfies $\mathrm{Fill}^1_{\frac{1}{2}}(\ell) = 2$ for all $\ell > 0$. In the Cayley Graph of Γ, as indeed in any geodesic metric space, we have $\mathrm{Fill}^1_0(\ell) = 2$ for all $\ell \in (0, \infty)$.

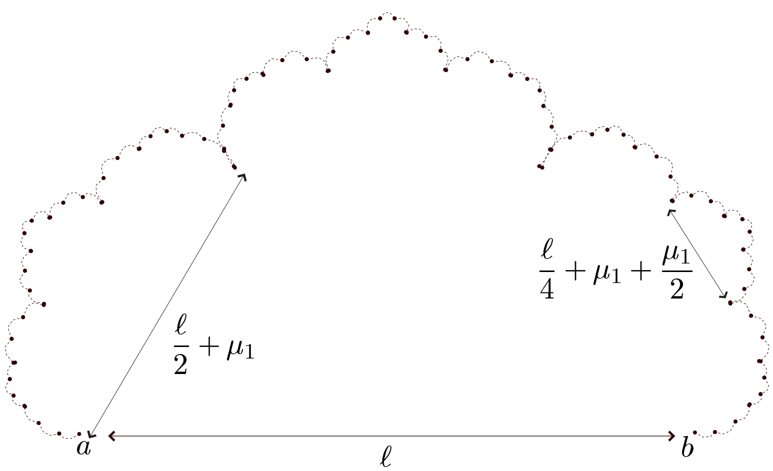

Figure 2: Coarse Koch Snowflake Curve.

3. Suppose there are $\mu_1 \geq 0$ and $K_1 \in \mathbb{N}$ such that for a metric space X we have $\mathrm{Fill}^1_{\mu_1}(\ell) \leq K_1$ for all $\ell \geq 0$. If we iteratively partition as constrained by $\mathrm{Fill}^1_{\mu_1}$ then we obtain a *coarse path* between any two points $a, b \in X$. Let $\ell := d(a, b)$. At worst this coarse path resembles a coarse Koch snowflake curve (see Figure 2): after r iterations we have a chain of at most ${K_1}^r + 1$ points starting at a and ending at b with the distance between adjacent points at most

$$\frac{\ell}{2^r} + \mu_1 + \frac{\mu_1}{2} + \ldots + \frac{\mu_1}{2^{r-1}}.$$

So when r is the least integer greater than or equal to $\log_2 \ell$, the distance between adjacent points is reduced to at most $2\mu_1 + 1$. We say the chain of points is an $(2\mu_1 + 1)$-***coarse path*** between a and b. The number of points making up this coarse path is at most ${K_1}^{1+\log_2 \ell} = K_1 \ell^{\log_2 K_1}$, a bound which is polynomial in ℓ. Also note that we can find a linear bounded in ℓ on the diameter of this coarse path by summing the series:

$$K_1 \sum_{i=1}^{r} \left(\frac{\ell}{2^i} + \mu_1 + \frac{\mu_1}{2} + \ldots + \frac{\mu_1}{2^{i-1}} \right) \ \leq \ K_1(\, \ell + 2\mu_1(1 + \log_2 \ell)\,).$$

(The bounds in this paragraph anticipate the higher dimensional isoperimetric and isodiametric functions we will prove in Theorem D.) If $\mu_1 = 0$ then the limit one obtains is a genuinely continuous but possibly non-rectifiable path (see the proofs of Proposition 4.1).

In anticipation of the forthcoming higher dimensional definitions we pause to express Definition 3.2 in alternative terms. Define C to be the combinatorial structure for the 1-disc that has just one 1-cell and has 0-skeleton $\mathbb{S}^0 = \{-1, 1\}$. Let γ be a map $C \to X$. A ***partition*** of the pair (C, γ) is a pair $(\bar{C}, \bar{\gamma})$ such that $\bar{C}$ is any finite combinatorial structure for the 1-disc (*i.e.* a concatenation of 1-cells), and $\bar{\gamma} : \bar{C}^{(0)} \to X$ is an extension of γ. In this case Definition 3.1 declares $\mathrm{mesh}(C, \gamma)$ to be $d(\gamma(-1), \gamma(1))$ and $\mathrm{mesh}(\bar{C}, \bar{\gamma})$ to be the maximum distance between the images under $\bar{\gamma}$ of the two vertices at the ends of each 1-cell in $\bar{C}$.

For $\gamma : C \to X$ and for $c \geq 0$ let $\mathcal{P}(\gamma, c)$ denote the set of all partitions $(\bar{C}, \bar{\gamma})$ of γ such that $\mathrm{mesh}(\bar{C}, \bar{\gamma}) \leq c$.

If $\mathcal{P}(\gamma, c)$ is non-empty then define

$$\mathcal{F}(\gamma, c) \ := \ \min\left\{ \#_1(\bar{C}) \,\middle|\, (\bar{C}, \bar{\gamma}) \in \mathcal{P}\left(\gamma, c\right) \right\},$$

and if $\mathcal{P}(\gamma, c)$ is empty then define $\mathcal{F}(\gamma, c) := \infty$. (Recall that $\#_1(\bar{C})$ denotes the number of 1-cells in $\bar{C}$.)

$$\mathrm{Fill}^1_{\mu_1}(\ell) \ = \ \sup\left\{ \mathcal{F}(\gamma, \frac{\ell}{2} + \mu_1) \,\middle|\, \gamma : C \to X \text{ with } \mathrm{mesh}(C, \gamma) \leq \ell \right\}.$$

We remark that the choice of the factor $\frac{1}{2}$ in the decrease of the mesh in the definition of the filling function is essentially arbitrary for our purposes: we will be concerned with when $\mathrm{Fill}^1_{\mu_1}$ is a bounded function and this property is independent of the choice of factor from the interval $(0, 1)$. The same comment applies to the factor of $\frac{1}{2}$ in the forthcoming definitions of the 2-dimensional and then the higher dimensional filling functions $\mathrm{Fill}^N_{\mathbf{R}, \boldsymbol{\mu}}$.

3.2 The definition of the 2-dimensional filling function $\mathrm{Fill}^2_{\mathbf{R}, \boldsymbol{\mu}}$

We now define the 2-dimensional filling function $\mathrm{Fill}^2_{\mathbf{R}, \boldsymbol{\mu}} : [0, \infty) \to \mathbb{N} \cup \{\infty\}$, with respect to real numbers μ_1, μ_2 with $0 \leq \mu_1 \leq \mu_2$ and an integer $R_1 \geq 3$. Recall that the notation $\mathbf{R} = (R_i)$ and $\boldsymbol{\mu} = (\mu_i)$ anticipates the forthcoming higher dimensional generalisation, but $R_2, R_3, \ldots$ and $\mu_3, \mu_4, \ldots$ are redundant for the purposes of $\mathrm{Fill}^2_{\mathbf{R}, \boldsymbol{\mu}}$.

Definition 3.4. Define $\mathrm{Sph}^1_{\mathbf{R}}$ to be the set of pairs (C, γ) such that $C \cong \mathbb{S}^1$ is a combinatorial complex homeomorphic to the 1-sphere with $\#_1(C) \leq R_1$, and $\gamma : C^{(0)} \to X$ is a map with domain the 0-skeleton of C.

It will be important later that the complexes used in the definition are also involved in the definition of $\mathbf{R}$-combinatorial complexes: they provide combinatorial structures for $\mathbb{S}^1$ when giving the maps for attaching 2-cells. Also notice that because $R_1 \geq 3$ these combinatorial 1-complexes may be the boundary of a 2-simplex. Roughly speaking $\mathrm{Fill}^2_{\mathbf{R}, \boldsymbol{\mu}}$ measures how readily such γ can be extended to some $\bar{\gamma} : \bar{C}^{(0)} \to X$ in a controlled manner, where $\bar{C}$ is a combinatorial structure for the 2-disc.

Consider $(C, \gamma) \in \mathrm{Sph}^1_{\mathbf{R}}$. For each (closed) 1-cell e of C the 1-dimensional filling function tells us about extending $\gamma\,|_e$. If $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}(\mathrm{mesh}(C,\gamma)) < \infty$ then we can refine each such e into a 1-complex $\bar{e}$ consisting of at most $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}(\mathrm{mesh}(C,\gamma))$ 1-cells, in such a way that there is an extension $\gamma_{\bar{e}} : \bar{e}^{(0)} \to X$ of $\gamma\,|_e$ with

$$\mathrm{mesh}(\bar{e}, \gamma_{\bar{e}}) \ \leq \ \frac{\mathrm{mesh}(C,\gamma)}{2} + \mu_1. \tag{2}$$

An ***essential edge partition*** of $(C, \gamma) \in \mathrm{Sph}^1_{\mathbf{R}}$ is any pair $(\hat{C}, \hat{\gamma})$ such that $\hat{C}$ is a refinement of C obtained by refining all the edges e of C into 1-complexes $\bar{e}$ as above, and $\hat{\gamma}^{(0)} : \hat{C} \to X$ is the extension of γ such that $\hat{\gamma}\,|_{\bar{e}} = \gamma_{\bar{e}}$ for every $\bar{e}$. Note that

$$\mathrm{mesh}(\hat{C}, \hat{\gamma}) \ = \ \max_e \{\mathrm{mesh}(\bar{e}, \gamma_{\bar{e}}) \mid e \text{ is a 1-cell of } C\}$$

and so also satisfies the bound (2).

A ***partition of*** (C, γ) ***subject to an essential edge partition*** $(\hat{C}, \hat{\gamma})$ is defined to be any pair $(\bar{C}, \bar{\gamma})$ for which $\bar{C}$ is an $\mathbf{R}$-combinatorial decomposition of the 2-disc $\mathbb{D}^2$ with $\partial\bar{C} = \hat{C}$ (as 1-complexes) and $\bar{\gamma} : \bar{C}^{(0)} \to X$ is an extension of $\hat{\gamma}$.

For each $(C, \gamma) \in \mathrm{Sph}^1_{\mathbf{R}}$, for each essential edge partition $(\hat{C}, \hat{\gamma})$ of (C, γ), and for each $c \geq 0$, let

$$\mathcal{P}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big)$$

denote the set of those partitions $(\bar{C}, \bar{\gamma})$ of (C, γ) subject to $(\hat{C}, \hat{\gamma})$ that have $\mathrm{mesh}(\bar{C}, \bar{\gamma}) \leq c$.

If $\mathcal{P}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big)$ is non-empty then define

$$\mathcal{F}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big) \ := \ \min\left\{ \#_2(\bar{C}) \,\middle|\, (\bar{C},\bar{\gamma}) \in \mathcal{P}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big) \right\},$$

and if $\mathcal{P}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big)$ is empty then define $\mathcal{F}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big) := \infty$.

The function $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\ell) : [0,\infty) \to \mathbb{N} \cup \{\infty\}$ controls the number of 2-cells required to produce a partition $(\bar{C}, \bar{\gamma})$ of any $(C, \gamma) \in \mathrm{Sph}^1_{\mathbf{R}}$ with $\mathrm{mesh}(C, \gamma) \leq \ell$ (subject to any essential edge partition), in such a way that the mesh is halved modulo the additive error term μ_2.

Definition 3.5. If $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}(\ell) = \infty$ then define $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\ell) := \infty$. Otherwise any $(C, \gamma) \in \mathrm{Sph}^1_{\mathbf{R}}$ with $\mathrm{mesh}(C, \gamma) \leq \ell$ admits an essential edge partition and we define

$$\begin{aligned}\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\ell) \ := \ \sup\Bigg\{ \mathcal{F}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ \ell/2 + \mu_2\Big) \,\Bigg|\, & (C,\gamma) \in \mathrm{Sph}^1_{\mathbf{R}} \text{ with } \mathrm{mesh}(C,\gamma) \leq \ell, \\ & \text{and } (\hat{C},\hat{\gamma}) \text{ is an essential edge partition of } (C,\gamma) \Bigg\}.\end{aligned}$$

In §5.2 we will reinterpret what the function means in the context of finitely generated groups. For a more *ad hoc* example of $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ in action consider the space X obtained by removing from the Euclidean plane an infinite collection of disjoint open balls B_n whose radii are n and whose centres are on the x-axis. This space, given the path metric, is a geodesic space and so has $\mathrm{Fill}^1_0(\ell) = 2$ for all $\ell \in (0, \infty)$. However if $\mu_1 := 0$ and $R_1 := 3$ then we find that $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\ell) = \infty$ whenever ℓ is sufficiently larger that μ_2. For instance take C to be a triangle and the image of $\gamma : C^{(0)} \to C$ to be three equally spaced points on the boundary of one of the holes B_n. Then the image of $\hat{\gamma}$, for any essential boundary partition $(\hat{C}, \hat{\gamma})$ of (C, γ), is six equally spaces points on the boundary of B_n. Then, assuming $n \gg \mu_2$, no partition $(\bar{C}, \bar{\gamma})$ of (C, γ) subject to $(\hat{C}, \hat{\gamma})$ exists.

For a more elaborate example start with the Euclidean plane with an infinite collection of disjoint open balls $B_{n,i}$ removed for all $n, i \in \mathbb{N} \smallsetminus \{0\}$ with $i \geq n$, where each $B_{n,i}$ has radius n and is centred on some vertical line $x = c_i$. Let $S^o_{n,i}$ be a Euclidian 2-sphere of radius i with a disc removed by cutting along a circle of perimeter $2\pi n$. Obtain the space Y by attaching each $S^o_{n,i}$ in the obvious way to the boundary of the hole where $B_{n,i}$ was removed. The space Y with the path metric is a geodesic space and it turns out that we can draw the same conclusions about $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}$ and $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ as we did for the space X. However on this occasion the reason $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ takes infinite values is not the non-existence of partitions $(\bar{C}, \bar{\gamma})$ but rather that there is no bound on the number of 2-cells in the $\bar{C}$.

3.3 The definition of the higher dimensional filling functions $\mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}$

The definition of 2-dimensional filling functions as set out above, readily generalises to higher dimensions. The N-dimensional filling function $\mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}} : [0, \infty) \to \mathbb{N} \cup \{\infty\}$ is defined recursively and is given with reference to real numbers $0 \leq \mu_1 \leq \ldots \leq \mu_N$ and integers $R_1, R_2, \ldots, R_{N-1}$ such that each $R_i \geq i + 2$. It is convenient to use the sequence notation $\mathbf{R} = (R_i)$ and $\boldsymbol{\mu} = (\mu_i)$, but the entries $R_N, R_{N+1}, \ldots$ and $\mu_{N+1}, \mu_{N+2}, \ldots$ are redundant.

Let us suppose we have defined the functions $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^{N-1}_{\mathbf{R},\boldsymbol{\mu}} : [0, \infty) \to \mathbb{N} \cup \{\infty\}$ and explain how then to define $\mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}$. First we need:

Definition 3.6. Let $\mathrm{Sph}^{N-1}_{\mathbf{R}}$ be the set of pairs (C, γ) such that $C \cong \mathbb{S}^{N-1}$ is an $\mathbf{R}$-combinatorial complex homeomorphic to the $(N-1)$-sphere with $\#_{N-1}(C) \leq R_{N-1}$, and γ is a map $C^{(0)} \to X$.

Note that the C referred to in this definition are precisely the complexes which provide the combinatorial structures for the $(N-1)$-spheres that are used to attach N-cells in $\mathbf{R}$-combinatorial structures. The insistence that each $R_i \geq i + 2$ ensures that the C of Definition 3.6 include the boundary of an N-simplex.

Consider $(C, \gamma) \in \mathrm{Sph}^{N-1}_{\mathbf{R}}$ and suppose that

$$\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}(\mathrm{mesh}(C, \gamma)),\ \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\mathrm{mesh}(C, \gamma)), \ldots,\ \mathrm{Fill}^{N-1}_{\mathbf{R},\boldsymbol{\mu}}(\mathrm{mesh}(C, \gamma)) \ < \ \infty.$$

An ***essential boundary partition*** of (C, γ) is any pair $(\hat{C}, \hat{\gamma}) := (C_{N-1}, \gamma_{N-1})$ that can be obtained from any sequence of pairs

$$(C, \gamma) \ = \ (C_0, \gamma_0), (C_1, \gamma_1), \ldots, (C_{N-1}, \gamma_{N-1}) \ = \ (\hat{C}, \hat{\gamma})$$

in the following way. Each C_k will be a refinement of C_{k-1} and each $\gamma_k : C_k^{(0)} \to X$ is an extension of γ_{k-1}; further

$$\mathrm{mesh}(C_k, \gamma_k) \ \leq \ \frac{\mathrm{mesh}(C, \gamma)}{2} + \mu_k.$$

Let $(C_0, \gamma_0) := (C, \gamma)$. For each (closed) 1-cell e^1 of C_0, take any refinement of e^1 into a 1-complex $\overline{e^1}$ with $\#_1(\overline{e^1}) \ \leq \ \mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}(\mathrm{mesh}(C_0, \gamma_0))$ such that there is an extension $\gamma_{\overline{e^1}}$:

$\overline{e^1}^{(0)} \to X$ of $\gamma_0\,|_{e^1}$ with

$$\mathrm{mesh}(\overline{e^1}, \gamma_{\overline{e^1}}) \;\leq\; \frac{\mathrm{mesh}(C_0, \gamma_0)}{2} + \mu_1. \tag{3}$$

Let C_1 denote the resulting refinement of C_0 and $\gamma_1 : C_1^{(0)} \to X$ the resulting extension of γ_0.

Next we refine each (closed) 2-cell e^2 of C_1 into any 2-complex $\bar{e^2}$ with $\#_2(\overline{e^2}) \leq \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\mathrm{mesh}(C,\gamma))$ in such a way that there is an extension $\gamma_{\overline{e^2}} : \overline{e^2}^{(0)} \to X$ of $\gamma_1\,|_{e^2}$ with

$$\mathrm{mesh}(\overline{e^2}, \gamma_{\overline{e^2}}) \;\leq\; \frac{\mathrm{mesh}(C, \gamma)}{2} + \mu_2. \tag{4}$$

Let C_2 be a refinement of C_1 obtained by refining all the 2-cells e^2 of C_1 in this way and let $\gamma_2 : C_2^{(0)} \to X$ be the resulting extension of γ_1. Note that $\partial \overline{e^2} = \partial e^2$ because the refining of e^2 to produce $\overline{e^2}$ is *subject to* the previously performed 1-cell refinements. Therefore it makes sense to build C_2 by assembling the refinements of the 2-cells of C_1. For a similar reason the definition of γ_2 makes sense.

Similarly we can produce C_3 by refining the 3-cells of C_2, each into at most $\mathrm{Fill}^3_{\mathbf{R},\boldsymbol{\mu}}(\mathrm{mesh}(C,\gamma))$ 3-cells, and we can extend γ_2 to $\gamma_3 : C_3^{(0)} \to X$. Continuing in the same manner through the dimensions we eventually arrive at a pair (C_{N-1}, γ_{N-1}) with

$$\mathrm{mesh}(C_{N-1}, \gamma_{N-1}) \;\leq\; \frac{\mathrm{mesh}(C, \gamma)}{2} + \mu_{N-1}.$$

Now a ***partition of*** (C,γ) ***subject to an essential boundary partition*** $(\hat{C}, \hat{\gamma})$ is defined to be any pair $(\bar{C}, \bar{\gamma})$ for which $\bar{\gamma} : \bar{C}^{(0)} \to X$ is an extension of $\hat{\gamma}$ and $\bar{C}$ is an $\mathbf{R}$-combinatorial decomposition of the N-disc $\mathbb{D}^N$ with $\partial \bar{C} = \hat{C}$ as $(N-1)$-complexes.

For each $(C,\gamma) \in \mathrm{Sph}^{N-1}_{\mathbf{R}}$, for each essential boundary partition $(\hat{C}, \hat{\gamma})$ of (C,γ), and for each $c \geq 0$, let

$$\mathcal{P}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big)$$

denote the set of all partitions $(\bar{C}, \bar{\gamma})$ of (C,γ) subject to $(\hat{C},\hat{\gamma})$ that have $\mathrm{mesh}(\bar{C},\bar{\gamma}) \leq c$.

When $\mathcal{P}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big)$ is non-empty define

$$\mathcal{F}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big) \;:=\; \min\Big\{\#_N(\bar{C}) \,\Big|\, (\bar{C},\bar{\gamma}) \in \mathcal{P}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big)\Big\},$$

and if $\mathcal{P}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big)$ is empty then define $\mathcal{F}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ c\Big) := \infty$.

We can now define $\mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}(\ell)$.

Definition 3.7. If $\mathrm{Fill}^k_{\mathbf{R},\boldsymbol{\mu}}(\ell) = \infty$ for some $k \in \{1, 2, \ldots, N-1\}$ then define $\mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}(\ell) := \infty$. Otherwise any $(C,\gamma) \in \mathrm{Sph}^{N-1}_{\mathbf{R}}$ with $\mathrm{mesh}(C,\gamma) \leq \ell$ admits an essential boundary partition and we define

$$\mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}(\ell) \;:=\; \sup\Big\{\mathcal{F}\Big((C,\gamma),\ (\hat{C},\hat{\gamma}),\ \ell/2 + \mu_N\Big) \Big| (C,\gamma) \in \mathrm{Sph}^{N-1}_{\mathbf{R}} \text{ with } \mathrm{mesh}(C,\gamma) \leq \ell, \text{ and } (\hat{C},\hat{\gamma}) \text{ is an essential boundary partition of } (C,\gamma)\Big\}.$$

4 Characterising metric spaces with highly connected asymptotic cones

Here is the characterisation:

Theorem A. *Let X be a metric space, let ω be a non-principal ultrafilter, and $N \geq 0$. The following are equivalent.*

- *The asymptotic cones* $\mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$ *are N-connected for all* $\mathbf{e}$ *and* $\mathbf{s}$.
- *There exist* $\mathbf{R}$, $\boldsymbol{\mu}$ *such that the filling functions* $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$ *are bounded.*

We will prove this theorem by induction on N, presenting first the case $N = 0$, which gives a characterisation of metric spaces with path-connected asymptotic cones. We will prove the case $N = 1$ in §4.2; that is, we will characterise metric spaces with 1-connected asymptotic cones. Then in §4.3 we will generalise the argument to higher dimensions, giving the induction step and thus completing the proof.

The condition of Theorem A that the $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ are N-connected is a quasi-isometry invariant. Consequently, by Corollary 2.6, the property of

$$\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}},\ \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}},\ \ldots,\ \mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$$

being bounded for some $\mathbf{R}$ and $\boldsymbol{\mu}$ is also a quasi-isometry invariant for metric spaces.

The approach we use in this section builds on [37] which, in turn, has origins in [25].

4.1 Characterising path connectedness

The first step towards Theorem A is the following characterisation of metric spaces with path connected cones. Recall that we use the notation $\mathrm{Fill}^1_{\mu_1}$ to denote the 1-dimensional filling function $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}$, as this function is defined only with reference to one constant, *i.e.* a number $\mu_1 \geq 0$.

Proposition 4.1. *Let X be a metric space and ω be a non-principal ultrafilter. The following conditions are equivalent.*

- *There exist $K_1 \in \mathbb{N}$ and $\mu_1 \geq 0$ such that in X:*

$$\forall \ell \geq 0, \quad \mathrm{Fill}^1_{\mu_1}(\ell) \leq K_1.$$

- *The asymptotic cones* $\mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$ *are path connected for all* $\mathbf{e}$ *and* $\mathbf{s}$.

Proof. First let us prove that boundedness of $\mathrm{Fill}^1_{\mu_1}$ implies that the asymptotic cones of X are path connected. Suppose we are given $f : \{-1, 1\} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ with $f(-1) = \mathbf{e}$. Let $\ell := \mathrm{mesh}(f) = d(f(-1), f(1))$. We seek to extend f to a continuous map $\bar{f} : \mathbb{D}^1 \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$.

Lemma 4.2. *Assume $K_1 < \infty$ is a bound on $\mathrm{Fill}^1_{\mu_1}$. Let γ be a map $C^{(0)} \to \mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$, where $C^{(0)} = C := \mathbb{S}^0 = \{-1, 1\}$. Then there is an extension $\bar{\gamma} : \bar{C}^{(0)} \to \mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$, where $\bar{C}$ is some combinatorial decomposition of $\mathbb{D}^1$ with K_1 1-cells and $\mathrm{mesh}(\bar{C}, \bar{\gamma}) \leq \frac{1}{2}\mathrm{mesh}(\gamma)$.*

Proof of Lemma 4.2. We can express $\gamma : C^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ as (γ_n), say, where each γ_n is a map $C^{(0)} \to X$. By hypothesis there exist $K_1 \in \mathbb{N}$ and $\mu_1 > 0$ such that we can find partitions $\bar{\gamma}_n : \bar{C}_n^{(0)} \to X$ of γ_n, with $\#_1(\bar{\gamma}_n) \leq K_1$ and $\mathrm{mesh}(\bar{C}_n, \bar{\gamma}_n) \leq \frac{1}{2}\mathrm{mesh}(\gamma_n) + \mu_1$. For simplicity we can take each $\bar{C}_n$ to have exactly K_1 1-cells. So each $\bar{C}_n$ may as well be $\bar{C}$, the unique (up to combinatorial isomorphism) combinatorial complex homeomorphic to $\mathbb{D}^1$ which has K_1 1-cells. Then $\bar{\gamma} := (\bar{\gamma}_n)$ is a well-defined map $\bar{C}^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ because for all $v \in \bar{C}^{(0)}$

$$d(\mathbf{e}, \bar{\gamma}(v)) \ \leq \ d(\mathbf{e}, \bar{\gamma}(v_0)) + d(\bar{\gamma}(v_0), \bar{\gamma}(v)),$$

where $v_0 \in C^{(0)}$, and

$$\begin{aligned} d(\bar{\gamma}(v_0), \bar{\gamma}(v)) \ &= \ \lim_\omega \frac{1}{s_n} d(\bar{\gamma}_n(v_0), \bar{\gamma}_n(v)) \ \leq \ \lim_\omega \frac{1}{s_n} K_1 \, \mathrm{mesh}(\bar{C}_n, \bar{\gamma}_n) \\ &\leq \ \lim_\omega \frac{1}{s_n} K_1 \left(\frac{1}{2}\mathrm{mesh}(\gamma_n) + \mu_1 \right) \ = \ \frac{K_1}{2}\mathrm{mesh}(\gamma). \end{aligned}$$

The final equality follows from the definition of distance in the cone as do the outer equalities in the following.

$$\mathrm{mesh}(\bar{C}, \bar{\gamma}) \ = \ \lim_\omega \frac{\mathrm{mesh}(\bar{C}_n, \bar{\gamma}_n)}{s_n} \ \leq \ \lim_\omega \frac{1}{s_n} \left(\frac{1}{2}\mathrm{mesh}(\gamma_n) + \mu_1 \right) \ = \ \frac{1}{2}\mathrm{mesh}(\gamma).$$

This completes the proof of the lemma.

Thus between any two points $\mathbf{a}, \mathbf{b} \in \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ we can find a chain of $K_1 + 1$ points $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{K_1}$, where $\mathbf{a}_0 = \mathbf{a}$, $\mathbf{a}_{K_1} = \mathbf{b}$ and each $d(\mathbf{a}_i, \mathbf{a}_{i+1}) \leq \frac{1}{2} d(\mathbf{a}, \mathbf{b})$. We iterate this procedure, and use the completeness of asymptotic cones to construct a path between $\mathbf{e} = f(-1)$ and $f(1)$.

Let $\mathcal{T}_0 := \mathbb{D}^1$ with the obvious cell structure of one 1-cell and two 0-cells. We successively refine the cell structure of $\mathcal{T}_0$ to produce cellular structures $\mathcal{T}_1, \mathcal{T}_2, \ldots$ for $\mathbb{D}^1$ as follows. Obtain $\mathcal{T}_n$ from $\mathcal{T}_{n-1}$ by refining every 1-cell of $\mathcal{T}_{n-1}$ into K_1 1-cells. So $\mathcal{T}_n$ is a cell decomposition of $\mathbb{D}^1$ with ${K_1}^n$ 1-cells.[7]

Define $f_0 : \mathcal{T}_0^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ by $f_0 := f$. Then using Lemma 4.2 inductively define $f_n : \mathcal{T}_n^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ to be an extension of $f_{n-1} : \mathcal{T}_{n-1}^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ such that $\mathrm{mesh}(\mathcal{T}_n, f_n) \leq \frac{1}{2}\mathrm{mesh}(\mathcal{T}_{n-1}, f_{n-1})$. Then $\mathrm{mesh}(\mathcal{T}_n, f_n) \leq \frac{1}{2^n}\ell$.

We now define $\bar{f} : \mathbb{D}^1 \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. Given $x \in \mathbb{D}^1$ choose $x_n \in \mathcal{T}_n^{(0)}$ such that x and x_n are in the same 1-cell of $\mathcal{T}_n$. Then define $\bar{f}(x) := \lim_{n \to \infty} f_n(x_n)$. Observe that $d(f_n(x_n), f_{n+1}(x_{n+1})) \leq K_1 \frac{\ell}{2^{n+1}}$, and so for $m > n$

$$d(f_m(x_m), f_n(x_n)) \ \leq \ \sum_{k=n}^{m-1} K_1 \frac{\ell}{2^{k+1}} \ \leq \ K_1 \frac{\ell}{2^n}.$$

Thus the sequence $(f_n(x_n))$ is Cauchy, and since $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ is complete, the limit $\lim_{n \to \infty} f_n(x_n)$ exists. A similar argument shows $\bar{f}(x)$ to be independent of the choice made in selecting each x_n.

[7]It would seem natural to take the 1-cells of $\mathcal{T}_n$ to correspond to intervals of equal length $2/{K_1}^n$ in $\mathbb{D}^1$. In fact this is not necessary and in the higher dimensional arguments we will not be able to assume such regularity.

Clearly $\bar{f}\big|_{\{-1,1\}} = f$. To complete the proof that $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ is path connected, all that remains to check is the continuity of $\bar{f}$. The following lemma suffices.

Lemma 4.3. *Suppose $C \subset \mathbb{D}^1$ is one of the 1-cells of $\mathcal{T}_n$. Then* $\mathrm{diam}\, \bar{f}(C) \leq (2K_1 + 1)\ell/2^n$.

Proof of Lemma 4.3. Consider $x, y \in C$. Then $\bar{f}(x) = \lim_{m\to\infty} f_m(x_m)$ and $\bar{f}(y) = \lim_{m\to\infty} f_m(y_m)$ where (x_n) and (y_n) can be taken to be sequences in $C \subset \mathbb{D}^1$. For $m > n$ we find

$$\begin{aligned} d(f_m(x_m), f_m(y_m)) &\leq d(f_m(x_m), f_n(x_n)) + d(f_n(x_n), f_n(y_n)) + d(f_n(y_n), f_m(y_m)) \\ &\leq K_1 \frac{\ell}{2^n} + \frac{\ell}{2^n} + K_1 \frac{\ell}{2^n}. \end{aligned}$$

Thus $d(\bar{f}(x), \bar{f}(y)) \leq (2K_1 + 1)\ell/2^n$ as required.

We now come to the proof of the reverse implication; so suppose that the asymptotic cones $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ are path connected for all $\mathbf{e}$ and $\mathbf{s}$. Let us assume that for all $\mu_1 \geq 0$ the function $\mathrm{Fill}^1_{\mu_1}$ is, in fact, unbounded. Then recalling the definition of $\mathrm{Fill}^1_{\mu_1}$ from §3.1 we find that for all $n \in \mathbb{N}$ there exist $a_n, b_n \in X$ for which there are no $a_n^0, a_n^1, ..., a_n^n \in X$ such that

$$\begin{aligned} a_n^0 = a_n \text{ and } a_n^n &= b_n, \text{ and} \\ d(a_n^i, a_n^{i+1}) &\leq \frac{1}{2} d(a_n, b_n) + n, \quad \text{for } i = 0, ..., n-1. \end{aligned} \tag{5}$$

Now we seek a contradiction.

Define a sequence of base points $\mathbf{e} = (e_n)$ by $e_n := a_n$ and scalars $\mathbf{s} = (s_n)$ by $s_n := d(a_n, b_n)$. Each s_n is at least n as otherwise the *error term* "$+n$" in (5) would render the existence of the $a_n^0, a_n^1, ..., a_n^n$ trivial. So $s_n \to \infty$ as $n \to \infty$. By hypothesis, $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ is path connected, and so there exists a continuous path $\gamma : [-1, 1] \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ with $\gamma(-1) = \mathbf{e} := (e_n)$ and $\gamma(1) = \mathbf{b} := (b_n)$. (Note that $\mathbf{b}$ is a well-defined point in $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ because of our definition of $\mathbf{s}$.) By uniform continuity on the compact set $[-1, 1]$, there exists $K \in \mathbb{N}$ such that when we subdivide the interval $[-1, 1]$ into K intervals I_i of equal length $2/K$, we find that $\mathrm{diam}(\gamma(I_i)) \leq 1/4$ for each i. Define $\mathbf{a}^i := \gamma(-1 + i2/K)$ for $i = 0, 1, \ldots, K$. Choose representatives $\mathbf{a}^i = (a_n^i)$ with $\mathbf{a}^0 = (a_n) = (e_n)$ and $\mathbf{a}^K = (b_n)$. By definition

$$\lim_\omega \frac{1}{s_n} d(a_n^i, a_n^{i+1}) \;=\; d(\mathbf{a}^i, \mathbf{a}^{i+1})$$

and so for ω-infinitely many n

$$\left| d(\mathbf{a}^i, \mathbf{a}^{i+1}) - \frac{1}{s_n} d(a_n^i, a_n^{i+1}) \right| \;\leq\; \frac{1}{4}. \tag{6}$$

The intersection of finitely many sets of ω-measure 1 itself has ω-measure 1. Thus there is an infinite set J of ω-measure 1 such that for all $n \in J$ the inequality (6) holds for $i = 0, 1, \ldots, K-1$. Now, since $s_n = d(a_n, b_n)$ and $d(\mathbf{a}^i, \mathbf{a}^{i+1}) \leq 1/4$, we can deduce that for $n \in J$, and for $i = 0, ..., K-1$

$$d(a_n^i, a_n^{i+1}) \;\leq\; \frac{1}{2} d(a_n, b_n).$$

Hence, referring back to 5), we see have a contradiction and the proof is complete. ■

Remark 4.4. When there exists $\mu_1 \geq 0$ such that $\mathrm{Fill}^1_{\mu_1}(\ell) \leq 2$ for all $\ell \geq 0$ it turns out that each $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ is a geodesic space. The path between $\mathbf{a}$ and $\mathbf{b}$ constructed in the proof of Proposition 4.1 is a geodesic. Indeed if the $\mathcal{T}_n$ in this proof are constructed from ${K_1}^n$ equal intervals then the resulting path will also be parametrised proportional to arc length.

4.2 Characterising 1-connectedness

Next we characterise metric spaces with 1-connected asymptotic cones. Recall that the 2-dimensional filling function $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ is defined with reference to the constants $0 \leq \mu_1 \leq \mu_2$ and $R_1 \in \mathbb{N}$.

Proposition 4.5. *Let X be a metric space and ω be a non-principal ultrafilter. Suppose the asymptotic cones $\mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$ are path connected for all $\mathbf{e}$ and $\mathbf{s}$. Fix $\mu_1 \geq 0$ and $K_1 \in \mathbb{N}$ such that in X,*

$$\forall \ell \geq 0, \quad \mathrm{Fill}^1_{\mu_1}(\ell) \leq K_1.$$

(Proposition 4.1 tells us that such μ_1 and K_1 exist.) The following are equivalent.

- *There exists $R_1, K_2 \in \mathbb{N}$ (with $R_1 \geq 3$) and $\mu_2 \geq \mu_1$ such that in X:*

$$\forall \ell \geq 0, \quad \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\ell) \leq K_2.$$

- *The asymptotic cones $\mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$ are 1-connected for all $\mathbf{e}$ and $\mathbf{s}$.*

Moreover we can, in fact, take $R_1 = 1 + K_1$.

Proof. Let us assume X is an unbounded metric space, else the asymptotic cones are points rendering the theorem trivial.

First we suppose there are $R_1 \geq 3$ and $\mu_2 \geq \mu_1$ such that $\mathrm{Fill}^1_{\mu_1}(\ell) \leq K_1$ for all $\ell \geq 0$ and we prove that the asymptotic cones of X are then all 1-connected. So we fix $\mathbf{e}$ and $\mathbf{s}$ and consider a closed loop f based at $\mathbf{e}$ in $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. This can be viewed as a continuous map $f : (\partial\mathbb{D}^2, \star) \to (\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}), \mathbf{e})$. The set $f(\partial\mathbb{D}^2)$ is compact and so has finite diameter L. Our objective is to show that f can be extended to a continuous map $\mathbb{D}^2 \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$.

We take a *tessellation* $\mathcal{T}_0$ of $\mathbb{D}^2$ by triangles whose vertices all lie in $\partial\mathbb{D}^2$. This is constructed by regarding the interior of $\mathbb{D}^2$ as the Klein model of the hyperbolic plane, inscribing an ideal triangle and then reflecting repeatedly in its edges to cover the plane – see the leftmost diagram of Figure 3. Let $\Delta_i \subset \mathbb{D}^2$ be the triangles obtained from the ideal triangles by including their ideal vertices. Then $\mathcal{T}_0$ is the tessellation of $\bigcup_i \Delta_i$ of $\mathbb{D}^2$. (Note that the tessellation only includes a countable dense subset of $\partial\mathbb{D}^2$.) We will appeal to the following properties of $\mathcal{T}_0$.

- The vertices $\mathcal{T}_0^{(0)}$ of $\mathcal{T}_0$ are dense in $\mathbb{S}^1$.
- With respect to the usual Euclidean metric d on $\mathbb{D}^2$, we find that for all $\kappa > 0$ only finitely many triangles $\Delta \subset \mathbb{D}^2$ of the tessellation $\mathcal{T}_0$ have $\mathrm{diam}(\Delta) > \kappa$.

Each triangle Δ in the tessellation $\mathcal{T}_0$ admits the combinatorial structure $\Delta \cong C$ of a 2-simplex C. And $\mathcal{T}_0$ is an infinite combinatorial structure, built up by joining the combinatorial structures admitted by the triangles across common edges. Now f restricts to the vertices of $\mathcal{T}_0$ to give a map $f_0 : \mathcal{T}_0^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. If, for any Δ in $\mathcal{T}_0$, we define $\gamma_\Delta : \Delta^{(0)} \to X$

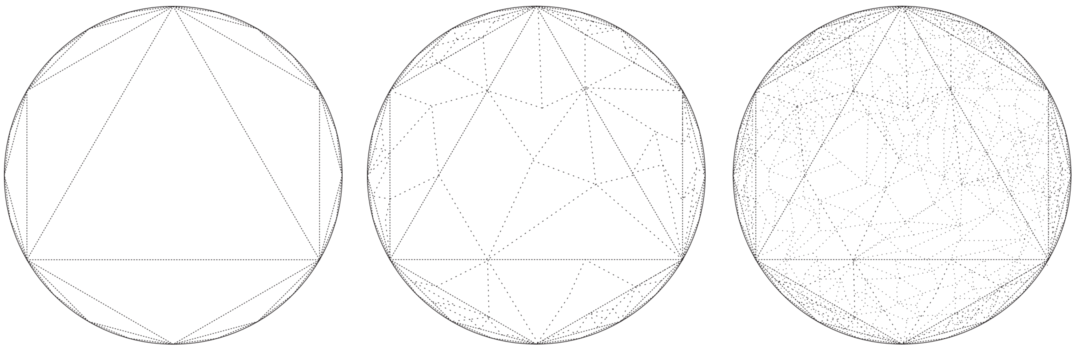

Figure 3: Tessellations $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$.

to be the restriction of f to the vertices of Δ then we find $\mathrm{mesh}(\Delta, \gamma_\Delta) \leq L$ because $L = \mathrm{diam} f(\partial \mathbb{D}^2)$. Furthermore (in the notation of §3.2) the pair $(\Delta, \gamma_\Delta) \in \mathrm{Sph}^1_{\mathbf{R}}$ since $R_1 \geq 3$.

We will now produce successive *refinements*[8] $\mathcal{T}_1, \mathcal{T}_2, \ldots$ of $\mathcal{T}_0$, as depicted in Figure 3 (in the case $R_1 = 4$). At the same time we shall define a sequence of maps $f_n : \mathcal{T}_n^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ such that each f_{k+1} extends f_k, and each 2-cell C of $\mathcal{T}_k$ is *refined* to some combinatorial 2-complex $\bar{C}$ in $\mathcal{T}_{k+1}$, in such a way that

$$\mathrm{mesh}(\bar{C}, f_{k+1}\,|_{\bar{C}^{(0)}}) \;\leq\; \frac{1}{2}\mathrm{mesh}(C, f_k\,|_{C^{(0)}}).$$

We define

$$\mathrm{mesh}(\mathcal{T}_k, f_k) \;:=\; \sup\left\{ d(f_k(a), f_k(b)) \mid a \text{ and } b \text{ are endpoints of an edge of } \mathcal{T}_k \right\}.$$

Then $\mathrm{mesh}(\mathcal{T}_0, f_0) \leq L$, and it will be the case that for all $k \geq 0$

$$\mathrm{mesh}(\mathcal{T}_{k+1}, f_{k+1}) \;\leq\; \frac{1}{2}\mathrm{mesh}(\mathcal{T}_k, f_k)$$

and so it will follow that $\mathrm{mesh}(\mathcal{T}_n, f_n) \leq L/2^n$ for all $n \geq 0$.

Fix $k \geq 0$. It suffices to describe the process of producing $(\mathcal{T}_{k+1}, f_{k+1})$ from $(\mathcal{T}_k, f_k)$. Choose a sequence of maps $f_{k,i} : \mathcal{T}_k^{(0)} \to X$ so that $(f_{k,i})_{i \in \mathbb{N}} = f_k$. First we make *essential edge partitions* $(\hat{\mathcal{T}}_{k,i}, \hat{f}_{k,i})$ of $(\mathcal{T}_k, f_{k,i})$. By hypothesis there are $\mu_1 \geq 0$ and $K_1 \in \mathbb{N}$ such that $\mathrm{Fill}^1_{\mu_1}(\ell) \leq K_1$ for all $\ell \geq 0$. So each 1-cell of $\mathcal{T}_k$ can be refined into at most K_1 1-cells to produce $\hat{\mathcal{T}}_{k,i}$ with

$$\mathrm{mesh}(\hat{\mathcal{T}}_{k,i}, \hat{f}_{k,i}) \;\leq\; \frac{1}{2}\mathrm{mesh}(\mathcal{T}_k, f_{k,i}) + \mu_1.$$

Next suppose that C is one of the 2-cells forming the tessellation $\mathcal{T}_k$, and $\hat{C}_i$ is its refinement in $\hat{\mathcal{T}}_{k,i}$. Then we can define a pair (C, γ_i), where $\gamma_i : C^{(0)} \to X$ is $f_{k,i}\,|_{C^{(0)}}$, and we consider its essential edge partition $(\hat{C}_i, \hat{\gamma}_i)$, where

$$\hat{\gamma}_i \;:=\; \hat{f}_{k,i}\Big|_{\hat{C}_i^{(0)}} \;:\; \hat{C}_i^{(0)} \to X.$$

[8]The closure of any 2-cell of the tessellation $\mathcal{T}_i$ is a finite combinatorial complex in the sense of §2.3. A refinement $\mathcal{T}_i$ is produced by refining (as defined in §2.3) all the 2-cells of $\mathcal{T}_i$ in a way that agrees across common 1-cells.

The assumption that $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ is bounded by K_2 allows us to deduce there we can find a partition $(\bar{C}_i, \bar{\gamma}_i)$ of (C, γ_i) subject to $(\hat{C}_i, \hat{\gamma}_i)$ satisfying:

$$\begin{aligned} \#_2(\bar{C}_i) &\leq K_2, \quad \text{and} \\ \mathrm{mesh}(\bar{C}_i, \bar{\gamma}_i) &\leq \frac{1}{2}\mathrm{mesh}(C_i, \gamma_i) + \mu_2. \end{aligned}$$

This process of partitioning is repeated for all of the 2-cells in $\mathcal{T}_{k,i}$, producing $\bar{f}_{k,i} : \bar{\mathcal{T}}_{k,i}^{(0)} \to X$. The refinements of the 2-cells agree across common boundaries of 2-cells in $\mathcal{T}_{k,i}$, as do the functions $\bar{\gamma}_i$. Hence we can collect them all together and produce $(\bar{\mathcal{T}}_{k,i}, \bar{f}_{k,i})$. It is reasonable to refer to $(\bar{\mathcal{T}}_{k,i}, \bar{f}_{k,i})$ as *a partition of* $(\mathcal{T}_k, f_{k,i})$ *subject to* $(\hat{\mathcal{T}}_{k,i}, \hat{f}_{k,i})$.

We would now like to define $f_{k+1} : \mathcal{T}_{k+1}^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ to be the ω-limit of $(\bar{f}_{k,i})$. To make sense of this we should first explain how we refine $\mathcal{T}_k$ to produce $\mathcal{T}_{k+1}$. This is a two-stage process: we first refine the 1-cells and then refine the 2-cells. Recall that a 1-cell e of $\mathcal{T}_k$ is refined in $\hat{\mathcal{T}}_{k,i}$ into a bounded number of 1-cells, and hence into one of finitely many combinatorial structures up to combinatorial equivalence; it follows that (up to combinatorial equivalence) exactly one of these combinatorial structures on e occurs for all i in some set of ω-measure 1. Define $\hat{\mathcal{T}}_k$ to be the refinement of $\mathcal{T}_k$ obtained by refining all the 1-cells accordingly. Similarly for a 2-cell C of $\mathcal{T}_k$ and $\gamma = (\gamma_i) : C^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ we find that $\hat{C}_i = \hat{C}$ up to combinatorial equivalence for ω-infinitely many i, where $\hat{C}_i$ is the refinement of C in $\hat{\mathcal{T}}_{k,i}$. Further the $\bar{C}_i$ are $\mathbf{R}$-combinatorial 2-complexes with at most K_2 2-cells and so there exists $\bar{C}$ such that for ω-infinitely many i, the 2-complex $\bar{C} = \bar{C}_i$ up to combinatorial equivalence, and $\partial\bar{C} = \hat{C}$. Refine each $\hat{C}$ to $\bar{C}$ to produce $\mathcal{T}_{k+1}$ from $\hat{\mathcal{T}}_k$. By construction

$$f_{k+1} \ := \ (\bar{f}_{k,i})_{i \in \mathbb{N}} : \ \mathcal{T}_{k+1}^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$$

is well defined, and

$$\mathrm{mesh}(\bar{C}, f_{k+1}\,|_{\bar{C}^{(0)}}) \ \leq \ \frac{1}{2}\mathrm{mesh}(C, f_k\,|_{C^{(0)}})$$

as required (*cf.* the proof of Lemma 4.2).

We are now ready to define $\bar{f} : \mathbb{D}^2 \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. On the boundary $\partial\mathbb{D}^2 = \mathbb{S}^1$ we let $\bar{f} := f$. For $x \in \mathbb{D}^2 \smallsetminus \partial\mathbb{D}^2$, let $x_n \in \mathbb{D}^2$ be a 0-cell of one of the 2-cells of $\mathcal{T}_n$ containing x, and then define $\bar{f}(x) := \lim_{n\to\infty} f_n(x_n)$.

Let us check that $\bar{f}$ is well defined. We prove that $(f_n(x_n))$ is a Cauchy sequence - then as the asymptotic cone is complete this sequence converges. The observation we use is

$$d(f_n(x_n), f_{n-1}(x_{n-1})) \ \leq \ 2\,K_2\,R_1\frac{L}{2^n}.$$

This holds because $R_1\,L/2^n$ is a bound on the distance between the images under f_n of any two vertices of any 2-cell in $\mathcal{T}_n$, and at most K_2 such 2-cells are used to fill a 2-cell in $\mathcal{T}_{n-1}$. The remaining factor of 2 on the right hand side of the inequality accounts for the possible non-uniqueness of the 2-cell in $\mathcal{T}_{n-1}$ from which x_{n-1} can be chosen. It follows that $(f_n(x_n))$ is Cauchy: for $m > n$ we find

$$d(f_m(x_m), f_n(x_n)) \ \leq \ \sum_{i=n+1}^{m} 2\,K_2\,R_1\frac{L}{2^i} \ < \ 2\,K_2\,R_1\frac{L}{2^n}. \tag{7}$$

A similar argument tells us that $\bar{f}$ is independent of the choice of x_n.

It remains to check that $\bar{f}$ is a *continuous* extension of f. This, in part, is the purpose of the following lemma.

Lemma 4.6. *Suppose $C \subset \mathbb{D}^2$ is one of the 2-cells of the tessellation $\mathcal{T}_n$. Then*

$$\operatorname{diam} \bar{f}(C) \;\leq\; R_1\,(4K_2+1)\,\operatorname{mesh}(C, f_n\,|_{C^{(0)}}\,).$$

Proof. Let $\kappa := \operatorname{mesh}(C, f_n\,|_{C^{(0)}}\,)$ and take $x, y \in C$. Then

$$\bar{f}(x) \;:=\; \lim_{n\to\infty} f_n(x_n)$$

and $\bar{f}(y) := \lim_{n\to\infty} f_n(y_n)$ for some sequences (x_n) and (y_n) of points in $C \subset \mathbb{D}^2$. For $k \geq n$

$$\operatorname{mesh}(C, f_k\,|_C\,) \;\leq\; \frac{1}{2^{k-n}}\operatorname{mesh}(C, f_n\,|_{C^{(0)}}\,),$$

and so for $m > n$

$$\begin{aligned} d(f_m(x_m), f_n(x_n)) \;&\leq\; \sum_{k=n+1}^{m} 2\,R_1\,K_2\,\operatorname{mesh}(C, f_k\,|_C\,) \\ &<\; 2\,R_1\,K_2\left(\frac{1}{2}+\frac{1}{2^2}+\ldots\right)\kappa \\ &=\; 2\,R_1\,K_2\,\kappa. \end{aligned}$$

(*cf.* (7) for the first inequality.) Therefore for $m > n$,

$$\begin{aligned} d(f_m(x_m), f_m(y_m)) \;&\leq\; d(f_m(x_m), f_n(x_n)) + d(f_n(x_n), f_n(y_n)) + d(f_n(y_n), f_m(y_m)) \\ &<\; 2\,R_1\,K_2\,\kappa + R_1\,\kappa + 2\,R_1\,K_2\,\kappa \\ &=\; R_1\,(4K_2+1)\,\kappa. \end{aligned}$$

The statement of the lemma then readily follows.

We now use Lemma 4.6 to prove continuity of $\bar{f}$. We treat the three cases $x \in \mathbb{D}^2 - \partial\mathbb{D}^2$, $x \in \partial\mathbb{D}^2 - \mathcal{T}_0^{(0)}$ and $x \in \mathcal{T}_0^{(0)}$ separately. In the following, $\mathbb{D}^2$ is equipped with its usual Euclidean metric. Take $\varepsilon > 0$.

Case $x \in \mathbb{D}^2 - \partial\mathbb{D}^2$. Note the bound $\operatorname{mesh}(C, f_n\,|_{C^{(0)}}\,) \leq L/2^n$ and apply Lemma 4.6 with n sufficiently large so that $R_1(4K_2+1)L/2^n < \varepsilon$. For all y in the 2-cells of $\mathcal{T}_n$ that contain x, we find $d(\bar{f}(x), \bar{f}(y)) < \varepsilon$.

Case $x \in \partial\mathbb{D}^2 - \mathcal{T}_0^{(0)}$. Uniform continuity of f tells us that there exists $\delta > 0$ such that for all $a, b \in \partial\mathbb{D}^2$ with $d(a,b) < \delta$,

$$d(f(a), f(b)) \;<\; \frac{\varepsilon}{2R_1(4K_2+1)}$$

(whence in particular $d(f(a), f(b)) < \varepsilon/2$). We say that a 2-simplex $\Delta \subset \mathbb{D}^2$ of the tessellation $\mathcal{T}_0$ is *δ-small* if $\operatorname{diam}(\Delta) < \delta$. It follows that any δ-small 2-simplex Δ of $\mathcal{T}_0$ satisfies

$$\operatorname{mesh}(\Delta, f\,|_{\Delta^{(0)}}\,) \;\leq\; \frac{\varepsilon}{2R_1(4K_2+1)},$$

and so by Lemma 4.6 satisfies $\operatorname{diam} \bar{f}(\Delta) < \varepsilon/2$.

Only finitely many 2-simplices of the tessellation $\mathcal{T}_0$ fail to be $\delta/2$-small. So if $x \in \partial\mathbb{D}^2 - \mathcal{T}_0^{(0)}$ then we can find a $\delta' < \delta/2$ such that the δ'-neighbourhood $B(x;\delta')$ of x in $\mathbb{D}^2$ meets only $\delta/2$-small 2-simplices of $\mathcal{T}_0$. So if $y \in B(x;\delta')$ then there is a $\delta/2$-small 2-simplex Δ of $\mathcal{T}_0$ such that $y \in \Delta$. If v is a 0-cell of Δ then $v \in \partial\mathbb{D}^2$ and

$$d(x,v) \;\leq\; d(x,y) + d(y,v) \;\leq\; \delta' \;+\; \delta/2 \;\leq\; \delta.$$

It follows that

$$d(\bar{f}(x),\bar{f}(y)) \;\leq\; d(\bar{f}(x),\bar{f}(v)) + d(\bar{f}(v),\bar{f}(y)) \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon,$$

establishing continuity of $\bar{f}$ at $x \in \partial\mathbb{D}^2 \smallsetminus \mathcal{T}_0^{(0)}$.

Case $x \in \mathcal{T}_0^{(0)}$. Continuity of $\bar{f}$ for $x \in \mathcal{T}_0^{(0)}$ follows from continuity of f and $\bar{f}\Big|_{\mathbb{D}^2 \smallsetminus \mathcal{T}_0^{(0)}}$.

We now come to proving that if the asymptotic cones of X are 1-connected then $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ is bounded. So assume that the asymptotic cones $\mathrm{Cone}_\omega(X,\mathbf{e},\mathbf{s})$ are 1-connected for all $\mathbf{e}$, $\mathbf{s}$. In particular the cones are path connected and so (by Proposition 4.1) there are $\mu_1 \geq 0$ and $K_1 \in \mathbb{N}$ such that $\mathrm{Fill}^1_{\mu_1}(\ell) \leq K_1$ for all $\ell \geq 0$. We seek to show that there are $R_1 \in \mathbb{N}$ and $\mu_2 \geq \mu_1$ such that $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ is bounded. We will in fact show that we can take R_1 to be any integer greater than or equal to $1 + K_1$ (thereby justifying the coda of the statement of the proposition). Note that $K_1 \geq 2$ because we assumed X to be unbounded, and therefore R_1 will be at least 3 as is required in the definition of $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$.

Fix $R_1 := 1 + K_1$ and suppose (for a contradiction) that for all $\boldsymbol{\mu}$ in which $\mu_2 \geq \mu_1$, the function $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ fails to be bounded. Then for all $n \in \mathbb{N}$ there exists $(C_n,\gamma_n) \in \mathrm{Sph}^1_{\mathbf{R}}$ with an essential edge partition $(\hat{C}_n,\hat{\gamma}_n)$ satisfying: if there is a partition $(\bar{C}_n,\bar{\gamma}_n)$ of (C_n,γ_n) subject to $(\hat{C}_n,\hat{\gamma}_n)$ such that

$$\mathrm{mesh}(\bar{C}_n,\bar{\gamma}_n) \;\leq\; \frac{1}{2}\mathrm{mesh}(C_n,\gamma_n) + n, \tag{8}$$

then $\#_2(\bar{C}_n) \geq n$.

Since $\#_1(C_n) \leq R_1$ for all n, one combinatorial structure C amongst the C_n must occur (up to combinatorial isomorphism) for all n is some set J of ω-measure 1. We may as well take C_n actually to be C for all $n \in J$.

Let $s_n := \mathrm{mesh}(C_n,\gamma_n)$. Now we claim that $s_n \to \infty$ as $n \to \infty$. This is true because if we obtain $(\bar{C}_n,\bar{\gamma}_n)$ by coning off $\hat{C}_n$ to one of its 0-cells then

$$\mathrm{mesh}(\bar{C}_n,\bar{\gamma}_n) \;\leq M\; \mathrm{mesh}(\hat{C}_n,\hat{\gamma}_n) \;\leq M\; \left(\frac{1}{2}\mathrm{mesh}(C_n,\gamma_n) + \mu_1\right),$$

for some constant $M > 1$. But $\#_2(\bar{C}_n)$ is bounded and so it must be the case that for large n,

$$M\;\left(\frac{1}{2}\mathrm{mesh}(C_n,\gamma_n) + \mu_1\right) \;>\; \frac{1}{2}\mathrm{mesh}\,(C_n,\gamma_n) + n$$

to avoid (8) holding, whence it follows that $s_n \to \infty$.

Define $\mathbf{e} = (e_n)$ by making some arbitrary chooses of $e_n \in \mathrm{Im}(\gamma_n)$. Then we define $\gamma := (\gamma_n)$ which is a map $(C^{(0)},\star) \to (\mathrm{Cone}_\omega(X,\mathbf{e},\mathbf{s}),\mathbf{e})$ for some $\star \in C^{(0)}$, and $\mathrm{mesh}(C,\gamma) = 1$. The reason this definition makes sense is that $C_n = C$ for all n in the set J that has ω-measure

1. Note that the images $\gamma(e)$ of vertices $e \in C$ are a finite distance from $\mathbf{e}$ in $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ for the same reason as in the corresponding part of the proof of Proposition 4.1.

Similarly $\hat{\gamma} := (\hat{\gamma}_n)$ is a well defined map $\hat{C}^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ with $\mathrm{mesh}(\hat{C}, \hat{\gamma}) \leq \frac{1}{2}$, where $\hat{C}$ is a refinement of C that is combinatorially equivalent to $\hat{C}_n$ for all n in some set $J_1 \subseteq \mathbb{N}$ of ω-measure 1.

Now in the manner of the proof of path connectedness of the asymptotic cones in Proposition 4.1 we can extend $\hat{\gamma}$ to a continuous map $f : (C, \star) \to (\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}), \mathbf{e})$. The 1-complex C is homeomorphic to the 1-sphere $\mathbb{S}^1$ and so, by hypothesis, we can extend f to a continuous map $\bar{f} : D \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ of f, where D is a 2-cell with $\partial D = C$.

We look for an $\mathbf{R}$-combinatorial refinement $\bar{C}$ of D with $\partial \bar{C} = \hat{C}$ as combinatorial complexes, such that we can express $\bar{f}\,|_{\bar{C}^{(0)}} : \bar{C}^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ as $\bar{f}\,|_{\bar{C}^{(0)}} = (\bar{\gamma}_n)$ for a sequence of maps $\bar{\gamma}_n : \bar{C}^{(0)} \to X$ with the following properties. For ω-infinitely many n, the pair $(\bar{C}, \bar{\gamma}_n)$ will be a partition of (C_n, γ_n) subject to its essential edge partition $(\hat{C}_n, \hat{\gamma}_n)$ and will satisfy $\mathrm{mesh}(\bar{C}, \bar{\gamma}_n) \leq \frac{1}{2}\mathrm{mesh}(C, \gamma_n)$. And every interior 1-cell e in $\bar{C}$ will satisfy

$$\mathrm{mesh}(e, f\,|_e) \;\leq\; \frac{1}{2} - \frac{1}{8}.$$

This will lead to a contradiction as we now explain. By definition of distance in $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$,

$$\lim_\omega \frac{1}{s_n}\mathrm{mesh}(e, \bar{\gamma}_n\,|_e) \;=\; \mathrm{mesh}(e, f\,|_e).$$

So for all $\nu > 0$

$$\omega \left\{ n \;\middle|\; \frac{1}{s_n}\mathrm{mesh}\,(e, \bar{\gamma}_n\,|_e) < \frac{1}{2} - \frac{1}{8} + \nu \right\} \;=\; 1,$$

and thus taking $\nu < \frac{1}{8}$

$$\omega \left\{ n \;\middle|\; \mathrm{mesh}\,(e, \bar{\gamma}_n\,|_e) \leq \frac{1}{2} s_n \right\} \;=\; 1.$$

But $s_n = \mathrm{mesh}(C, \gamma_n)$ and there are only finitely many 1-cells e in the interior of $\bar{C}$. So there will be a set $J_2 \subseteq \mathbb{N}$ of ω-measure 1 such that for all 1-cells e in the interior of $\bar{C}$,

$$\forall n \in J_2, \quad \mathrm{mesh}\,(e, \bar{\gamma}_n\,|_e) \;\leq\; \frac{1}{2}\mathrm{mesh}(C, \gamma_n).$$

The remaining 1-cells e of $\bar{C}$ are in $\partial \bar{C}$ and satisfy

$$\forall n \in J_1, \quad \mathrm{mesh}(e, \bar{\gamma}_n\,|_e) \;=\; \mathrm{mesh}(e, \hat{\gamma}_n\,|_e) \;\leq\; \mathrm{mesh}(\hat{C}, \hat{\gamma}_n) \;\leq\; \frac{1}{2}\mathrm{mesh}(C, \gamma_n) + \mu_1.$$

So for $n \in J_1 \cap J_2$, which will be a set of ω-measure 1 and hence will be infinite, we will find $\mathrm{mesh}(\bar{C}, n) \leq \frac{1}{2}\mathrm{mesh}(C, \gamma_n) + \mu_1$. Therefore when n is greater than $\#_2(\bar{C})$ and μ_1, we will have our contradiction with (8).

It remains to explain how to find such a $\bar{C}$. Via a choice of homeomorphism between D and the standard Euclidean 2-disc, D inherits a metric d. Uniform continuity allows us to find an $\varepsilon > 0$ such that for $a, b \in D$

$$d(a, b) < \varepsilon \quad\Rightarrow\quad d(\bar{f}(a), \bar{f}(b)) \leq \frac{1}{2} - \frac{1}{8}.$$

This would make it easy to find our $\mathbf{R}$-combinatorial refinement $\bar{C}$ of D if we overlooked the requirement that $\partial\bar{C} = \hat{C}$. We give a special care to the construction of $\bar{C}$ in a neighbourhood of the boundary of D to remedy this.

We fix a constant $\alpha > 0$ such that $3\,\alpha \leq \varepsilon$ and for $a, b \in D$

$$d(a,b) < \alpha \quad \Rightarrow \quad d(\bar{f}(a), \bar{f}(b)) \leq \frac{1}{16}.$$

Again such α exists by uniform continuity of $\bar{f}$ on D.

The idea is to repeatedly take essential edge partitions (as is the proof of Proposition 4.1), refining the boundary of D into 1-cells of length at most α. Roughly speaking, these 1-cells are then projected a distance at most α into the interior of D as depicted in Figure 4. The resulting innermost edges then have length at most ε. The innermost region of D can then be triangulated and then the construction close to the boundary together with this triangulation form $\bar{C}$.

More explicitly we first recall that $f : C \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ is constructed by the means used in the proof of path-connectedness of the asymptotic cones in Proposition 4.1 . This amounts to repeatedly partitioning $(e, \gamma\,|_e)$ for every 1-cell e in C. (Recall that we achieve this by expressing $\gamma\,|_e$ as an ω-limit of maps $\rho_n : e^{(0)} \to X$, then partitioning each (e, ρ_n) within the constraints of $\mathrm{Fill}^1_{\mu_1}(\mathrm{mesh}(e, \rho_n))$, and then retrieving a map into the cone by taking an ω-limit). The first partition of the 1-cells refines C to $\hat{C}$ and extends $\gamma : C^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ to $\hat{\gamma} : \hat{C}^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. We define $(C^0, \gamma^0) := (C, \gamma)$ and $(C^1, \gamma^1) := (\hat{C}, \hat{\gamma})$. Subsequent partitions of all the 1-cells give us (C^2, γ^2), $(C^3, \gamma^3), \ldots$ such that each 1-cell of C^i is refined in C^{i+1} into at most K_1 1-cells (where K_1 is the minimal upper bound on $\mathrm{Fill}^1_{\mu_1}$) and γ^{i+1} is an extension of γ^i. Recall that $\mathrm{mesh}(C^0, \gamma^0) = 1$ and $\mathrm{mesh}(C^{i+1}, \gamma^{i+1}) \leq \frac{1}{2}\mathrm{mesh}(C^i, \gamma^i)$ for $i \geq 0$, so in particular $\mathrm{mesh}(C^k, \gamma^k) \leq 1/2^k$ for all $k \geq 0$.

The C^i provide combinatorial structures $\phi_i : C^i \overset{\cong}{\to} \partial D$ for ∂D. For $i = 1, 2, \ldots, r-1$ the 2-complex C^{i+1} is a refinement of C^i. So there is a natural homeomorphism $C^i \to C^{i+1}$ that embeds the 0-skeleta $C^{i,(0)}$ into $C^{i+1,(0)}$, and is such that the composition $C^i \to C^{i+1} \overset{\phi_{i+1}}{\to} \partial D$ is equal to the homeomorphism $C^i \overset{\phi_i}{\to} \partial D$. Recall that f is then defined to be a limit (which is proved to exist using an appeal to the completeness of the cone) of the sequence of maps $\gamma^i \circ \phi_i^{-1}$ (restricted to $\phi_i(C^{i,(0)})$), and is, in fact, an extension of each of these maps.

The images under ϕ_i of the 1-cells of C^i are subsets of ∂D and hence define rectifiable paths with respect to the metric d. So we can assume (for simplicity) that in each of the successive refinements C^i to C^{i+1}, any 1-cell e in C^i is refined into 1-cells e_j in C^{i+1} with $\mathrm{length}(e_j) \leq \frac{1}{2}\mathrm{length}(e)$. It follows that there is some r such that the 1-cells of C^r have length at most α.

We construct (singular) *annular* 2-complexes $A_1, A_2, \ldots, A_r$ such that, with respect to some embedding in the Euclidean plane, the *outer boundary* of A_i is combinatorially isomorphic to C^1 and the *inner boundary* is combinatorially isomorphic to C^i. We define $A_1 := C^1$ and construct each A_{i+1} from A_i as follows.

Let e^1 be a 1-cell in C^i, the inner boundary of A_i. Then e^1 is refined to a 1-complex $\overline{e^1}$ in C^{i+1}, and $\#_1\left(\overline{e^1}\right) \leq K_1$. Let e^2 be a 2-cell whose boundary is given a combinatorial structure with one more 1-cell than $\overline{e^1}$ (so $\#(\partial e^2) \leq K_1 + 1 = R_1$). Attach e^2 to e^1 by identifying one of the 1-cells of e^2 with e^1. Attach 2-cells in this way to every 1-cell in the inner boundary of A_i to produce A_{i+1}.

Now, by construction, the 1-skeleton $A_r^{(1)}$ of A_r can be regarded as $\bigcup_{i=1}^{r} C^i$ with C^i meeting C^j $(i \neq j)$ only at 0-cells. Also recall from above that there are homeomorphisms $\phi_i : C^i \to \partial D$. Now we can find an embedding $\psi : A_r \to D$ in such a way that $\psi|_{C^1} = \phi_1$ and

$$d(\phi_i(x), \psi(x)) \leq \alpha$$

for all $i = 2, 3, \ldots, r$ and all points $x \in C^i \subseteq A_r$. In effect, we are pushing C^i a distance at most α away from ∂D. An example of the result is illustrated in Figure 4.

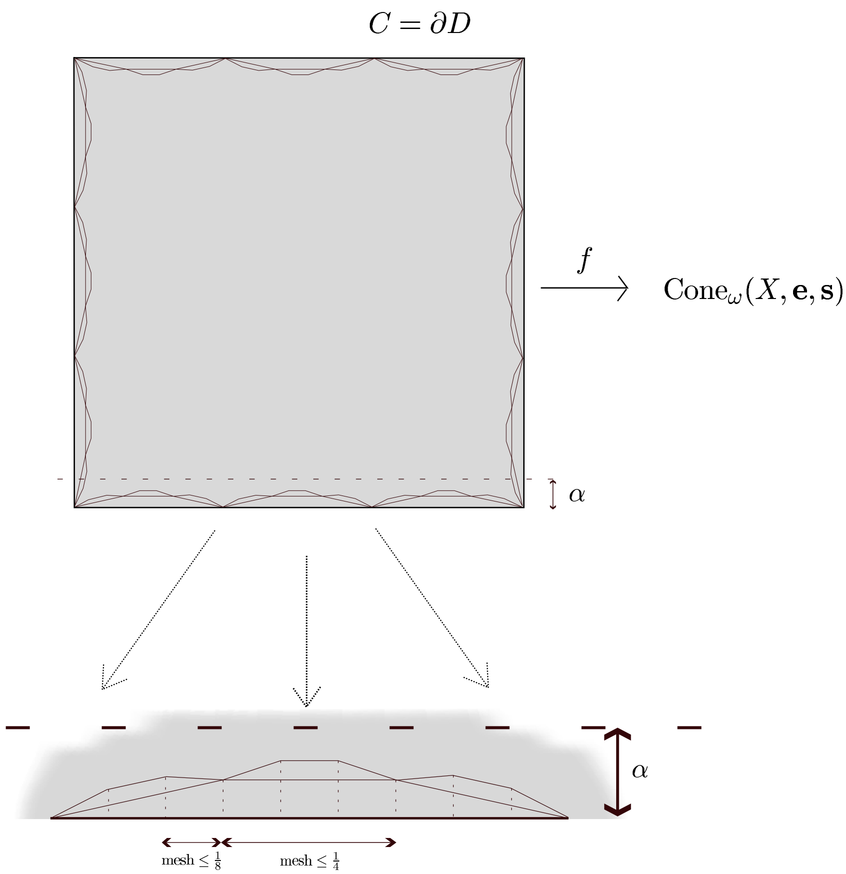

Figure 4: Partitioning γ at the boundary (with $K_1 = 3$, $R_1 = 4$ and $r = 3$).

Consider a 1-cell e in $C^i \subseteq A_r$ when $2 \leq i \leq r$. Let a and b be the endpoints of e. Then

$$\begin{aligned} \max\{\, d(\psi(a), \phi_i(a)),\ d(\psi(b), \phi_i(b)) \,\} &\leq \alpha, \quad \text{and} \\ d(\bar{f}(\phi_i(a)), \bar{f}(\phi_i(b))) \leq \mathrm{mesh}(C^i, \gamma^i) &= \frac{1}{2^i} \leq \frac{1}{4}, \end{aligned}$$

and we can now deduce that

$$\begin{aligned} d(\bar{f}(\psi(a)), \bar{f}(\psi(b))) &\leq d(\bar{f}(\psi(a)), \bar{f}(\phi_i(a))) + d(\bar{f}(\phi_i(a)), \bar{f}(\phi_i(b))) + d(\bar{f}(\phi_i(b)), \bar{f}(\psi(b))) \\ &\leq \frac{1}{16} + \frac{1}{4} + \frac{1}{16} \\ &= \frac{1}{2} - \frac{1}{8}. \end{aligned}$$

Moreover, if e is a 1-cell in C^r then

$$\begin{aligned} d(\psi(a), \psi(b)) &\leq d(\psi(a), \phi_r(a)) + d(\phi_r(a), \phi_r(b)) + d(\phi_r(b), \psi(b)) \\ &\leq 3\,\alpha \\ &\leq \varepsilon. \end{aligned}$$

So we can now triangulate $D \smallsetminus \operatorname{Im}\psi$ with 2-simplices of diameter at most ε in such a way that we produce the combinatorial structure $\bar{C}$ for D that we seek. As discussed earlier this leads to a contradiction as required. ■

4.3 Characterising higher connectedness

In this section we complete the proof of Theorem A. We establish the characterisation given in that theorem by presenting the two implications separately. Our arguments are generalisations to higher dimensions of those used in §4.2. Recall that to prove 1-connectedness of the asymptotic cones in Proposition 4.5 we used a tessellation of the 2-disc $\mathbb{D}^2$ by 2-simplices whose vertices are all in $\partial\mathbb{D}^2$. We will need a higher dimensional analogue: a tessellation $\mathcal{T}_0$ of an $(N+1)$-disc $\mathbb{D}^{N+1}$ by $(N+1)$-simplices with vertices in $\partial\mathbb{D}^{N+1}$. For $N \geq 3$ the ideal tessellation of $\mathbb{D}^{N+1}$ (viewed as the Klein disc model of $\mathbb{H}^{N+1}$) cannot be constructed by repeated reflection in the faces of an ideal $(N+1)$-simplex as we did in dimension 2. However we do not require such regularity. All we need is the following property defined with respect to the standard Euclidean metric on $\mathbb{D}^{N+1}$.

> *Given $\delta > 0$, only finitely many $(N+1)$-simplices $\Delta \subset \mathbb{D}^{N+1}$ in the tessellation $\mathcal{T}_0$ have $\operatorname{diam}(\Delta) > \delta$.*

It is possible to provide some *ad hoc* argument to demonstrate the existence of such tessellations. Alternatively, the following results about hyperbolic manifolds suffice. The existence of open, complete, hyperbolic $(N+1)$-manifolds of finite volume in all dimensions follows from results of Millson [32] for example (see also [39, page 571]). Then Epstein and Penner prove in [15] that such a manifold is obtained from a finite collection of ideal polyhedra by identifying faces. The lifts of these polyhedra give a tessellation of the universal cover $\mathbb{H}^{N+1}$ by finitely many types of ideal polyhedra. Identify $\mathbb{D}^{N+1}$ with the Klein model for $\mathbb{H}^{N+1}$. We can decompose the ideal polyhedra into ideal simplices in a consistent manner (although not necessarily into hyperbolic ideal simplices else some may have zero volume), to produce $\mathcal{T}_0$.

A compactness argument tells us that only finitely many of the ideal $(N+1)$-simplices meet a ball of a given radius $R > 0$ about the origin in $\mathbb{D}^{N+1} = \mathbb{H}^{N+1}$. Thus the condition displayed above is satisfied.

Now recall that the $(N+1)$-dimensional filling function $\mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$ is defined with reference to the constants $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{N+1}$ and $R_1, R_2, \ldots, R_N \in \mathbb{N}$ (with each $R_i \geq i+2$). We are ready to prove one direction of Theorem A.

Proposition 4.7. *Let X be a metric space, let ω be a non-principal ultrafilter, and $N \geq 0$. Suppose there exist $\mathbf{R}$, $\boldsymbol{\mu}$ such that the filling functions $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$ are bounded. Then the asymptotic cones $\mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$ are N-connected for all $\mathbf{e}$ and $\mathbf{s}$.*

Proof. We fix $\mathbf{e}$ and $\mathbf{s}$ and prove that under the hypotheses of the proposition, $\pi_N \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}) = 0$. We follow closely the method used in the two dimensional case – that is, in the proof of Proposition 4.5 .

Consider a continuous map $f : (\mathbb{S}^N, \star) \to (\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}), \mathbf{e})$ from the boundary $\mathbb{S}^N = \partial\mathbb{D}^{N+1}$ of a Euclidean $(N+1)$-disc $\mathbb{D}^{N+1}$ to X. Let $L := \mathrm{diam} f(\mathbb{S}^N)$, which is finite because $f(\mathbb{S}^N)$ is compact. We seek to extend f to a continuous map $\bar{f} : (\mathbb{D}^{N+1}, \star) \to (\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}), \star)$.

As described at the start of this section, take a tessellation $\mathcal{T}_0$ of $\mathbb{D}^{N+1}$ by $(N+1)$-simplices whose vertices all lie in $\partial\mathbb{D}^{N+1}$. The vertices $\mathcal{T}_0^{(0)}$ of $\mathcal{T}_0$ form a dense subset of $\partial\mathbb{D}^{N+1}$. Define $f_0 : \mathcal{T}_0^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ to be $f\Big|_{\mathcal{T}_0^{(0)}}$. So $\mathrm{mesh}(\mathcal{T}_0, f_0) \leq L$.

The number of N-cells in the boundary of an $(N+1)$-simplex Δ is $N+2$. By definition $R_N \geq N+2$ and so it follows that the standard combinatorial structure of $\partial\Delta$ is amongst the structures of N-spheres that can be used to attach $(N+1)$-cells when constructing $\mathbf{R}$-combinatorial $(N+1)$-complexes. Therefore for each $(N+1)$-simplex Δ of $\mathcal{T}_0$ it is the case that $(\Delta, f_0\,|_{\Delta^{(0)}}) \in \mathrm{Sph}^N_{\mathbf{R}}$.

Generalising the 2-dimensional argument (*i.e.* the proof of Proposition 4.5) we now produce successive *refinements* $\mathcal{T}_1, \mathcal{T}_2, \ldots$ of $\mathcal{T}_0$, and define a sequence maps $f_n : \mathcal{T}_n^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ such that each f_{k+1} extends f_k. In the manner allowed by $\mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}$, each N-cell C of $\mathcal{T}_k$ will be refined to some combinatorial N-complex $\bar{C}$ in $\mathcal{T}_{k+1}$, and

$$\mathrm{mesh}(\bar{C}, f_{k+1}\,|_{\bar{C}^{(0)}}) \;\leq\; \frac{1}{2}\mathrm{mesh}(C, f_k\,|_{C^{(0)}}).$$

So in particular $\mathrm{mesh}(\mathcal{T}_{k+1}, f_{k+1}) \leq \frac{1}{2}\mathrm{mesh}(\mathcal{T}_k, f_k)$, from which it will follow that $\mathrm{mesh}(\mathcal{T}_n, f_n) \leq \frac{L}{2^n}$ for all n.

Fix $k \geq 0$. It suffices to describe the process of *refining* $f_k : \mathcal{T}_k^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ to $f_{k+1} : \mathcal{T}_{k+1}^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. Choose $f_{k,i} : \mathcal{T}_k^{(0)} \to X$ so that $(f_{k,i})_{i\in\mathbb{N}} = f_k$.

First for each fixed i we make essential boundary partitions (as defined in §3.3 above) of $(e^{N+1}, f_{k,i}\,|_{e^{N+1}})$ for all the closed $(N+1)$-cells e^{N+1} of $\mathcal{T}_k$. Recall that an essential boundary partition of $(e^{N+1}, f_{k,i}\,|_{e^{N+1}})$ is made by first partitioning the 1-cells, then the 2-cells (*subject to* the partitions of the 1-cells), then the 3-cells (*subject to* the partitions of the 2-cells), and so on until finally partitioning the N-cells. It follows that we can take essential boundary partitions of the $(e^{N+1}, f_{k,i}\,|_{e^{N+1}})$ in a way that agrees on any j-cell ($j = 1, 2, \ldots, N$) common to two $(N+1)$-cells of $\mathcal{T}_k$. The result is a refinement $\hat{\mathcal{T}}_{k,i}$ of $\mathcal{T}$ and an extension $\hat{f}_{k,i} : \hat{\mathcal{T}}_{k,i}^{(0)} \to X$ of $f_{k,i}$. This satisfies

$$\mathrm{mesh}(\hat{\mathcal{T}}_{k,i}, \hat{f}_{k,i}) \;\leq\; \frac{1}{2}\mathrm{mesh}(\mathcal{T}_k, f_{k,i}) + \mu_N.$$

It follows from the hypothesis that $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}$ are all bounded, that each N-cell of $\mathcal{T}_k$ is refined into a bounded number of N-cells to produce $\hat{\mathcal{T}}_{k,i}$.

Suppose C is one of the $(N+1)$-simplices forming the tessellation $\mathcal{T}_k$. Let $\hat{C}_i$ be its refinement in $\hat{\mathcal{T}}_{k,i}$. Consider the pair (C, γ_i), where $\gamma_i := f_{k,i}\,|_{C^{(0)}}$ and consider its essential boundary partition $(\hat{C}_i, \hat{\gamma}_i)$, where $\hat{\gamma}_i := \hat{f}_{k,i}\Big|_{\hat{C}_i^{(0)}} : \hat{C}_i^{(0)} \to X$. By hypothesis we can find a partition $(\bar{C}_i, \bar{\gamma}_i)$ of (C, γ_i) subject to $(\hat{C}_i, \hat{\gamma}_i)$, where $\bar{C}_i$ is some $\mathbf{R}$-combinatorial $(N+1)$-disc with $\hat{C}_i = \partial\bar{C}_i$, with $\#(\bar{C}_i) \leq K_{N+1}$, and $\mathrm{mesh}(\bar{C}_i, \bar{\gamma}_i) \leq \frac{1}{2}\mathrm{mesh}(C, \gamma_i) + \mu_{N+1}$.

This process of refinement is repeated over all of the $(N+1)$-cells in $\hat{\mathcal{T}}_{k,i}$, producing $\bar{f}_{k,i} : \bar{\mathcal{T}}_{k,i}^{(0)} \to X$. The partition agrees across N-cells common to two $(N+1)$-cells of $\hat{\mathcal{T}}_{k,i}$ because the partitions of the (C, γ_i) are constructed *subject to* essential boundary partitions.

We would now like to define $f_{k+1} : \mathcal{T}^{(0)}_{k+1} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ to be an ω-limit of $(\bar{f}_{k,i})$. The tessellation $\mathcal{T}_{k+1}$ is obtained by refining $\mathcal{T}_k$: first refining the 1-cells, then the 2-cells and so on through the dimensions until finally refining the $(N+1)$-cells. A given 1-cell e^1 of $\mathcal{T}_k$ is refined in $\hat{\mathcal{T}}_{k,i}$ into one of finitely many combinatorial structures. So, up to combinatorial equivalence, one of these combinatorial structure occurs for all i in some set of ω-measure 1. Refine all the 1-cells of $\mathcal{T}_k$ accordingly. For a given 2-cell e^2 in $\mathcal{T}_k$, there is a set S_{e^2} of ω-measure 1 such that for all $i \in S_{e^2}$ there is one combinatorial structure (up to combinatorial equivalence) into which e^2 is refined in $\hat{\mathcal{T}}_{k,i}$, and further this refinement agrees with the refinements of the boundary 1-cells. So next refine all the 2-cells of $\mathcal{T}_k$ accordingly. Proceed similarly through the dimensions until finally the $(N+1)$-cells have been refined.

Thus

$$f_{k+1} \ := \ (\bar{f}_{k,i})_{i \in \mathbb{N}} : \mathcal{T}^{(0)}_{k+1} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$$

is well defined. Also, as required, we see that for every $(N+1)$-cell C of $\mathcal{T}_k$, $\mathrm{mesh}(\bar{C}, f_{k+1}|_{\bar{C}^{(0)}}) \leq \frac{1}{2}\mathrm{mesh}(C, f_k|_{C^{(0)}})$ where $\bar{C}$ is the refinement of C in $\mathcal{T}^{k+1}$.

We are now in a position to define $\bar{f} : \mathbb{D}^{N+1} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. On the boundary $\partial\mathbb{D}^{N+1} = \mathbb{S}^N$ we set $\bar{f} := f$. Given $x \in \mathbb{D}^{N+1} - \partial\mathbb{D}^{N+1}$, let x_n be a 0-cell on the boundary of one of the $(N+1)$-cells of $\mathcal{T}_n$ containing x. Define $\bar{f}(x) := \lim_{n\to\infty} f_n(x_n)$.

Since the asymptotic cone is complete, to prove $\lim_{n\to\infty} f_n(x_n)$ exists it is enough to show that $(f_n(x_n))$ is a Cauchy sequence. The argument we use is the same as that of the 2-dimensional case. The key observation is that

$$d(f_n(x_n), f_{n-1}(x_{n-1})) \ \leq \ 2K_{N+1}.R\frac{L}{2^n},$$

where $R := \prod_{i=1}^N R_i$. Note that R bounds the number of 1-cells in a combinatorial structure for an N-sphere used in attaching $(N+1)$-cells in an $\mathbf{R}$-combinatorial complex. We also note that $\bar{f}$ is independent of the choice of (x_n).

The following lemma will be useful for proving continuity of $\bar{f}$.

Lemma 4.8. *Suppose $C \subset \mathbb{D}^{N+1}$ is one of the closed $(N+1)$-cells constituting $\mathcal{T}_n$. Then*

$$\mathrm{diam}\,\bar{f}(C) \ \leq \ R(4K_{N+1}+1).\,\mathrm{mesh}(C, f_n|_{C^{(0)}}).$$

The proof of this lemma runs just as that of Lemma 4.6 . Continuity of $\bar{f}$ can also be proved by a similar means to the 2-dimensional argument. The case that requires attention is proving continuity at $x \in \partial\mathbb{D}^{N+1}$. But notice that the observation discussed at the start of this section (that given $\delta > 0$, the tessellation $\mathcal{T}_0$ of $\mathbb{D}^{N+1}$ includes only finitely many $(N+1)$-simplices that fail to be δ-small) is precisely what is required for the argument to go through with the dimension increased from 2 to $N+1$. ∎

The following proposition provides the other implication required to complete the proof of Theorem A.

Proposition 4.9. *Let X be a metric space, let ω be a non-principal ultrafilter, and $N \geq 0$. Suppose the asymptotic cones $\mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$ are N-connected for all $\mathbf{e}$ and $\mathbf{s}$. Then there exist $\mathbf{R}$ and $\boldsymbol{\mu}$ such that the filling functions $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$ are bounded.*

Moreover we can, in fact, take $\mathbf{R} = (R_k)$ to be any sequence satisfying the following recursive condition. Assume $R_1, R_2, \ldots, R_{k-1}$ are defined and there are $\mu_k \geq \mu_{k-1} \geq \ldots \geq$

$\mu_1 \geq 0$ and $K_k \in \mathbb{N}$ such that $\mathrm{Fill}^k_{\mathbf{R},\boldsymbol{\mu}}(\ell) \leq K_k$ for all $\ell > 0$. Then R_k can be any integer greater than or equal to:

$$\max \left\{ k+2, \quad 1+R_{k-1}+K_k, \quad 1+R_{k-1}+\prod_{i=1}^{k+1} i, \quad 1+R_{k-1}+\prod_{i=1}^{k-1} R_i \right\}.$$

Observe that, in the statement of this proposition, R_k is at least $k+2$, which is the number of k-cells in the boundary of an $(k+1)$-simplex. So the k-dimensional $\mathbf{R}$-combinatorial complexes include the triangular complexes. Also R_k is at least $1+R_{k-1}$ greater than each of the following:

- K_k,
- $\prod_{i=1}^{k+1} i$, that is, the number of k-simplices in the first barycentric subdivision of a k-simplex,
- $\prod_{i=1}^{k-1} R_i$, which is, a sufficient number of k-simplices to triangulate any closed k-cell e^k in an $\mathbf{R}$-combinatorial structure (*i.e.* any k-cell e^k whose boundary ∂e^k is a $(k-1)$-combinatorial complex and $\#_{j-1}(\partial e^j) \leq R_{j-1}$ for every closed j-cell e^j in ∂e^k (for $j = 2, 3, \ldots, k$).)

Notice that R_k is defined in terms of k and the constants $K_1, K_2, \ldots, K_k$ and $R_1, R_2, \ldots, R_{k-1}$, so it makes sense to define $\mathrm{Fill}^{k+1}_{\mathbf{R},\boldsymbol{\mu}} : [0,\infty) \to \mathbb{N} \cup \{\infty\}$ *with respect to* the sequence $\mathbf{R}$ (and $\boldsymbol{\mu}$).

Proof of Proposition 4.9. We prove the proposition by induction on N. We have established the cases $N = 0, 1$ in Propositions 4.1 and 4.5. (We appeal here to the observation made in the proof of Proposition 4.5 that R_1 can be taken to be any integer such that $\mathrm{Fill}^1_{\mu_1}(\ell) \leq R_1 - 1$ for all $\ell \geq 0$.) Let us now address the induction step.

Suppose $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ is N-connected for all $\mathbf{e}$ and $\mathbf{s}$. In particular $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ is $(N-1)$-connected for all $\mathbf{e}$ and $\mathbf{s}$. So by induction hypothesis we can assume that there are some $R_1, R_2, \ldots, R_{N-1}$ satisfying the recursive condition in the proposition, and $\mu_N \geq \mu_{N-1} \geq \ldots \mu_1 \geq 0$ such that $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}$ are bounded. Take any integer R_N satisfying the condition in the proposition. Suppose that, for all choices of $\mu_{N+1} \geq \mu_N$, the $(N+1)$-dimensional filling function $\mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$ fails to be bounded. Then given any $n \in \mathbb{N}$, there exists $(C_n, \gamma_n) \in \mathrm{Sph}^N_{\mathbf{R}}$ with an essential boundary partition $(\hat{C}_n, \hat{\gamma}_n)$ satisfying: if there is a partition $(\bar{C}_n, \bar{\gamma}_n)$ of (C_n, γ_n) subject to $(\hat{C}_n, \hat{\gamma}_n)$ such that

$$\mathrm{mesh}(\bar{C}_n, \bar{\gamma}_n) \;\leq\; \frac{1}{2}\mathrm{mesh}(C_n, \gamma_n) + n, \tag{9}$$

then $\#_{N+1}(\bar{C}_n) \geq n$.

There is some combinatorial complex C that is combinatorially equivalent to C_n for ω-infinitely many n, because C_n can only take one of finitely many combinatorial types by virtue of (C_n, γ_n) being in $\mathrm{Sph}^N_{\mathbf{R}}$.

Let $s_n := \mathrm{mesh}(C_n, \gamma_n)$. Then $s_n \to \infty$ for the same reason as given in the proof of Proposition 4.5. Let e_n be chosen in $\mathrm{Im}(\gamma_n)$. Then $\gamma := (\gamma_n)$ is a well defined map $(C^{(0)}, \star) \to (\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}), \mathbf{e})$ for some vertex $\star$ of C, and $\mathrm{mesh}(C, \gamma) = 1$.

Also there is a $\hat{C}$ such that $\hat{C}_n$ is a combinatorial structure equivalent to $\hat{C}$ for ω-infinitely many n. So $\hat{\gamma} := (\hat{\gamma}_n)$ is a well defined map $\hat{C}^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ with $\mathrm{mesh}(\hat{C}, \hat{\gamma}) \leq \frac{1}{2}$.

Now because of the hypothesis that $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ is N-connected, we can extend γ along first the 1-cells then the 2-cells and eventually the $(N-1)$-cells to a continuous map $f : C \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$. And because $C \cong \mathbb{S}^N$ we can extend f to a continuous map $\bar{f} : D \to X$ where D is an $(N+1)$-cell with $\partial D = \hat{C}$. As we will discuss later it will be important that the extensions of γ producing f are made by the methods used to prove $(N-1)$-connectedness of the cones. That is, we use repeated refinements of 1-cells, the 2-cells, *etc.* , in the manner constrained by the bounds on $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}$.

As in the 2-dimensional argument (the proof of Proposition 4.5) we produce an $\mathbf{R}$-combinatorial decomposition $\bar{C}$ of D with $\partial\bar{C} = \hat{C}$ in a way that pulls back to produce partitions of ω-infinitely (C_n, γ_n) subject to $(\hat{C}_n, \hat{\gamma}_n)$, with $\mathrm{mesh}(\hat{C}_n, \bar{\gamma}_n) \leq \frac{1}{2}\mathrm{mesh}(C_n, \gamma_n) + \mu_N$. This will then provide the required contradiction. Again it will be enough to find $\bar{C}$ with $\mathrm{mesh}(\bar{C}, \bar{f}\,|_{\bar{C}^{(0)}}) \leq \frac{1}{2} - \frac{1}{8}$.

We give D a metric d inherited from a choice of homeomorphism to the Euclidean $(N+1)$-disc $\mathbb{D}^{N+1}$ and then uniform continuity of $\bar{f} : D \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ tells us that there exists $\varepsilon > 0$ such that

$$d(a, b) < \varepsilon \;\Rightarrow\; d(\bar{f}(a), \bar{f}(b)) \;\leq\; \frac{1}{2} - \frac{1}{8}.$$

Further, we shall need a constant $\alpha > 0$ such that $3\alpha \leq \varepsilon$ and for $a, b \in D$

$$d(a, b) < \alpha \quad\Rightarrow\quad d(\bar{f}(a), \bar{f}(b)) \leq \frac{1}{16};$$

again such an α exists by uniform continuity of $\bar{f}$ on D.

The obvious approach is to exploit ε to produce $\bar{C}$. However as in the 2-dimensional case we have to do some work to ensure we can find $\bar{C}$ with $\partial\bar{C} = \hat{C}$. In the 2-dimensional case we needed *annuli* $A_1, A_2, \ldots, A_r$. In this more general setting the $A_1, A_2, \ldots, A_r$ will be $(N+1)$-complexes that topologically are non-uniform thickenings of $\mathbb{S}^N$.

In the construction of $f : C \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ above we used the hypothesis that $\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ is $(N-1)$-connected. But more particularly we extend γ to f by repeated use of the filling functions $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^N_{\mathbf{R},\boldsymbol{\mu}}$, via the same means as are used to show that the boundedness of these filling functions imply the cones are $(N-1)$-connected. It is necessary for us to monitor exactly how this works.

Recall that $\gamma : C^{(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ was expressed as $\gamma = (\gamma_n)$ where each γ_n is a map $C_n^{(0)} \to X$. In the first place the essential boundary partitions $(\hat{C}_n, \hat{\gamma}_n)$ of each of the (C_n, γ_n) give (in the ω-limit) a refinement $\hat{C}$ of C and an extension $\hat{\gamma}$ of γ. Define $(C^0, \gamma^0) := (C, \gamma)$ and $(C^1, \gamma^1) := (\hat{C}, \hat{\gamma})$. Successive refinements C^k and extensions $\gamma^k : C^{k,(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$ are produced as follows. Given C^k and γ^k, express γ^k as $(\gamma^k_n)_{n\in\mathbb{N}}$, where each γ^k_n is a map $C^{k,(0)} \to X$. Then for every closed 1-cell e^1 of C^k partition $(e^1, \gamma^k_n\,|_{e^1})$ as allowed by $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}$. So e^1 is refined into at most $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}(\mathrm{mesh}(C^k, \gamma^k_n\,|_{e^1}))$ 1-cells. Next partition the 2-cells as per $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$, and then the 3-cells and so on until the N-cells have been partitioned. The result is a refinement C^{k+1} of C^k, where C^{k+1} is the combinatorial structure that occurs for ω-infinitely many n. Also this produces $\gamma^{k+1} : C^{k+1,(0)} \to \mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s})$, an extension of γ^k. It will be the case that $\mathrm{mesh}(C^{k+1}, \gamma^{k+1}) \;\leq\; \frac{1}{2}\mathrm{mesh}(C^k, \gamma^k)$ and thus

$$\mathrm{mesh}(C^r, \gamma^r) \;\leq\; \frac{1}{2^r}\mathrm{mesh}(C, \gamma) \;=\; \frac{1}{2^r}.$$

The C^i provide combinatorial structures $\phi_i : C^i \overset{\cong}{\to} \partial D$ for ∂D. The N-complex C^{i+1} is a refinement of C^i. So there is a homeomorphism $C^i \overset{\cong}{\to} C^{i+1}$ such that the composition

$C^i \overset{\cong}{\to} C^{i+1} \overset{\phi_{i+1}}{\to} \partial D$ is the homeomorphism $C^i \overset{\phi_i}{\to} \partial D$. Then f is defined using a limit and is an extension of each of each of the maps $\gamma^i \circ \phi_i^{-1}$ (restricted to $\phi_i(C^{i,(0)})$).

The approach we would now like to take is to find r sufficiently large that the N-cells of C^r have diameter (with respect to d) at most ε. This method worked in the 2-dimensional case. However in higher dimensions there is a small added complication. Recall that in the 2-dimensional argument as we refined each 1-cell of C^k, we were able to assume that its length was at least halved. But in higher dimensions, as we produce $C^1, C^2, \ldots$, we cannot assume that the N-cells (in the successively refined combinatorial structure for ∂D) decrease in diameter.

However Lemma 4.8 applies to the N-cells of $C^1, C^2, \ldots$ and allows us to deduce that if e^N is an N-cell of C^r then

$$\mathrm{diam} f(e^N) \ \leq \ R(4K_N + 1).\mathrm{mesh}(C^r, \gamma^r) \ \leq \ R(4K_N + 1).\frac{1}{2^r}.$$

Take r sufficiently large that $R(4K_N + 1)/2^r \leq 1/4$. Then for every N-cell of C^r we find $\mathrm{diam} f(e^N) \leq 1/4$. Now

$$\#_N(C^r) \ \leq \ \#_N(C).{K_N}^r \ \leq \ R_N {K_N}^r.$$

Each N-cell of C^r can be triangulated. This can be achieved by first triangulating its 2-cells, then the 3-cells, *etc.* , and then finally the N-cells. This requires fewer than $\prod_{i=1}^{N-1} R_i$ N-simplices. Let C^{r+1} be the resulting refinement of C^r. This is a combinatorial structure for ∂D via a homeomorphism $\phi_{r+1} : C^{r+1} \overset{\cong}{\to} \partial D$. The images $\phi_{r+1}(\Delta^N)$ of each of the N-simplices Δ^N of C^{r+1} are bi-Lipschitz homeomorphic to a standard N-simplex. Repeated barycentric subdivision decomposes a standard Euclidean N-simplex into N-simplices of arbitrarily small diameter. Let $C^{r+2}, C^{r+3}, \ldots$ be refinements of C^{r+1} obtained through successive barycentric subdivision. Again these are combinatorial structures for ∂D via maps $\phi_{r+i} : C^{r+i} \overset{\cong}{\to} \partial D$. Further there will be some $s \geq 1$ such that the N-simplices that make up C^{r+s} each have diameter at most α.

Now we construct the $(N+1)$-complexes A_i for $i = 1, 2, \ldots, r+s$ with *outer* boundary C^1 and *inner* boundary C^i. Define $A_1 := C^1$ and obtain A_{i+1} by attaching cells to inner boundary of A_i as follows. A 1-cell e^1 in $C^i \subset A_i$ is refined to some 1-complex $\overline{e^1}$ in C^{i+1}. Let e^2 be an abstract closed 2-cell with $\#_1(\partial e^2) = \#_1(\overline{e^1}) + 1$. Attach e^2 to e^1 by identifying one of the 1-cells of ∂e^2 with $e^1 \subset A_i$. In this way attach 2-cells to all 1-cells of $C^i \subset A_i$. Next consider how a 2-cell e^2 of $C^i \subset A_i$ is refined to some 2-complex $\overline{e^2}$ in C^{i+1}. A copy of $\partial\overline{e^2}$ can be found in the boundary of the 2-cells we attached to the 1-cells in $\partial e^2 \subseteq A_i$. We can therefore glue in a copy of $\overline{e^2}$ accordingly; this leaves a 3-cell hole which we fill by gluing in a 3-cell e^3 with

$$\#_2(\partial e^3) \ = \ 1 + \#_1(e^2) + \#_2(\overline{e^2}).$$

Repeat this process for every two cell of $C^i \subset A_i$. Then a similar operation is performed for every refinement of a 3-cell in $C^i \subset A_i$, and so on. The general step is that a j-cell e^j of $C^i \subset A_i$ is refined to some j-complex $\overline{e^j}$ in C^{i+1}. We find a copy of $\partial\overline{e^j}$ in the boundary of the j-cells attached during the previous stage. Attach $\overline{e^j}$ accordingly. this leaves a $(j+1)$-disc hole which is filled by gluing in a $(j+1)$-cell e^{j+1} with

$$\#_j(\partial e^{j+1}) \ = \ 1 + \#_{j-1}(e^j) + \#_j(\overline{e^j}).$$

Now

$$\begin{aligned}
\#_{j-1}(e^j) &\leq R_{j-1}, \quad \text{and} \\
\#_j(\overline{e^j}) &\leq \max\left\{ K_j, \ \prod_{i=1}^{j+1} i, \ \prod_{i=1}^{j-1} R_i \right\}.
\end{aligned}$$

So for $j = 1, 2, \ldots, N$ the number of j-cells in the boundary of any $(j+1)$-cell of A_{r+s} is at most R_j, and therefore A_{r+s} is an $\mathbf{R}$-combinatorial complex.

Now there exists an embedding $\psi : A_r \to D$ such that $\psi\,|_{C^1} = \phi_1$ and

$$d(\phi_i(x), \psi(x)) \leq \alpha$$

for $i = 2, 3, \ldots, r+s$ and for all $x \in C^i \subseteq A_r$. Roughly speaking this embedding is the result of projecting the C^i a distance at most α away from ∂D towards the centre of D. (Recall that the metric on D was inherited from a choice of homeomorphism of D to the standard Euclidean $(N+1)$-disc.)

Consider a 1-cell e in $C^i \subseteq A_{r+s}$ when $2 \leq i \leq r+s$. Let a and b be the endpoints of e. Then just as in the proof of Proposition 4.5

$$\begin{aligned}
\max\{ \ d(\psi(a), \phi_i(a)), \ d(\psi(b), \phi_i(b)) \ \} &\leq \alpha, \\
d(\ \bar{f}(\phi_i(a)), \ \bar{f}(\phi_i(b))\,) &\leq \frac{1}{4}, \quad \text{and} \\
d(\ \bar{f}(\psi(a)), \ \bar{f}(\psi(b))\,) &\leq 1 - \frac{1}{8}.
\end{aligned}$$

And if e is a 1-cell in C^{r+s} then

$$d(\psi(a), \psi(b)) \leq \varepsilon.$$

So we can now triangulate $D \smallsetminus \mathrm{Im}\psi$ with $(N+1)$-simplices of diameter at most ε. This produces the $\mathbf{R}$-combinatorial complex $\bar{C}$ with $\partial\bar{C} = \hat{C}$, that provides a combinatorial structure for D and leads to a contradiction we seek. ∎

5 Groups with simply connected asymptotic cones

In 5.F$''_1$ of [25] Gromov proved that a necessary condition for a group Γ to have simply connected asymptotic cones is that Γ satisfies a polynomial isoperimetric inequality. (See also Druţu [10]; in addition R. Handel [27] did some early work pertaining to this – essentially he proved the implication *1* $\Rightarrow$ *2* of Theorem B of this article.) Gromov asked in 5.F$_2$ of [25] whether this was a sufficient condition, a question which Bridson answered in the negative in [5] by giving examples of groups that satisfy polynomial isoperimetric inequalities but not linear isodiametric inequalities – in Theorem C we will see that satisfying a linear isodiametric inequality is another necessary condition for the asymptotic cones of a group to be simply connected.

Known examples of groups with simply connected asymptotic cones include nilpotent groups (Pansu [36]) and groups with quadratically bounded Dehn functions (Papasoglu [37]). Groups with quadratically bounded Dehn function include: hyperbolic groups, finitely generated abelian groups, automatic groups, fundamental groups of compact non-positively curved spaces, $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 4$, certain nilpotent groups including integral Heisenberg groups of dimension greater than 3 (see [1]), some non-uniform lattices in rank 1 Lie groups that have these nilpotent groups as cusp groups – including lattices in $\mathrm{SO}(n,1)$ and for $n>2$ in $\mathrm{SU}(n,1)$ (this list is taken from [6]; see also [7]). Recently Guba [26] has proved that Thompson's group F has a quadratic Dehn function.

In this section we will interpret the 2-dimensional filling function $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ in the context of finitely generated groups Γ. This will lead to a characterisation of finitely generated groups with simply connected asymptotic cones in Theorem B. The approach is to *partition* null-homotopic words in Γ into null-homotopic words of at most half the length. (Recall from §2.5 that *null-homotopic* words are those that evaluate to 1 in Γ, or equivalently are those that define edge-circuits in the Cayley graph $C(\Gamma)$ of Γ.)

5.1 Interpreting $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ for geodesic metric spaces

When X is a *geodesic metric space* (defined in §2.4) pairs of vertices have midpoints and so $\mathrm{Fill}^1_0(\ell)=2$ for all $\ell>0$ (as already observed in Examples 3.3). The purpose of the following proposition is to reinterpret the condition of Proposition 4.5 that concerns $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ being bounded, in the particular circumstance when X is a geodesic space. We will be able to do away with the notion of an *essential edge partitions* used the definition of $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ in §3.2 and in its place will make choices of geodesics between vertices.

Proposition 5.1. *Suppose X is a geodesic metric space. Let Δ be a 2-simplex. The following condition is necessary and sufficient for all the asymptotic cones of X to be simply connected.*

> *There exist $K_2'\in\mathbb{N}$ and $\mu_2'\geq 0$ such that: for all $\ell>0$, and for all geodesic triangles $\gamma:\partial\Delta\to X$ with edge lengths at most ℓ, there is a partition $\bar{\gamma}:\bar{\Delta}^{(1)}\to X$ of γ, with $\mathrm{mesh}(\bar{\Delta},\bar{\gamma})\leq\frac{\ell}{2}+\mu'_2$, and $\#_2(\bar{\Delta})\leq K_2'$.*

Here, by a ***partition*** $\bar{\gamma}:\bar{\Delta}^{(1)}\to X$ of γ we mean an extension of γ where $\bar{\Delta}$ is a finite *triangulation* of Δ. For each edge e of $\bar{\Delta}$ the map $\bar{\gamma}\,|_e$ defines a geodesic in X. The mesh of $(\bar{\Delta},\bar{\gamma})$ is the length of the longest of these geodesics.

Proof of Proposition 5.1. It is possible to prove this proposition by adapting the proof of Proposition 4.5. We take the alternative route of fixing $R_1:=1+K_1$ and showing that the condition in this proposition is equivalent to the existence of $K_2\in\mathbb{N}$ and $\mu_2\geq\mu_1=0$ such that $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\ell)\leq K_2$ for all $\ell\geq 0$. Proposition 4.5 tells us that this will suffice.

Firstly we show that if there exist $R_1,K_2\in\mathbb{N}$ ($R_1\geq 3$) and $\mu_2>0$ such that $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\ell)\leq K_2$ for all $\ell\geq 0$, then the condition of the proposition holds. Suppose we have a geodesic triangle $\gamma:\partial\Delta\to X$. The pair $(\Delta,\gamma\,|_{\partial\Delta^{(0)}})$ is in $\mathrm{Sph}^1_{\mathbf{R}}$ because $R_1=3$. An essential edge partition $(\hat{\Delta},\gamma\,\big|_{\partial\hat{\Delta}^{(0)}})$ of $(\Delta,\gamma\,|_{\partial\Delta^{(0)}})$ is obtained by splitting each of the sides of the geodesic triangle into two 1-cells, the extra vertices being added at the midpoints of the geodesic edges. The hypothesis that $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ is bounded by K_2 then gives us a partition $(\bar{\Delta},\bar{\gamma})$ subject to $(\hat{\Delta},\gamma\,\big|_{\partial\hat{\Delta}^{(0)}})$. Now, $\bar{\Delta}$ is a *triangular* complex since $R_1=3$. Extend $\bar{\gamma}:\bar{\Delta}^{(0)}\to X$ to a map $\bar{\Delta}^{(1)}\to X$ by that restricts to γ on $\partial\bar{\Delta}$ and by mapping the 1-cells in the interior of $\bar{\Delta}$ to (choices of) geodesics between the images of their end vertices. Deduce that the condition of the proposition holds with $K_2'=K_2$ and $\mu_2'=\mu_2$.

To prove the converse we assume the condition of the proposition holds and we show that if we take $R_1=3$, $\mu_1=0$ and $\mu_2=\mu_2'$ then $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}(\ell)\leq 4K_2'$ for all $\ell\geq 0$.

So suppose we have $(\partial\Delta, \gamma) \in \mathrm{Sph}^1_{\mathbf{R}}$. As $R_1 = 3$ we may take Δ to be a 2-simplex (the mono-gon and bi-gon being degenerate cases). Further, suppose we have an essential edge partition $(\hat{\Delta}, \hat{\gamma})$: that is, in effect, just (choices of) midpoints $m_1, m_2, m_3 \in X$ between $\gamma(e_i^\iota)$ and $\gamma(e_i^\tau)$ for the each of the three edges e_1, e_2, e_3 of Δ (with e_i^ι and e_i^τ denoting the two end vertices of e_i). Extend $\hat{\gamma}$ to a map $\partial\Delta \to X$ by choosing geodesics edges through these midpoints. Then let $\bar{\gamma} : \bar{\Delta}^{(1)} \to X$ be a partition as per hypothesis. In Figure 5 the darker shaded 2-complex depicts $\bar{\Delta}$.

One might now try to conclude the argument by restricting $\bar{\gamma}$ to the 0-skeleton $\bar{\Delta}^{(0)}$ and thereby provide the partition required for $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$. However this overlooks the requirement that $\partial\bar{\Delta} = \hat{\Delta}$ as complexes. The 1-cells e_1, e_2, e_3 of Δ are each split into two 1-cells in $\hat{\Delta}$, but their refinements $\bar{e}_1, \bar{e}_2, \bar{e}_3$ in $\partial\bar{\Delta}$ can be completely different combinatorial complexes to their refinements in $\hat{\Delta}$ – in the definition of a *partition* following the statement of Proposition 5.1 there is no restriction on the location of 0-cells in $\partial\bar{\Delta}$. This technicality can be overcome by enlarging the complex $\bar{\Delta}$ to a complex $\boldsymbol{\Delta}$ by attaching the lighter shaded 2-cells as shown in Figure 5. We add 0-cells v_1, v_2, v_3, one for each of $\bar{e}_1, \bar{e}_2, \bar{e}_3$, and we *cone off* each $\bar{e}_i$ to v_i: that is we have a 1-cell from v_i to each vertex of $\bar{e}_i$ and 2-cells as shown.

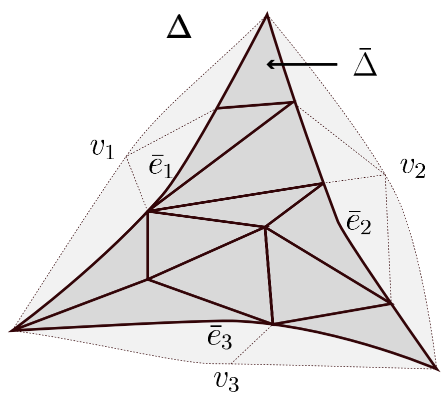

Figure 5: Converting between two types of partition.

Define a map $\boldsymbol{\gamma} : \boldsymbol{\Delta}^{(0)} \to X$ by making $\boldsymbol{\gamma}$ equal to $\bar{\gamma}\,|_{\bar{\Delta}^{(0)}}$ on $\bar{\Delta}^{(0)}$ and by mapping v_1, v_2, v_3 to m_1, m_2, m_3 respectively. Notice that the distance between the images of two vertices of $\boldsymbol{\Delta}$ at the end of an edge in $\boldsymbol{\Delta} \smallsetminus \bar{\Delta}$ is at most $\frac{1}{2}\mathrm{mesh}(\Delta, \gamma)$, because the m_i are midpoints of the geodesics $\hat{\gamma}(e_i)$. So $\mathrm{mesh}(\boldsymbol{\Delta}, \boldsymbol{\gamma}) \leq \frac{1}{2}\mathrm{mesh}(\Delta, \gamma) + \mu_2'$.

The number of 2-cells in this enlarged complex is at most $4K_2'$ as estimated as follows. The number of 2-cells in $\bar{\Delta}$ is at most K_2'. As each 2-cell is triangular, the number of 1-cells in $\bar{\Delta}$ is at most $3K_2'$, which is therefore an upper bound for the number of 2-cells in $\boldsymbol{\Delta} \smallsetminus \bar{\Delta}$. ■

5.2 Interpreting $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$ for finitely generated groups

We now give a characterisation of finitely generated groups with simply connected asymptotic cones. The implication *1* ⇒ *2* of Theorem B was proved by R. Handel in [27] and subsequently by Gromov [25], §5.F. There is an exegesis of Gromov's proof by Druţu [10]. The reverse implication is used by Papasoglu [37, page 793] in showing that groups satisfying a quadratic isoperimetric inequality have simply connected asymptotic cones. Our arguments here and in §4 are developments of those given by Gromov, Papasoglu and Druţu.

Note that it follows from Corollary 2.7 that condition *1* in this theorem does not depend on the choice of generating set $\mathcal{A}$. Also notice that we specify the sequence of base points to be $\mathbf{1}$, the constant sequence at $1 \in \Gamma$. This does not represent a serious restriction since we learnt in Lemma 2.2 that the choice of sequence of basepoints is immaterial in the definition of an asymptotic cone of a quasi-homogeneous space. Also recall Corollary 2.8 that tells us that the asymptotic cone $\mathrm{Cone}_\omega(\Gamma, \mathbf{1}, \mathbf{s})$ of a group Γ with finitely generating set $\mathcal{A}$ is the same as the asymptotic cone $\mathrm{Cone}_\omega(C(\Gamma, \mathcal{A}), \mathbf{1}, \mathbf{s})$ of its Cayley graph $C(\Gamma, \mathcal{A})$.

Theorem B. *Let Γ be a group with finite generating set $\mathcal{A}$. Fix any non-principal ultrafilter ω. The following are equivalent.*

1. *The asymptotic cones $\mathrm{Cone}_\omega(\Gamma, \mathbf{1}, \boldsymbol{s})$ of Γ are simply connected for all $\boldsymbol{s} = (s_n)$ with $s_n \to \infty$.*

2. *There exist $K, L \in \mathbb{N}$ such that for all null-homotopic words w of length $\ell(w) \geq L$ there is an equality*

$$w = \prod_{i=1}^{K} u_i w_i u_i^{-1} \tag{10}$$

in the free group $F(\mathcal{A})$ for some words u_i and w_i such that the w_i are null-homotopic and $\ell(w_i) \leq \ell(w)/2$ for all i.

Proof. First, take $\mathcal{R}_w$ to be the set of null-homotopic words of length at most $\ell(w)/2$ in Lemma 2.10 (van Kampen's Lemma) to prove the following.

Lemma 5.2. *Condition 2 of Theorem B is equivalent to:*

3. *There exist $K, L \in \mathbb{N}$ such that for all null-homotopic words w of length $\ell(w) \geq L$ there exists a diagram D_w with at most K 2-cells and with the boundary circuit of each 2-cell of D_w labelled by a null-homotopic word of length at most $\ell(w)/2$.*

Let D_w be the diagram of the lemma above. Call a point v of the diagram D_w a ***branching vertex*** if a small neighbourhood of the 1-skeleton of D_w about v has at least three connected components when we remove v. ***Arcs*** connect the branching vertices and D_w has at most K ***faces***. We may assume that a word read along any edge in the interior of D_w represents a geodesic in $C(\Gamma, \mathcal{A})$ – otherwise we could replace some of the w_i in (1) by shorter null-homotopic words. The following is essentially a lemma of Papasoglu in [37].

Lemma 5.3. *Let V and F be the number of branching vertices and of faces (respectively) in any topological disc portion of the diagram D_w. Then $V \leq 2(F-1)$.*

Proof. This is an Euler characteristic calculation. At least 3 arcs meet at each vertex (here we use the hypothesis that the we have a topological disc) and so the number of arcs E in the topological disc satisfies $E \geq \frac{3V}{2}$. So, as $V - E + F = 1$, we find $V - 3V/2 + F \geq 1$, and thus $V \leq 2(F-1)$ as required. ∎

To complete the proof of Theorem B it is enough to show that for the Cayley graph $C(\Gamma, \mathcal{A})$ of Γ (which is a geodesic metric space) the condition of Proposition 5.1 is equivalent to condition *3* of Lemma 5.2.

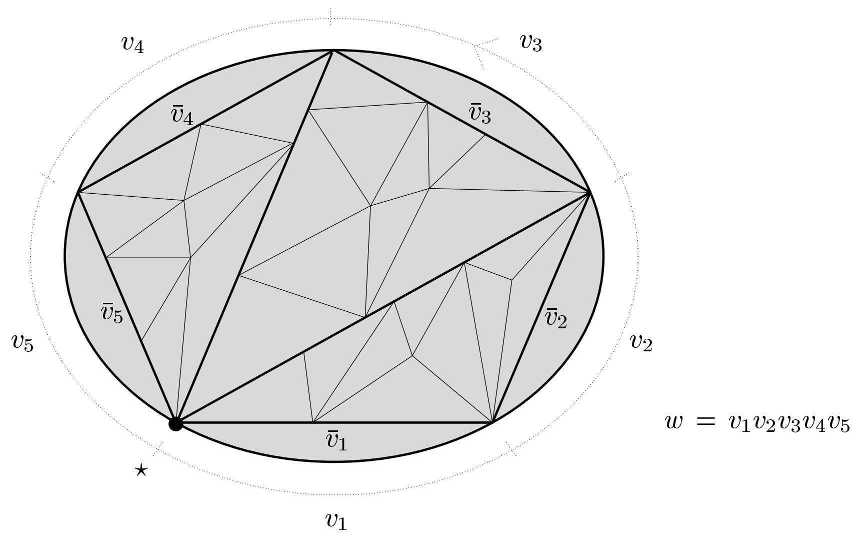

Figure 6: Constructing a van Kampen diagram D_w for w.

First we prove that the condition of Proposition 5.1 implies condition *3*. We are given a null-homotopic word w and (provided $\ell(w)$ is sufficiently large) we shall explain how to produce a diagram D_w for w in which each 2-cell has boundary length at most $\ell(w)/2$. The method is illustrated in Figure 6.

Let $n := \ell(w)$. We start by expressing the word w as the concatenation of five subwords $w = v_1v_2v_3v_4v_5$ in such a way that each word v_i has length between $(n/5) - 1$ and $(n/5) + 1$. Choose geodesic words $\bar{v}_i$ in Γ that equal the v_i. Thus we produce a diagram with boundary label w (read anticlockwise from a base point $\star$), by inscribing a geodesic pentagon in a n-sided polygon. When $n \geq 4$ the five outermost loops in this diagram have length at most $n/2$. We now partition this geodesic pentagon. Start by adding two diagonal geodesics to triangulate the pentagon. The mesh (*i.e.* the length of the longest side) of the three resulting geodesic triangles is at most $2((n/5) + 1)$, which is less than $n/2$ when $n \geq 4$. The condition of Proposition 5.1 allows us to repeatedly *partition* these geodesic triangles, each time halving the mesh modulo a possible error μ_2'. One partition reduces the mesh to at most $(n/4) + \mu_2'$ and then a second reduces it to at most $(n/8) + \mu_2'/2 + \mu_2'$. It is enough to partition until the mesh is $n/6$ as then the circumference of the triangles will be at most $n/2$ as required. So provided $n/8 + \mu_2'/2 + \mu_2' \leq n/6$ (that is, $n \geq 36\mu_2'$) two partitions, and therefore $5 + 3{K_2}^2$ triangles, will suffice.

The result is a diagram D_w having at most $5 + 3{K_2}^2$ 2-cells.

We now prove that if there are $K, L \in \mathbb{N}$ such that any null-homotopic word w admits a diagram D_w as per condition *3* of Lemma 5.2, then the criterion displayed in Proposition 5.1 holds. We take $\hat{\mu}_2 = 4L/3$. We aim to prove that there exists $\hat{K}_2$ such that all geodesic triangles can be partitioned into $\hat{K}_2$ geodesic triangles, achieving a halving of the mesh modulo a possible error $\hat{\mu}_2$.

First observe that it is sufficient to restrict our attention to geodesic triangles in $C(\Gamma, \mathcal{A})$ with vertices at 0-cells of $C(\Gamma, \mathcal{A})$. Such a geodesic triangle Δ of mesh ℓ defines a null-homotopic word $w \in \Gamma$ of length at most 3ℓ. Apply condition *3* of Lemma 5.2 to w to give a diagram for w in which the boundary words of the faces each have length at most

$\max\{3\ell/2, L\}$. Applying condition *3* twice more produces a diagram D_w for w in which the edge-circuits w_i of the 2-cells have length $\ell(w_i) \leq \max\{3\ell/8, L\}$, and the number of 2-cells in D_w is at most K^3.

We can assume these 2-cells to be non-singular *polygons* with arcs between adjacent branching vertices labelled by geodesic words. Recall that D_w is a tree-like arrangement of topological discs and 1-dimensional arcs. Thus the number of sides of such each *polygon* is bounded by the number of vertices in any topological disc portion of D_w. By Lemma 5.3 this is at most $2(K^3-1)+3$ (that is, at most $2(K^3-1)$ branching vertices and the 3 original vertices of Δ).

Now triangulate the *polygons* by adding choices of geodesics for the diagonals, partitioning each m-sided polygon into $m-2$ geodesic triangles. The resulting geodesic triangles have mesh at most the circumference of the polygons and so less than $\ell/2$. Further $m \leq 2(K^3-1)+3$. So $(2(K^3-1)+1)K^3$ such geodesic triangles are used to partition the faces.

The diagram D_w consists of (at most K^3) 2-dimensional discs joined by 1-dimensional arcs. We have shown above that the 2-dimensional discs can be decomposed into at most $(2(K^3-1)+1)K^3$ geodesic triangles. The 1-dimensional arcs are part of the sides of the original geodesic triangle Δ. So they are all geodesic arcs of length at most ℓ, except one which may have a single branching point. There are at most K^3+1 such geodesic arcs, each of which can be considered to be two (degenerate) geodesic triangles of mesh at most $\ell/2$. The possible tripod section can be considered to be at most six such degenerate triangles.

Conclude that the condition of Proposition 5.1 holds with $\hat{K}_2 := (2(K^3-1)+1)K^3 + 2(K^3+1)+6$ and $\hat{\mu}_2 := L$. ∎

5.3 Upper bounds for filling functions

In this section we show that Theorem B leads to bounds on three important invariants ("filling functions") of finitely presentable groups. We will see that if Γ is a finitely generated group with simply connected asymptotic cones then Γ is finitely presentable and we will give upper bounds for the (first order) *Dehn function* $\mathrm{Area} : \mathbb{N} \to \mathbb{N}$, the *minimal isodiametric function* $\mathrm{Diam} : \mathbb{N} \to \mathbb{N}$ and the *filling length function* $\mathrm{FL} : \mathbb{N} \to \mathbb{N}$. These were all defined and discussed in §2.5.3.

The polynomial bound (11) of the following theorem is the result of Gromov ($5F_1''$ in [25]) which sparked off this whole area of investigation. Druţu, in Theorem 5.1 of [10], has also provided a proof that Area is polynomially bounded. The isodiametric inequality (12) appears as a remark of Papasoglu at the end of [37]. The constants K and L in this theorem are those arising in Theorem B.

Theorem C. *Suppose that the asymptotic cones* $\mathrm{Cone}_\omega(\Gamma, \mathbf{1}, \boldsymbol{s})$ *of a finitely generated group* Γ *are simply connected for all sequences of scalars* $\boldsymbol{s}$ *with* $s_n \to \infty$. *Then there exists a finite presentation* $\langle \mathcal{A} \mid \mathcal{R} \rangle$ *for* Γ *with respect to which, for all* $n \in \mathbb{N}$ *the Dehn function, the minimal isodiametric function, and the filling length function satisfy*

$$\begin{aligned} \mathrm{Area}(n) &\leq Kn^{\log_2(K/L)}, && (11)\\ \mathrm{Diam}(n) &\leq (K+1)n, && (12)\\ \mathrm{FL}(n) &\leq 2(K+1)n, && (13) \end{aligned}$$

for some constants $K, L > 0$. *Further given a null-homotopic word* w *with* $\ell(w) = n$, *there is a van Kampen diagram* D_w *on which these three bounds are realised simultaneously.*

Proof. First notice that the bound (12) follows immediately from (13) since for any finitely presentable group Diam $\leq \frac{1}{2}$FL. This is proved in by noting that for any shelling of a van Kampen diagram D and for any vertex v of D, at some stage the contracting boundary loop of the shelling passes through v; this loop provides two paths from v to the basepoint.

Let $\mathcal{A}$ be a generating set for Γ. (It follows from Corollary 2.7, that 1-connectivity of the cones of Γ does not depend on the choice of generating set.)

Let w be any null-homotopic word in Γ. By Lemma 5.2, there exist $K, L \in \mathbb{N}$ such that if $\ell(w) \geq L$ then we can find a diagram D_1 for w with at most K 2-cells with boundary words w_i of length $\ell(w_i) \leq \ell(w)/2$. This procedure can be iterated.

Next each w_i for which $\ell(w_i) > L$ has a (possibly singular) 2-disc diagram C_{w_i} with boundary w_i. This provides a means of producing a new diagram D_2 from D_1: each 2-cell $e^2_{w_i}$ in D_1 is replaced by C_{w_i}. Repeating we get diagrams D_k for $k = 1, 2, \ldots$, with boundary word w and with at most K^k 2-cells.

Take k sufficiently large that $\ell(w)/2^k \leq L$ and define $\mathcal{R}$ to be the finite set

$$\mathcal{R} \; := \; \{\text{words } r \mid r = 1 \text{ in } \Gamma \text{ and } \ell(r) \leq L\}.$$

When $\ell(w)/2^k \leq L$ we have a diagram $D_k = D_w$ for w in which every 2-cell has boundary circuit in $\mathcal{R}$. So D_w is a van Kampen diagram for w over the finite presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ and it follows that $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$.

Now for $\ell(w)/2^k$ to be less than or equal to L, it is enough for k to be the least integer greater than or equal to $\log_2(\ell(w)/L)$. So the number of 2-cells in D_w is at most $K^{1+\log_2(\ell(w)/L)}$. Thus we have the bound on the Dehn function:

$$\text{Area}(n) \;\leq\; K^{1+\log_2(n/L)} \;=\; K n^{\log_2(K/L)}$$

as required.

The 1-skeleta of the diagrams $D_1, D_2, \ldots$ do not necessarily embed in $D_w^{(1)}$ because the diagrams inscribed in the 2-cells of D_i to produce D_{i+1} may be singular. However to each D_i we can associate a 2-disc diagram D'_i whose 1-skeleton is the image of the natural map of $D_i^{(1)}$ into D_w.

Claim. For any open 1-cell e^1 in the boundary of a diagram D_w constructed as above, there exists a shelling of D_w to $\partial D_w \smallsetminus e^1$ in which the boundary circuit has length at most $2(K+1)n$.

Proof by induction on $n = \ell(w)$. When $n \leq L$ the diagram D_w is just a 2-cell with boundary label w and the result is immediate.

For the induction step, take e^1 in the boundary of a diagram D_w for a null-homotopic word w with $\ell(w) = n > L$. Let us describe the *shelling* of D_w. Start with any shelling of D'_1 to $\partial D'_1 \smallsetminus e^1$. The total number of 1-cells in D'_1 is at most $(K+1)\ell(w)/2$, that is, at most $K\ell(w)/2$ in the 2-dimensional portions of D'_1 and at most $\ell(w)/2$ in the 1-dimensional portions. So the contracting boundary loop in the shelling of D'_1 has length at most $(K+1)\ell(w)$.

Now we see that a shelling of D_w to $\partial D_w \smallsetminus e^1$ can be made from the shelling of D'_1 together with shellings of the subdiagrams D_{w_i} of D_w that fill the 2-cells of D'_1. The shellings of the D_{w_i} are performed one at a time in the order dictated by the collapsing of 2-cells in the shelling of D'_1. Now, the boundary word w_i on each of the D_{w_i} has length $\ell(w_i) \leq \ell(w)/2$. So by induction hypothesis each D_{w_i} can be shelled to $\partial D_w \smallsetminus e^1_{w_i}$ and the boundary circuit in the shelling of D_{w_i} has length always less that or equal to $2(K+1)\ell(w)/2$. Deduce that length of the boundary circuit the shelling of D_w always remains at most $(K+1)\ell(w)+(K+1)\ell(w)$ and the claim is proved.

To complete the shelling of D_w all that is required is to collapse $\partial D_w \setminus e^1$ to the base vertex $\star$, and this involves no increase in filling length. We deduce that $\mathrm{FL}(n) \leq 2(K+1)n$ as required. ∎

Open problems 5.4. In 5.F_2 of [25] Gromov asked whether satisfying a polynomial isoperimetric function was a sufficient condition for a group to have simply connected asymptotic cones, and this was answered negatively by Bridson. One can now ask whether the bounds found in Theorem C are sufficient.

It is actually not known whether the polynomial bound on the Dehn function together with the linear isodiametric function are sufficient. It is possible that the linear upper bound on the filling length function follows from these other two bounds. Indeed, it is an open problem due to Gromov [25, page 100], whether for a general finitely presented group $\mathrm{FL} \preceq \mathrm{Diam}$.

5.4 Applications

Certain families of groups are known to have simply connected cones. Here we draw attention to the significance of the inequalities of Theorem C for these groups.

As we mentioned earlier, in [37] Papasoglu proves that the asymptotic cones of a group satisfying a quadratic (first order) isoperimetric inequality are all simply connected. So it follows that:

Corollary 5.5. *If the Dehn function of a finitely presented group* Γ *admits a quadratic bound then there is a linear bound on its filling length.*

The use of asymptotic cones would appear a circuitous route to this result – it would seem that an analysis of Papasoglu's methods in [37] would yield a direct proof. One reason this is of interest is because it tells us that the filling length function FL and the optimal isodiametric function Diam for Γ are $\simeq$-equivalent, answering Gromov's question (mentioned in Open Questions 5.4 above) positively for one important class of groups.

A particular instance of this is Thompson's group F, which was recently proved by Guba [26] to have a quadratic minimal isoperimetric function. (The author is grateful to Steve Gersten for drawing his attention to this result.)

Corollary 5.6. *The filling length function of Thompson's group* F *admits a linear bound.*

Pansu proves in [36] that virtually nilpotent groups have simply connected (indeed contractible) asymptotic cones. Therefore another corollary is

Corollary 5.7. *The filling length function of a finitely generated virtually nilpotent group admits a linear upper bound (and hence so does the optimal isodiametric function).*

This linear bound is observed by Gromov [25, page 101] for simply connected nilpotent Lie Groups, by analysing the geometry of Carnot-Caratheodory spaces.

We mention that Corollary 5.7 precipitated a further result of the author in collaboration with Gersten and Holt in [17]. A search was made for a direct combinatorial proof of Corollary 5.7, and this turned out to be possible via a induction argument on the nilpotency class c

of the group. A crucial feature of this induction argument is to keep track of an isoperimetric function at the same time as a linear bound on the filling length function. The result is that we learn that a null-homotopic word of length n admits a van Kampen diagram that not only has filling length bounded linearly in n but also has area bounded by a polynomial in n of degree $c+1$. Thus, in particular, we proved that finitely generated nilpotent groups admit a polynomial isoperimetric function of degree one greater than their class, resolving a long-standing conjecture.

6 Higher order isoperimetric and isodiametric functions of groups

In §2.6 we will prove Theorem D, which supplies N-th order isoperimetric and isodiametric functions of groups Γ whose asymptotic cones are N-connected. In this section we supply the requisite definitions: the finiteness properties $\mathcal{F}_k$, the notion of k-presentations, the higher order combinatorial isoperimetric functions (also known as higher order Dehn functions) and the higher order isodiametric functions.

6.1 Type $\mathcal{F}_{k+1}$ and k-presentations

Our account in this section draws heavily on that of Bridson in [8], where in particular k-presentations are introduced. The following definition amounts to saying that a group Γ is of ***type*** $\mathcal{F}_k$ when it admits an Eilenberg-MacLane space $\mathrm{K}(\Gamma,1)$ with finite k-skeleton. (See [9, page 470].)

If a group Γ is finitely generated it is said to be of type $\mathcal{F}_1$. Given a finite generating set $\mathcal{A}$ for such a Γ (with $\mathcal{A}\cap\mathcal{A}^{-1}=\phi$), we can construct its ***rose***: the wedge $\mathcal{K}^1 := \bigvee_{a\in\mathcal{A}}(\mathbb{S}^1,\star)$ of finitely many oriented circles, each labelled by a generator. The group Γ is said to be of ***type*** $\mathcal{F}_2$ when it is finitely presentable. Recall from 2.5 that given a finite set of defining relations we can attach finitely many 2-discs to the rose (using the relators to describe the attaching maps) to produce the standard compact 2-complex $\mathcal{K}^2$ such that $\pi_1\mathcal{K}^2=\Gamma$. The universal cover of $\mathcal{K}^2$ is the Cayley 2-complex associated to the given presentation (or "1-***presentation***" in the terminology used below) of Γ, and the 1-skeleton of the Cayley complex is its Cayley graph.

Higher finiteness properties concern enlarging $\mathcal{K}^2$ to make its universal cover highly connected. We say that Γ is of type $\mathcal{F}_3$ when $\pi_2\mathcal{K}^2$ is finitely generated as a Γ-module. In this event there is a finite set of continuous maps $f_i^2:(\mathbb{S}^2,\star)\to(\mathcal{K}^2,\star)$, whose homotopy classes generate the Γ-module $\pi_2\mathcal{K}^2$. These f_i^2 attach 3-discs to $\mathcal{K}^2$, killing π_2 of its universal cover $\widetilde{\mathcal{K}^2}$.

The homotopy class of a continuous map $(\mathbb{S}^2,\star)\to(\mathcal{K}^2,\star)$ necessarily includes a *singular* combinatorial map $(S_i,\star)\to(\mathcal{K}^2,\star)$ where $S_i\cong\mathbb{S}^2$ is some combinatorial complex. So the f_i^2 can, in general, be taken to be *singular* combinatorial maps (as defined in §2.3). (We cannot, in general, take the f_i^2 to be combinatorial maps.) A choice of 1-presentation together with the singular combinatorial attaching maps $f_i^2:(S_i,\star)\to(\mathcal{K}^2,\star)$ is referred to as a 2-***presentation*** for Γ. Let $\mathcal{K}^3$ be the complex arrived at by attaching the S_i to $\mathcal{K}^2$ via the singular combinatorial attaching maps f_i^2.

The process of enlarging the complex $\mathcal{K}^3$ further, through successive dimensions, leads us to a recursive definition of $\mathcal{F}_{k+1}$ and of k-***presentations*** as follows. Suppose Γ is of type $\mathcal{F}_k$ and we have a $(k-1)$-presentation. Consider attaching $(k+1)$-discs to $\mathcal{K}^k$ to kill $\pi_k\mathcal{K}^k$. If finitely many $(k+1)$-discs suffice (that is, $\pi_k \mathcal{K}^k$ is finitely generated as a Γ-module) we say Γ is of type $\mathcal{F}_{k+1}$. Call the resulting $(k+1)$-complex $\mathcal{K}^{k+1}$; by construction its universal cover $\widetilde{\mathcal{K}^{k+1}}$ is k-connected. The finite set of singular combinatorial attaching maps $f_i : S_i^k \to \mathcal{K}^k$, where each S_i^k is a combinatorial structure for the k-sphere, together with a $(k-1)$-presentation then make up a k-presentation.

Observe that for $k \geq 2$, the 0-skeleton of $\widetilde{\mathcal{K}^k}$ can be identified with Γ and so inherits the word metric. This metric agrees with the path metric on the 1-skeleton of $\widetilde{\mathcal{K}^k}$ where each 1-cell is given length 1.

We say Γ is of type $\mathcal{F}_\infty$ when it is of type $\mathcal{F}_N$ for all N.

6.2 Higher order isoperimetric and isodiametric functions

The filling functions defined in this section concern the combinatorial volume and diameter of fillings of combinatorial N-spheres. Suppose Γ is of type $\mathcal{F}_{N+1}$ and so admits a finite N-presentation. Construct a compact singular combinatorial $(N+1)$-complex $\mathcal{K}^{N+1}$ as described above. Consider a singular combinatorial map $\gamma : (S^N, \star) \to (\widetilde{\mathcal{K}^N}, \star)$, where $S^N \cong \mathbb{S}^N$ is some combinatorial structure on the N-sphere. Since $\widetilde{\mathcal{K}^{N+1}}$ is N-connected, we can ***fill*** $\gamma : (S^N, \star) \to (\widetilde{\mathcal{K}^N}, \star)$ by giving a singular combinatorial extension $\bar{\gamma} : D^{N+1} \to \widetilde{\mathcal{K}^{N+1}}$ with respect to some combinatorial decomposition $D^{N+1} \cong \mathbb{D}^{N+1}$ of the $(N+1)$-disc such that $S^N = \partial D^{N+1}$ as N-complexes.

We define the ***combinatorial N-volume*** (or *mass*) of γ as follows: let $\mathrm{Vol}_N(\gamma)$ be the number of N-cells e^N in C^N such that $\gamma\,|_{e^N}$ is a homeomorphism. Similarly define $\mathrm{Vol}_{N+1}(\bar{\gamma})$ to be the number of $(N+1)$-cells e^{N+1} in D^{N+1} such that $\gamma\,|_{e^{N+1}}$ is a homeomorphism. We define the ***filling volume*** $\mathrm{FVol}(\gamma)$ to be the minimum amongst all $\mathrm{Vol}_{N+1}(\bar{\gamma})$ such that $\bar{\gamma}$ fills γ.

Incidentally, this definition has an algebraic interpretation. It is explained in §5 of [8] that $\mathrm{FVol}(\gamma)$ can be reinterpreted as being the least M such that there is an equality

$$[\gamma] \;=\; \sum_{i=1}^{M} g_i \cdot [\partial e_{j(i)}^{N+1}]$$

in the Γ-module $\pi_N(\mathcal{K}^N, \star)$, where $\partial e_{j(i)}^{N+1}$ are the attaching maps of the $(N+1)$-cells $e_{j(i)}^{N+1}$ used to enlarge $\mathcal{K}^N$ to $\mathcal{K}^{N+1}$.

Similarly we can define the diameter and the filling diameter of $\gamma : (S^N, \star) \to (\widetilde{\mathcal{K}^N}, \star)$. We endow the 1-skeleton of S^N with a pseudo metric by giving each edge that collapses to a single vertex under γ length 0, and length 1 otherwise. Then the ***diameter*** of γ is defined by

$$\mathrm{Diam}(\gamma) \;:=\; \max\left\{ \; d(\star, v) \;\;|\;\; v \in S^{N,(0)} \; \right\}.$$

If $\bar{\gamma} : D^{N+1} \to \widetilde{\mathcal{K}^{N+1}}$ is a filling function for γ then we define the diameter of $\bar{\gamma}$ in the same way, via a pseudo-metric on D^{N+1}. We define the ***filling diameter*** $\mathrm{FDiam}(\gamma)$ of γ to be the minimum of $\mathrm{Diam}(\bar{\gamma})$ amongst all $\bar{\gamma}$ that fill γ.

Another natural way to define the diameter of γ (and similarly $\bar{\gamma}$) is as the diameter of the image of γ: that is, $\max\left\{d(\star, \gamma(v)) \mid v \in S^{N,(0)}\right\}$, where d is the combinatorial metric on the 1-skeleton of $\widetilde{\mathcal{K}^N}$. Our definition of the previous paragraph is an upper bound for the diameter of the image of γ.

We now give the definitions of some higher order filling functions. Let Ω_N be the set of singular combinatorial maps $(C^N, \star) \to (\widetilde{\mathcal{K}^{N+1}}, \star)$ where $C^N \cong \mathbb{S}^N$ is a combinatorial structure on the N-sphere.

Definition 6.1. ***The N-th order (combinatorial) Dehn function*** $\delta^{(N)} : \mathbb{N} \to \mathbb{N}$ is defined by

$$\delta^{(N)}(n) \ := \ \sup_{\gamma} \{ \mathrm{FVol}(\gamma) \mid \gamma \in \Omega_N \text{ with } \mathrm{Vol}_N(\gamma) \leq n \} .$$

This definition agrees with those in [2] and [8]. However it will not suffice for our purposes as we now explain.

The way we will obtain bounds on the filling volume of singular combinatorial maps of spheres $\gamma : (S^N, \star) \to (\widetilde{\mathcal{K}^N}, \star)$ will be to *cone off* to the basepoint $\star \in S^N$. This fills S^N with *rods*, that is, cones over the N-cells of S^N. We fill each of these rods in a way that agrees across common faces. (In fact we only need to fill the rods that arise from the $\mathrm{Vol}_N(\gamma)$ non-collapsing N-cells in S^N.) It turns out that we can bound the volume of each rod by a function of the diameter of the image of its 0-skeleton. This length is at most the diameter of the image of γ in $\widetilde{\mathcal{K}^N}$ – that is, $\max \left\{ d(\star, \gamma(v)) \mid v \in S^{N,(0)} \right\}$, which is, in turn, at most $\mathrm{Diam}(\gamma)$. We can then find an upper bound for the $(N+1)$-volume of an $(N+1)$-disc filling γ by multiplying the bound on the volume of the fillings of the rods by the combinatorial N-volume of γ. It follows that we get an upper bound on $\mathrm{FVol}_{N+1}(\gamma)$ in terms of two variables: diameter ℓ and N-volume n. This motivates us to define a two-variable minimal isoperimetric function as follows.

Definition 6.2. ***An N-th order two-variable minimal (combinatorial) isoperimetric function*** generalises the function $\delta^{(N)}(n)$ to take account of diameter. It is a function $(\mathbb{N} \cup \{\infty\})^2 \to \mathbb{N} \cup \{\infty\}$ defined by

$$\delta^{(N)}(n, \ell) \ := \ \sup_{\gamma} \left\{ \ \mathrm{FVol}(\gamma) \ \middle| \ \gamma \in \Omega_N \text{ with } \mathrm{Vol}_N(\gamma) \leq n \text{ and } \mathrm{Diam}(\gamma) \leq \ell \right\} .$$

Note that $\delta^{(N)}(n) = \delta^{(N)}(n, \infty)$. This type of isoperimetric function has been used in related contexts by Epstein *et al.* [14, Theorem 10.2.1] (*"mass times diameter estimate"*) and Gromov [23] (the "*cone inequality*").

We will also wish to monitor the filling diameter. So we define a two-variable minimal isodiametric function as follows.

Definition 6.3. ***The N-th order minimal (combinatorial) isodiametric function*** $\eta^{(N)} : (\mathbb{N} \cup \{\infty\})^2 \to \mathbb{N} \cup \{\infty\}$ for Γ is defined by

$$\eta^{(N)}(n, \ell) := \ \sup_{\gamma} \left\{ \ \mathrm{FDiam}(\gamma) \ \middle| \ \gamma \in \Omega_N \text{ with } \mathrm{Vol}_N(\gamma) \leq n \text{ and } \mathrm{Diam}(\gamma) \leq \ell \right\} .$$

Remark 6.4. Let us consider how one might attempt to use $\delta^{(N)}(n,\ell)$ to bound $\delta^{(N)}(n)$ by controlling diameter ℓ in terms of N-volume n.

When $N = 1$ we see $\max\{d(\star,\gamma(e^0)) \mid e^0 \in S^{1,(0)}\} \leq \mathrm{Vol}_1(\gamma)$ and hence $\delta^{(N)}(n) = \delta^{(N)}(n,n)$ and $\eta^{(N)}(n) \leq \eta^{(N)}(n,n)$. But when $N \geq 2$ it is possible for ℓ to grow arbitrarily large, independently of n. For example consider filling a singular combinatorial 2-sphere γ in $\mathcal{K}^2$ when Γ is the free abelian group of rank 3. It is possible that the image of $\gamma : S \to \widetilde{\mathcal{K}^2}$ is a *dumbbell*: two 2-spheres joined by an *arc* (a concatenation of 1-cells). There is *a priori* no bound in terms of n on the length of the path between the two 2-spheres. This allows us to find $\gamma : S \to \widetilde{\mathcal{K}^2}$ such that S includes 2-cells e^2 whose cones e^3 in $\bar{S}$ have $\mathrm{mesh}(e^3, \gamma\,|_{e^{3,(0)}})$ growing arbitrarily large, independently of $n = \mathrm{Vol}_2(\gamma)$.

The next strategy one might try is to decompose the *singular* combinatorial map $\gamma : (S^N,\star) \to (\widetilde{\mathcal{K}^N},\star)$ into a sum of *non-singular* combinatorial maps. This works in dimension $N = 2$: collapse the cells in S^2 that collapse under γ, to produce a complex $\hat{S}^2$ which is comprised of combinatorial 2-spheres intersecting along simple paths, or joined by simple paths (see [38]). Thus we factor γ though $\hat{S}^2$:

$$S^2 \to \hat{S}^2 \to \mathcal{K}^2,$$

and thereby see that to fill γ it is sufficient to fill non-singular (*i.e.* genuinely cellular) maps $\gamma_i : S_i^N \to \widetilde{\mathcal{K}^N}$ with each S_i^N a combinatorial structure for $\mathbb{S}^N$. Further $\sum \mathrm{Vol}_N\gamma_i = \mathrm{Vol}_N\gamma$ and for each i

$$\max\left\{d(\star,\gamma(e^0)) \mid e^0 \in S_i^{N,(0)}\right\} \;\leq\; C_N \mathrm{Vol}_N(\gamma_i),$$

where C_N is the maximum number of 1-cells in (closed) N-cells in $\mathcal{K}^N$. It follows that if $n \mapsto \delta^{(2)}(n,n)$ is bounded above by a superadditive[9] function $\hat{\delta}^2$ then for all n

$$\delta^{(2)}(n) \;\leq\; \hat{\delta}^{(2)}(n).$$

The obstacle to this method working in dimension $N \geq 3$ is determining whether singular combinatorial N-spheres can be decomposed into non-singular N-spheres.

In §3.3 of [23] Gromov proves the Federer-Fleming inequality for closed submanifolds V in $\mathbb{R}^N$: for some universal constant C_N,

$$\mathrm{FillVol}(V \subset \mathbb{R}^N) \leq C_N \mathrm{Vol}(V)^{\frac{N+1}{N}}.$$

His proof uses the *cone inequality*

$$\mathrm{FillVol}(V \subset \mathbb{R}^N) \leq \frac{1}{N+1}\mathrm{Diam}(V)\,\mathrm{Vol}(V)$$

together with an estimate $\mathrm{Diam}V \leq D_N\,\mathrm{Vol}(V)^{1/N}$, for some universal constant D_N, that comes from decomposing V into "essentially round pieces". It is not clear that this can be adapted to our combinatorial context to express the diameter term ℓ in $\delta^{(N)}(n,\ell)$ in terms of N-volume n.

We now prove that $\delta^{(1)}$ and $\eta^{(1)}$ agree with the functions Area and Diam of §5.3.

[9]A function $\psi : \mathbb{N} \to \mathbb{R}$ is *superadditive* when $\psi(r+s) \geq \psi(r) + \psi(s)$ for all $r, s \in \mathbb{N}$.

Proposition 6.5. *For all n,*

$$\begin{aligned}\delta^{(1)}(n) &= \text{Area}(n), \\ \eta^{(1)}(n,\infty) &= \text{Diam}(n).\end{aligned}$$

Proof. If $\gamma : (C^1, \star) \to (\widetilde{\mathcal{K}^2}, \star)$ is a singular combinatorial map then γ defines a null-homotopic word w of length $\ell(w) = \text{Vol}_1(\gamma)$ in the alphabet $\mathcal{A}^{\pm 1}$. A van Kampen diagram $\gamma_w : D_w \to \widetilde{\mathcal{K}^2}$ for w can be used to obtain a filling $\bar{\gamma} : D^2 \to \widetilde{\mathcal{K}^2}$ for γ as follows. It is a requirement of the definition of a filling that D^2 be homeomorphic to $\mathbb{D}^2$. However D_w can have 1-dimensional portions. So we thicken the diagram by attaching an annular neighbourhood of ∂D_w to ∂D_w. That is, we attach $\ell(w)$ rectangular 2-cells all of which will collapse under $\bar{\gamma}$, to produce a new 2-complex D'_w. Now it is not necessarily the case that $\partial D'_w = C^1$, on account of some of the 1-cells of C^1 collapsing under the map γ; but this is rectified by inserting extra 1-cells into ∂D_w and extra 2-simplices (as necessary) into D'_w all of which will collapse under $\bar{\gamma}$. Call the resulting diagram D^2. The number of 2-cells of D^2 that do not collapse under $\bar{\gamma}$ is $\#_2(D_w)$. It follows that $\delta^{(1)}(n) \leq \text{Area}(n)$.

We now prove the reverse inequality: $\text{Area}(n) \leq \delta^{(1)}(n)$. A null-homotopic word w defines a combinatorial map $\gamma : C^1 \to \widetilde{\mathcal{K}^2}$ where C^1 is a combinatorial complex homeomorphic to $\mathbb{S}^1$ with $\ell(w)$ 1-cells. Let $\bar{\gamma} : D^2 \to \widetilde{\mathcal{K}^2}$ be a filling of γ with at most $\delta^{(1)}(\ell(w))$ 2-cells. (So $D^2 \cong \mathbb{D}^2$ and $\bar{\gamma}$ is a singular combinatorial map.) Collapsing all the cells of D^2 that do not map homeomorphically onto their images produces a van Kampen diagram for w.

The inequalities obtained in the above two paragraphs combine to give $\delta^{(1)}(n) = \text{Area}(n)$. Similarly it follows from the constructions above and Remark 6.4 that $\eta^{(1)}(n,\infty) = \eta^{(1)}(n,n) = \text{Diam}(n)$. ∎

Remark 6.6. The functions $\delta^{(N)}$ are referred to as *combinatorial* Dehn functions to distinguish them from their *geometric* counterparts. The N-th order geometric Dehn function concerns Γ acting properly discontinuously and cocompactly on an N-connected Riemannian manifold M. It gives the infimal bound on the $(N+1)$-volume of discs filling maps of Lipschitz N-spheres into M. Such functions are used in [14, page 221], [23], [28] for example.

One might hope that the combinatorial functions $\delta^{(N)}$ are $\simeq$-equivalent to the geometric functions. It may be necessary to restrict the scope of the geometric Dehn function to fillings of N-spheres whose Lipschitz constant is within some bound, for otherwise it is not clear that the higher order geometric Dehn functions take finite values.

6.3 Higher order filling functions and quasi-isometry

It is a consequence of Theorem 1 of Alonso, Wang and Pride in [2] that if Γ is of type $\mathcal{F}_{N+1}$ then $\delta^{(N)}(n)$ takes finite values for all $n \in \mathbb{N}$. (More particularly, Theorem 1 in [2] that any $\widetilde{\mathcal{K}^{N+1}}$ for Γ is *"N-Dehn"* – see the next paragraph.) It follows that $\delta^{(N)}(n,\ell)$ also takes finite values for all $n, \ell \in \mathbb{N}$. It is also the case that if Γ is of type $\mathcal{F}_{N+1}$ then $\eta^{(N)}(n,\ell)$ takes finite values for all $n, \ell \in \mathbb{N}$. Essentially the proof relies on the local finiteness of $\widetilde{\mathcal{K}^{N+1}}$ and the fact that it admits a cocompact action of Γ; a careful treatment of the many technical details can be found in [2].

The definitions of $\delta^{(N)}(n)$ and $\delta^{(N)}(n,\ell)$ generalise readily to any singular combinatorial complex X. In this generality they may take infinite values on account of there being a sequence of N-spheres of N-volume at most n with unbounded filling volume. Or $\delta^{(N)}$ may be ill-defined because of X not being N-connected. However the criterion of being *"N-Dehn"* defined in [2] precludes these eventualities in all dimensions up to N. Define X to be ***N-Dehn*** when the following all hold.

(a). X is N-connected,

(b). $\delta^{(k)}(n) < \infty$ for $k = 1, 2, \ldots, N$ and for all $n \in \mathbb{N}$,

(c). the $(N+1)$-cells are attached to the complex via singular combinatorial maps $S^N \to X^{(N)}$; the N-spheres S^N must be one of only finite many combinatorial types.

It follows from Theorem 2 in [2] that if two complexes X and Y are N-Dehn and are quasi-isometric (with respect to the combinatorial metrics on their 1-skeletons) then their higher order Dehn functions $\delta_X^{(k)}(n)$ and $\delta_Y^{(k)}(n)$ are $\simeq$-equivalent for $k = 1, 2, \ldots, N$.

One sees that the two-variable isoperimetric and isodiametric functions for X and Y are similarly related: there exists $C > 0$ such that for all n and ℓ,

$$\begin{array}{rcl} \delta_X^{(N)}(n, \ell) & \leq & C\, \delta_Y^{(N)}(C\, n\, ,\, C\, \ell) + C\, n + C, \\ \eta_X^{(N)}(n, \ell) & \leq & C\, \eta_Y^{(N)}(C\, n\, ,\, C\, \ell) + C\, \ell + C. \end{array}$$

We will use these inequalities in §8.

Here is an outline of the proof of these inequalities. Our assumptions are that X and Y are both N-Dehn singular combinatorial complexes and there is a quasi-isometry f between them. We may as well take f to be a map $X^{(0)} \to Y^{(0)}$.

Suppose that we have a singular combinatorial map $\gamma_X : (S_X^N, \star) \to X$ with $\mathrm{Vol}_N(\gamma_X) = n$, where S_X^N is a combinatorial N-sphere. Then $f \circ \gamma_X \Big|_{S_X^{N,(0)}}$ is a map from the 0-skeleton of S_X^N to the 0-skeleton of Y. The idea is to extend to a singular combinatorial map $\gamma_Y : (S_Y^N, \star) \to Y$ where S_Y^N is a refinement of S_X^N. This map γ_Y is constructed from $f \circ \gamma_X \Big|_{S_X^{N,(0)}}$ by building through the dimensions. First of all, a filling is made of each $f \circ \gamma_X \mid_{e^1}$ for every 1-cell e^1 of S_X^N as per $\delta_Y^{(1)}$ and the 1-cells of S_X^N are refined accordingly. Then a filling of the 2-cells is made as per $\delta_Y^{(2)}$, and then the 3-cells, and so on. On finally filling the N-cells in accordance with $\delta_Y^{(N)}$, the resulting complex S_Y^N has N-volume at most $C\, n$ and diameter at most $C\, n$ for some constant C. So the filling volume of S_Y^N is at most $\delta_Y^{(N)}(C\, n\, ,\, C\, \ell)$ and the filling diameter is at most $\eta_Y^{(N)}(C\, n\, ,\, C\, \ell)$. However we need to *pull-back* a filling $\bar{\gamma}_Y : (D_Y^{N+1}, \star) \to Y$ of $\gamma_Y : (S_Y^N, \star) \to Y$ to get a filling of γ_X.

There is a quasi-isometry $g : Y^{(0)} \to X^{(0)}$ such that there is a constant k such that $d(gf(v), v) \leq k$ for all $v \in X^{(0)}$ (see §8.16 of [9]). Now $g \circ \bar{\gamma}_Y \mid_{D_Y^{N+1,(0)}}$ is a map from the 0-skeleton of D_Y^{N+1} to $X^{(0)}$. Extend this map to a singular combinatorial map $\gamma_X' : D_Y^{N+1} \to X$ (this introduces the multiplicative constant C in the two inequalities). Then *homotop* γ_X to γ_X' to get the filling of S_Y^N realising the two inequalities (this *homotopy* is the reason for the linear terms in the two inequalities).

It follows from Corollary 4 of [2] that, up to $\simeq$-equivalence, the function $n \mapsto \delta^{(N)}(n)$ depends only on Γ and not on the choice of $\widetilde{\mathcal{K}^{N+1}}$. Similarly two different constructions of $\widetilde{\mathcal{K}^{N+1}}$ give two different functions $\delta^{(N)}(n, \ell)$ (and similarly $\eta^{(N)}(n, \ell)$) that are related by the two inequalities given above. (Any construction of $\widetilde{\mathcal{K}^{N+1}}$ admits a cocompact Γ-action and hence any two choices of $\widetilde{\mathcal{K}^{N+1}}$ are quasi-isometric.)

7 Groups with highly connected asymptotic cones

In this section we prove type $\mathcal{F}_{N+1}$ finiteness and N-th order isoperimetric and isodiametric functions for groups Γ with N-connected asymptotic cones. These results are consequences of the boundedness of the filling functions $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$ established in Theorem A for such groups Γ.

Theorem D. *Let Γ be a finitely generated group with a word metric. Suppose that the asymptotic cones of Γ are all N-connected ($N \geq 1$). Then Γ is of type $\mathcal{F}_{N+1}$.*

Further, fix any finite $(N+1)$-presentation for Γ. There exist $a_N, b_N \in \mathbb{N}$ and $\alpha_N > 0$ such that for all $n \in \mathbb{N}$ and $\ell \geq 0$,

$$\delta^{(N)}(n,\ell) \;\leq\; a_N\, n\, \ell^{\alpha_N}, \tag{14}$$

$$\eta^{(N)}(n,\ell) \;\leq\; b_N\, \ell, \tag{15}$$

These bounds are always realisable simultaneously.

Proof. Our proof is by induction on N. The case $N = 1$ follows from Theorem C, since $\delta^{(1)}(n,\ell) \leq \delta^{(1)}(n) = \mathrm{Area}(n)$ and $\eta^{(1)}(n,\ell) \leq \eta^{(1)}(n,n) = \mathrm{Diam}(n)$ by Proposition 6.5. (Alternatively the $N = 1$ case can be proved along the lines of the argument for the induction step set out below.)

We now prove the induction step. So assume that Γ is a group of type $\mathcal{F}_N$. Fix any finite N-presentation for Γ and let $\mathcal{K}^N$ be the associated compact N-complex.

Suppose $\gamma : (S^N, \star) \to (\widetilde{\mathcal{K}^N}, \star)$ is a singular combinatorial map in which the combinatorial complex S^N is homeomorphic to $\mathbb{S}^N$. We seek an extension of γ to a singular combinatorial map $\bar{\gamma} : (D^{N+1,(N)}, \star) \to (\widetilde{\mathcal{K}^N}, \star)$, in which $D^{N+1,(N)}$ denotes the N-skeleton of a combinatorial complex D^{N+1} that is homeomorphic to $\mathbb{D}^{N+1}$ and has boundary $\partial D^{N+1} = S^N$. We will bound the combinatorial type of the $(N+1)$-cells in D^{N+1} (independently of γ) and thereby show that only finitely many $(N+1)$-cells need be attached to $\mathcal{K}^N$ to produce a complex $\mathcal{K}^{N+1}$ with N-connected universal cover.

Let $n := \mathrm{Vol}_N(\gamma)$, the combinatorial volume of γ – that is, the number of open N-cells of S that map homeomorphically onto their images. Let $\ell := \mathrm{Diam}(\gamma)$, the diameter of γ.

As the asymptotic cones of Γ are N-connected, Theorem A tells us that there are $\mathbf{R}$, $\boldsymbol{\mu}$ and $K_1, K_2, \ldots, K_{N+1} > 0$ such that $\mathrm{Fill}^k_{\mathbf{R},\boldsymbol{\mu}}(\ell) \leq K_k$ for $k = 1, 2, \ldots, N+1$. Moreover, as Γ is a finitely generated group with a word metric, we take $\mu_1 := 1/2$ (as in Examples 3.3 (2)). Essentially we will produce $\bar{\gamma}$ and D^{N+1} by repeatedly taking partitions of γ known to exist on account of the upper bounds on the functions $\mathrm{Fill}^k_{\mathbf{R},\boldsymbol{\mu}}$. However recall that these functions concern maps from the 0-skeleta of combinatorial complexes of controlled combinatorial type, and notice that we have no *a priori* control on the combinatorial type of S^N. So our first step is to *cut up* S^N into *pieces* of controlled combinatorial type – we use the *cone* $\hat{S}^N$ of S^N: let

$$\hat{S}^N \;:=\; (S^N \times [0,1])/(S^N \times \{1\}),$$

which is a $(N+1)$-complex, inheriting a combinatorial structure from S^N. So in $\hat{S}^N$ there is one $(N+1)$-cell corresponding to each N-cell in S^N. Refer to any $(k+1)$-cell in $\hat{S}^N$ that is the cone over a k-cell in S^N as a *rod*. Use γ to define a map $\hat{\gamma} : \hat{S}^{N,(0)} \to \widetilde{\mathcal{K}^N}^{(0)} = \Gamma$ in which each 0-cell e^0 of $S^N \times \{0\}$ is mapped to $\gamma(e^0)$ and the cone vertex $S^N \times \{1\}$ is mapped to $\star$. Then $\mathrm{mesh}(\hat{S}^N, \hat{\gamma}) = \ell$.

As $\gamma : S^N \to \widetilde{\mathcal{K}^N}$ is a singular combinatorial map, it maps some of the cells e^k in S^N into the image of their boundary. Thus we can factor γ

$$\gamma : S^N \;\to\; S^N/\sim \;\to\; \widetilde{\mathcal{K}^N}$$

through a complex $S^N/\sim$ where: $a \sim b$ if and only if a and b are in the same open cell in S^N and $\gamma(a) = \gamma(b)$. This collapsing extends across $\hat{S}^N = (S^N \times [0,1])/(S^N \times \{1\})$, to give a complex $\hat{S}^N/\sim$ in which $(a, \alpha) \sim (b, \beta)$ if and only if $\alpha = \beta$, the points a and b are in the same open cell in S^N, and $\gamma(a) = \gamma(b)$.

Filling of the rods in $\hat{S}^N/\sim$ will be a process that builds through dimensions. We will first partition $(e^1, \hat{\gamma}\,|_{e^1})$ for each of the 1-cells in $\hat{S}^N/\sim$, in accordance with the bound on $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}$. Then we partition the $(e^2, \hat{\gamma}\,|_{e^2})$ subject to the partitions of each $(e^1, \hat{\gamma}\,|_{e^1})$, and in accordance with the bound on $\mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}$. We continue through the dimensions until we have partitioned the $(e^{N+1}, \hat{\gamma}\,|_{e^{N+1}})$ subject to the lower dimensional partitions.

However before we can use the filling functions in this way recall that for $\mathrm{Fill}^{k+1}_{\mathbf{R},\boldsymbol{\mu}}$ to apply to a pair (C_0, θ_0) such that θ_0 is a map $C_0^{(0)} \to \Gamma$, the complex C_0 must be $\mathbf{R}$-combinatorial and homeomorphic to $\mathbb{S}^k$. Now the function $\mathrm{Fill}^{k+1}_{\mathbf{R},\boldsymbol{\mu}}$ is defined with reference to $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^k_{\mathbf{R},\boldsymbol{\mu}}$ in that partitions are constructed subject to essential boundary partitions which are built up through successive dimensions within the bounds on $\mathrm{Fill}^1_{\mathbf{R},\boldsymbol{\mu}}, \mathrm{Fill}^2_{\mathbf{R},\boldsymbol{\mu}}, \ldots, \mathrm{Fill}^k_{\mathbf{R},\boldsymbol{\mu}}$. Proposition 4.9 gives us some freedom in the values of $R_1, R_2, \ldots, R_N$: we can take each R_k to be any integer greater than or equal to

$$\max\left\{\; k+2, \quad 1 + R_{k-1} + K_k, \quad 1 + R_{k-1} + \prod_{i=1}^{k+1} i, \quad 1 + R_{k-1} + \prod_{i=1}^{k-1} R_i \;\right\}.$$

Therefore we are able to ensure that each R_k is also at least the number of k-cells in the boundary of a cone on a k-cell in $\mathcal{K}^k$. In particular R_k can be taken to be at least the number of k-cells in the boundary of a rod of a k-cell in $\hat{S}^N/\sim$. So the pairs (C_0, θ_0) are then indeed within the scope of $\mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$.

We now focus on filling one of the *rods* of $\hat{S}^N$. Let e^N be one of the closed N-cells of S^N, that is, a closed N-cell whose boundary combinatorial structure is that of the $(N-1)$-sphere of its attaching map to $S^{N,(N-1)}$. Let the $(N+1)$-complex C_0 be the rod in $\hat{S}^N$ that is the cone over e^N, and let $\theta_0 := \hat{\gamma}\,\Big|_{C_0^{(0)}}$. We will take repeated partitions of the pairs (C_0, θ_0), in a manner constrained by the bound on $\mathrm{Fill}^{N+1}_{\mathbf{R},\boldsymbol{\mu}}$.

A first partition refines C_0 to a complex C_1 with $\#_{N+1}(C_1) \leq K_{N+1}$, and extends θ_0 to a map $\theta_1 : C_1 \to \Gamma = \widetilde{\mathcal{K}^N}^{(0)}$ with

$$\mathrm{mesh}(C_1, \theta_1) \;\leq\; \frac{1}{2}\mathrm{mesh}(C, \theta_0) + \mu_{N+1} \;\leq\; \frac{\ell}{2} + \mu_{N+1}.$$

Next partitioning each of the $(N+1)$-cells in C_1 produces (C_2, θ_2). Continuing we get successive refinements $C_0, C_1, C_2, \ldots$ and successive extensions $\theta_0, \theta_1, \theta_2, \ldots$ where for each M, the map $\theta_M : C_M^{(0)} \to \Gamma = \widetilde{\mathcal{K}^N}^{(0)}$ satisfies

$$\begin{aligned}
\mathrm{mesh}(C_M, \theta_M) &\leq \frac{1}{2^M}\mathrm{mesh}\,(C_0, \theta_0) + \mu_{N+1} + \frac{\mu_{N+1}}{2} + \ldots + \frac{\mu_{N+1}}{2^{M-1}} \\
&\leq \frac{\ell}{2^M} + 2\mu_{N+1}
\end{aligned}$$

and $\#_{N+1}(C_M) \leq {K_{N+1}}^M$.

However in order that we will be able to assemble the filled rods C_0 into a combinatorial $(N+1)$-disc D^{N+1} filling S^N it is crucial that the cell e^N in C_0 is preserved in C_M, that is, it escapes refinement in the partitioning process. Recall that the filling functions $\mathrm{Fill}^k_{\mathbf{R},\boldsymbol{\mu}}$ concern a halving of the mesh on partitioning, modulo an error term μ_k, and these error terms satisfy: $\mu_{N+1} \geq \mu_N \geq \ldots \geq \mu_1 = 1/2$. The partition (C_1, θ_1) of (C_0, θ_0) is constructed subject to any choice of essential boundary partition. Assume that in the process of taking an essential boundary partition, the minimal number of cells is used every time a cell is refined. Now $\mathrm{mesh}(e^N, \theta_0 \mid_{e^{N,(0)}}) = 1$ and so when one comes to refining the cells of e^N in C_0 they can, in fact, be assumed to be left undisturbed. For the same reason the cell e^N is left undisturbed in the boundary of the subsequent refinements $C_1, C_2, \ldots, C_M$.

Now we give bounds on the number of $(N+1)$-cells in C_M, and the diameter of the image of θ_M. Let M be the least integer such that $\ell/2^M \leq 1$. So $M \leq 1 + \log_2 \ell$. Then

$$\#_{N+1}(C_M) \;\leq\; {K_{N+1}}^M \;\leq\; {K_{N+1}}^{1+\log_2 \ell} \;=\; K_{N+1}\ell^{\log_2 K_{N+1}}. \tag{16}$$

Further, the observation that for each k

$$\mathrm{diam}(\mathrm{Im}\,\theta_k) \leq K_{N+1}\mathrm{mesh}(C_k, \theta_k) + \mathrm{diam}(\mathrm{Im}\,\theta_{k-1})$$

leads us to the bound:

$$\begin{aligned}
\mathrm{diam}(\mathrm{Im}\,\theta_M) &\leq K_{N+1}\,(\mathrm{mesh}(C_1, \theta_1) + \mathrm{mesh}(C_2, \theta_2) + \ldots + \mathrm{mesh}(C_M, \theta_M)) \\
&\leq K_{N+1}\left(\left(\frac{\ell}{2} + \mu_{N+1}\right) + \left(\frac{\ell}{2^2} + \mu_{N+1} + \frac{\mu_{N+1}}{2}\right) + \ldots \right.\\
&\qquad \left. + \left(\frac{\ell}{2^M} + \mu_{N+1} + \frac{\mu_{N+1}}{2} + \ldots + \frac{\mu_{N+1}}{2^{M-1}}\right)\right) \\
&\leq K_{N+1}(\ell\,+\,2M\mu_{N+1}) \\
&\leq K_{N+1}(\ell\,+\,2\mu_{N+1}(1 + \log_2 \ell)).
\end{aligned} \tag{17}$$

Now $\theta_M : C_M^{(0)} \to \widetilde{\mathcal{K}^N}^{(0)}$ is only a map to the 0-skeleton of $\widetilde{\mathcal{K}^N}$. We need to extend it to a singular combinatorial map from the N-skeleton of an $(N+1)$-complex to $\widetilde{\mathcal{K}^N}$. Let $\hat{C}_0 := C_M$ and $\hat{\theta} := \theta_M : \hat{C}_0^{(0)} \to \widetilde{\mathcal{K}^N}$. From $\hat{\theta}$ we will obtain a singular combinatorial map $\hat{\theta}_N : \hat{C}_N^{(N)} \to \widetilde{\mathcal{K}^N}$, where $\hat{C}_N$ will be a refinement of $\hat{C}_0$ and $\hat{\theta}_N \Big|_{\partial \hat{C}_N} = \theta$.

Firstly, refine every 1-cell e of $\hat{C}_0$ as follows. Let e_i and e_t be the two vertices of e. Refine e into a chain of $d(\hat{\theta}_0(e_i), \hat{\theta}_0(e_t))$ edges. (That is, at most $2\mu_{N+1}+1$ edges.) Call the resulting complex $\hat{C}_1$. Then extend $\hat{\theta}_0$ to a combinatorial map $\hat{\theta}_1 : \hat{C}_1^{(1)} \to \widetilde{\mathcal{K}^N}$ in the natural way. Now the number of 1-cells in the boundary of each 2-cell of $\hat{C}_0$ is at most R_1. Define

$$n_1 \;:=\; \ell_1 \;:=\; R_1(2\mu_{N+1} + 1).$$

So the length (*i.e.* 1-volume) of each 2-cell in $\hat{C}_1$ is at most n_1, and ℓ_1 is a bound on the diameter of the image $\hat{\theta}_2$ of the boundary of any 2-cell in $\hat{C}_1$.

Next in accordance with $\delta^{(1)}(n_1, \ell_1)$ extend $\hat{\theta}_1$ and refine $\hat{C}_1$ to produce a singular combinatorial map $\hat{\theta}_2 : \hat{C}_2^{(2)} \to \widetilde{\mathcal{K}^N}$. So each of the 2-cells e^2 in $\hat{C}_1$ is refined to a combinatorial 2-disc $\bar{e}^2$ in $\hat{C}_2$ and the number of 2-cells that do not collapse under the map $\hat{\theta}_2$ is at most $\delta^{(1)}(n_1, \ell_1)$. Let

$$n_2 \ := \ R_2\, \delta^{(1)}(n_1, \ell_1),$$

which is a bound on the number of non-collapsing 2-cells in the boundary of any 3-cell in $\hat{C}_2$. Further the diameter of each $\bar{e}^2$ satisfies

$$\max\left\{ d(\theta_2(\star_{\bar{e}^2}), \theta_2(v)) \mid \text{0-cells } v \text{ of } \bar{e}^2 \right\} \ \leq \ \eta^{(1)}(n_1, \ell_1),$$

where $\star_{\bar{e}^2}$ is any choice of base vertex in $\bar{e}^2$. Let

$$\ell_2 \ := \ R_2\, \eta^{(1)}(n_1, \ell_1),$$

which is a bound on the diameter of the image under $\hat{\theta}_2$ of the boundary of any 3-cell in $\hat{C}_2$.

Continue similarly through the dimensions inductively defining

$$n_{k+1} \ := \ R_{k+1}\, \delta^{(k)}(n_k, \ell_k) \quad \text{and} \quad \ell_{k+1} \ := \ R_{k+1}\, \eta^{(k)}(n_k, \ell_k).$$

Eventually one produces a combinatorial map $\hat{\theta}_N : \hat{C}_N^{(N)} \to \widetilde{\mathcal{K}^N}$. The number of N-cells in the boundary of each of the $(N+1)$-cells in $\hat{C}_N$, that do not collapse under $\hat{\theta}_N$ is at most n_N and the diameter of their images under $\hat{\theta}_N$ is at most ℓ_N. And crucially, both n_N and ℓ_N are independent of γ. Notice also that the N-cell e^N in C_M remains undisturbed in the refinement $\hat{C}_N$.

Let $\hat{D}^{N+1}$ be the $(N+1)$-complex obtained from filling all the rods C_0 in $\hat{S}^N/\sim$ as described above: that is, each rod C_0 in $\hat{S}^N$ is refined to an $(N+1)$-complex $\hat{C}_N$ and assembled. Note that because the refinements of the rods are built up through the dimensions, the common faces of adjacent rods can be assumed to agree and hence fit together. Indeed, for the same reason, the rods can be assumed to fit together to fill not just $\hat{S}^N$ but $\hat{S}^N/\sim$. Moreover the singular combinatorial maps $\hat{\theta}_N : \hat{C}_N^{(N)} \to \widetilde{\mathcal{K}^N}$ for each rod can be assembled to give a singular combinatorial map $\hat{D}^{N+1,(N)} \to \widetilde{\mathcal{K}^N}$. Let D^{N+1} be the combinatorial $(N+1)$-disc obtained from $\hat{D}^{N+1}$ by *pulling back* the composition $\hat{S}^N \ \to \ \hat{S}^N/\sim \ \stackrel{\cong}{\to} \ \hat{D}^{N+1}$. That is, D^{N+1} is the refinement of $\hat{S}^N$ in which each cell of $\hat{S}^N$ that does not collapse in $\hat{S}^N$ is refined to have the combinatorial structure of the cell it maps to in $\hat{D}^{N+1}$. We then define a singular combinatorial map $\bar{\gamma} : D^{N+1,(N)} \to \widetilde{\mathcal{K}^N}$ to be the composition $D^{N+1,(N)} \to \hat{D}^{N+1,(N)} \to \widetilde{\mathcal{K}^{(N)}}$.

We now claim that only finitely many $(N+1)$-cells need to be attached to $\mathcal{K}^N$ in order construct a complex $\mathcal{K}^{N+1}$ such that every singular combinatorial map $\gamma : (S^N, \star) \to (\widetilde{\mathcal{K}^N}, \star)$ can be extended to a singular combinatorial map $\bar{\gamma} : (\bar{D}^{N+1}, \star) \to (\widetilde{\mathcal{K}^{N+1}}, \star)$. Attach one $(N+1)$-cell to $\mathcal{K}^N$ for every singular combinatorial maps $\tau : (S^N, \star) \to (\widetilde{\mathcal{K}^N}, \star)$ such that $\mathrm{Vol}_N(\tau) \leq n_N$ and $\mathrm{Diam}_N(\tau) \leq \ell_N$. There are only finitely many such τ because $\widetilde{\mathcal{K}^N}$ has bounded local geometry. Call the resulting finite complex $\mathcal{K}^{N+1}$. Then filling in each of the $(N+1)$-cells in $\bar{\gamma}$ gives an extension $D^{N+1} \to \widetilde{\mathcal{K}^{N+1}}$ of $\gamma : S^N \to \widetilde{\mathcal{K}^N}$ with $\partial D^{N+1} \to S^N$. Deduce that Γ is of type $\mathcal{F}_{N+1}$.

It remains to prove the bounds on $\delta^{(N)}(n,\ell)$ and $\eta^{(N)}(n,\ell)$. From the N-presentation of Γ form any $(N+1)$-presentation. So let $\mathcal{K}^{N+1}$ be *any* finite complex with N-connected universal cover, that can be obtained by attaching $(N+1)$-cells to $\mathcal{K}^N$.

Above we constructed the singular combinatorial map $\bar{\gamma} : D^{N+1,(N)} \to \widetilde{\mathcal{K}^N}$. We now fill each of the $(N+1)$-cells of D^{N+1} with $\delta^{(N)}(n_N,\ell_N)$ $(N+1)$-cells, thereby extending $\bar{\gamma}$ to a singular combinatorial map $\bar{\bar{\gamma}} : \bar{D}^{N+1} \to \widetilde{\mathcal{K}^{N+1}}$.

Let

$$\begin{aligned} \alpha_N &:= 1+\log_2 K_{N+1}, \\ a_N &:= \delta^{(N)}(n_N,\ell_N), \quad \text{and} \\ b_N &:= b'_N\,\eta^{(N)}(n_N,\ell_N), \end{aligned}$$

where $b'_N > 0$ is sufficiently large that

$$K_{N+1}(\ell + 2\mu_{N+1}(1+\log_2 \ell)) \le b'_N \ell$$

for all positive integers $\ell \ge 1$.

From (16) we know that each of the rods C_0 over one of the n non-collapsing N-cells in $\gamma : S^N \to \widetilde{\mathcal{K}^N}$ is refined in C_M into at most ℓ^{α_N} non-collapsing $(N+1)$-cells e^{N+1}. Each of these $(N+1)$-cells is then refined further into at most a_N $(N+1)$-cells in a complex $\bar{e}^{N+1}$. This proves:

$$\delta^{(N)}(n,\ell) \le a_N\, n\, \ell^{\alpha_N}.$$

The diameter of the image of each $\bar{e}^{N+1}$ is at most b_N so it follows from (17) that

$$\eta^{(N)}(n,\ell) \;\le\; \eta^{(N)}(\infty,\ell) \;\le\; b_N\,\ell.$$

■

Recall that in Remark 6.4 we discussed decomposing singular combinatorial 2-spheres into combinatorial 2-spheres. Assuming that the number of 1-cells in the boundary of a each 2-cell is bounded above by some constant, then the diameter of each of the combinatorial 2-spheres is bounded above by its volume (up to a multiplicative constant). So our discussion in Remark 6.4 together with the theorem above give:

Corollary 7.1. *Suppose the asymptotic cones of a group Γ are all 2-connected. Then the second order Dehn function $\delta^{(2)}(n)$ admits a polynomial bound.*

One would like to draw the same conclusion about $\delta^{(N)}(n)$ for $N > 2$ but it is unclear whether singular combinatorial N-spheres can be decomposed in a way that allows the same argument to work. However we do get:

Corollary 7.2. *Suppose the asymptotic cones of a group Γ are all N-connected. Let $\hat{\delta}^{(N)}$ be a filling function defined similarly to $\delta^{(N)}$, except by quantifying only over* combinatorial *N-spheres rather than over* singular combinatorial *N-spheres. Then $\hat{\delta}^{(N)}(n)$ satisfies a polynomial bound.*

8 Polycyclic groups

Recall that a group is ***polycyclic*** if it admits a normal series terminating at the trivial group for which all the factor groups are cyclic. A group is ***virtually*** polycyclic or ***virtually*** nilpotent when it has a finite index subgroup that is polycyclic or nilpotent (respectively).

Theorem E. *Let Γ be a virtually polycyclic group and let ω be any non-principal ultrafilter. The following are equivalent.*

(i). Γ *is virtually nilpotent.*

(ii). $\mathrm{Cone}_\omega(\Gamma, \mathbf{1}, \boldsymbol{s})$ *is contractible for all sequences of scalars* $\mathbf{s}$.

Proof. Pansu proves in [36] that the asymptotic cones of a virtually nilpotent group Γ are all nilpotent Lie Groups with Carnot-Caratheodory metrics and hence are contractible. (Indeed he proves that the sequence $(\Gamma, \frac{1}{s_n}d)$ converges in the Gromov-Hausdorff topology and so the cone is independent of the sequence of scalars and ultrafilter.) This establishes the implication *(i)* $\Rightarrow$ *(ii)*.

It is a recent result of Harkins [28] that a polycyclic group Γ is automatic if and only if it is virtually abelian. The strategy of his proof is as follows. Wolf proved in [45] that the growth function of a polycyclic group is either polynomial or strictly exponential. Gromov's famous result (in [25]) that groups of polynomial growth are virtually nilpotent, together with the fact that a virtually nilpotent group is automatic if and only if it is virtually abelian (Theorem 8.2.8 of [14]) deal with the polynomial growth case. Harkins shows that if Γ has strictly exponential growth then one of its higher order *geometric*[10] Dehn functions is strictly exponential. Hence on account of the polynomial bounds on the geometric higher order Dehn functions of automatic groups, Γ cannot be automatic – see Theorem 10.2.1 of [14]. We will adapt Harkins' argument to show that if Γ has strictly exponential growth then the higher order isoperimetric and isodiametric inequalities of Theorem D cannot hold. This suffices to establish the implication *(ii)* $\Rightarrow$ *(i)* because it follows from Theorem D that the asymptotic cones of Γ cannot all be contractible.

Harkins proves (using results of Mostow) that if Γ is virtually polycyclic then it is quasi-isometric to a co-compact lattice $\hat{\Gamma}$ in some simply connected, connected solvable Lie group G for any choice of left invariant Riemannian metric on G. Moreover G has the form $G = M]\mathbb{R}^n$, an extension of $\mathbb{R}^n$ by M, where $M \trianglelefteq G$ is the nil radical of G, and G is diffeomorphic to $\mathbb{R}^l$ for some l and hence is contractible. Now assume Γ is not virtually nilpotent. Then Γ has strictly exponential growth by Wolf [45], whence $\hat{\Gamma}$ also has strictly exponential growth. Harkins then constructs a nilpotent Lie subgroup N of M that is exponentially distorted in G.

Assume, for a contradiction, that the inequalities of Theorem D for Γ hold in all dimensions. These are *combinatorial* rather that *geometric* isoperimetric and isodiametric inequalities so (unlike in Harkins' argument) we will make use of a $\hat{\Gamma}$-invariant simplicial triangulation τ of G (which exists by Theorem 10.3.1 of [14]). Let $\sigma \subset G$ be a subcomplex of τ that is a fundamental domain for the action of $\hat{\Gamma}$. If we give the 1-skeleton of τ the path metric in which each edge has length 1 then $\hat{\Gamma}$, and hence Γ, are quasi-isometric to $\tau^{(1)}$. So, by the inequalities in §6.3, the two-variable combinatorial isoperimetric and isodiametric functions

[10]Geometric Dehn functions were discussed in Remark 6.6.

$\delta_\tau^{(k)}(n,\ell)$ and $\eta_\tau^{(k)}(n,\ell)$ for τ also satisfy the bounds of Theorem D for each dimension k (after the constants a_k and b_k have been altered suitably).

Now let us focus on the situation in which $M = \mathbb{R}^p$ for some p, for then Harkins' argument is more straight-forward. In this case $N = \mathbb{R}^m \leq \mathbb{R}^p$ for some m, and we will prove that the bound on $\delta_\tau^{(m-1)}(n,\ell)$ fails.

Define the $(m-1)$-cycle c_{m-1} to be the boundary of the standard Euclidean m-dimensional cube $c_m := [-\ell,\ell]^m$ in $N = \mathbb{R}^m$ with vertices having co-ordinates each $\pm\ell$. Fix a vertex u in the fundamental domain σ. Given a vertex v of c_{m-1} there is some (not necessarily unique) $\gamma \in \hat{\Gamma}$ such that v is in the translate $\gamma\sigma$ of σ. Let $v_\sigma = \gamma u$, a vertex of $\gamma\sigma$. Then $d(v, v_\sigma)$ is at most the diameter of σ.

We construct another $(m-1)$-cycle, given by a singular combinatorial map $\gamma_{m-1} : C_{m-1} \to \tau^{(m-1)}$ for some combinatorial $(m-1)$-sphere C as follows. First define C_0 to be the standard combinatorial structure for the boundary of an m-dimensional cube, and define $\gamma_0 : C_0^{(0)} \to \tau^{(0)}$ by mapping the vertices of C_0 to the vertices v_σ of τ obtained by perturbing the vertices v of c_{m-1} as discussed in the previous paragraph. Then the distance in G between the images of vertices at the ends of an edge in C is $\preceq \log\ell$ because N is exponentially distorted in G. Then extend γ_0 to a singular combinatorial map $\gamma_1 : C_1^{(1)} \to \tau^{(1)}$ by refining $C_0^{(1)}$ so that edges connecting vertices in C_0 are now mapped to geodesics in $\tau^{(1)}$. Next extend across the 2-cells to $\gamma_2 : C_2^{(2)} \to \tau^{(2)}$ by filling in accordance with the bounds on the two-variable isoperimetric and isodiametric functions $\delta_\tau^{(1)}$ and $\eta_\tau^{(1)}$, and then extend by filling across 3-cells similarly, and so on through the dimensions until we have $\gamma_{m-1} : C_{m-1} \to \tau^{(m-1)}$.

Now we claim that the singular combinatorial $(m-1)$-volume of γ_{m-1} is $\preceq (\log\ell)^{L_{m-1}}$ for some $L_{m-1} > 0$. One sees inductively that for $1 \leq k \leq m-1$, the k-skeleton of the cube has both singular combinatorial k-volume and diameter $\preceq (\log\ell)^{L_k}$ for some $L_{m-1} > 0$. In the case $k = 1$ this is trivially true. For the induction step observe that by Theorem D, the filling providing the $(k+1)$-cells in the cube has $(k+1)$-volume $\preceq \delta_\tau^{(k)}((\log\ell)^{L_k}, (\log\ell)^{L_k}) \preceq (\log\ell)^{L_k}((\log\ell)^{L_k})^{\alpha_k}$ and diameter $\preceq \eta_\tau^{(k-1)}((\log\ell)^{L_k}, (\log\ell)^{L_k}) \preceq (\log\ell)^{L_k}$.

So the isoperimetric inequality for $\delta_\tau^{(m-1)}$ gives us a bound on the combinatorial filling m-volume of γ_m of $\preceq (\log\ell)^{L_m}$ for some constant L_m. Combinatorial filling m-volume is an upper bound for geometric filling volume up to a multiplicative constant equal to the maximum geometric m-volume of an m-cell in τ. So the geometric filling m-volume of γ_m is $\preceq (\log\ell)^{L_m}$.

Following Harkins we find a lower bound $\sim \ell^m$ on the geometric filling volume of γ_{m-1}, and this differs exponentially from the upper bound $\preceq (\log\ell)^{L_m}$, whence we will have the required contradiction. Harkins proves that G admits an exact left-invariant m-form $\omega = d\pi$ on G. Exactness implies that the norm $||\omega||$ is constant. We assemble a filling $\gamma_m : C_m \to \tau^{(m)}$ of $\gamma_{m-1} : C_{m-1} \to \tau^{(m-1)}$ by attaching an m-chain filling $\gamma_{m-1} - c_{m-1}$ to the standard $c_m = [-\ell,\ell]^m$ cube in $\mathbb{R}^m$. By Stokes' Theorem

$$\int_{\gamma_m(C_m)} \omega = \int_{\gamma_{m-1}(C_{m-1})} \pi$$

and so the geometric filling volume of $\gamma_m(C_m)$ does not depend on the particular construction of γ_m. We can construct the m-chain filling $\gamma_{m-1} - c_{m-1}$ by assembling pairs of m-chains, such that the two chains in each pair are related by a translation and their contributions to $\int_{\gamma_m(C_m)} \omega$ cancel as they are equal but have different signs due to their opposite orientations.

So $\left|\int_{\gamma_m}\omega\right|$ is the volume of the m-cube $c_m = [-\ell,\ell]^m$, which is $\sim \ell^m$. Thus, as $||\omega||$ is constant, the minimal filling volume of $\gamma_m(C_m)$ is at least

$$\frac{1}{||\omega||}\left|\int_{\gamma_m}\omega\right| \sim \ell^m.$$

In the case of general M, Harkins demonstrates the existence of an exponentially distorted Lie subgroup N in $G = M]\mathbb{R}^n$ and shows that one of the higher dimensional isoperimetric inequalities of dimension at most $m := \dim N/[N,N]$ grows strictly exponentially. The method described above applies to the Lie group $M/[M,M]\,]\mathbb{R}^n$, and Harkins deals with the added complications of *pulling back* the constructions to G. ■

A Non-principal ultrafilters and ultralimits

Let I be a non-empty set. A ***filter*** on I is a map $\omega : \mathcal{P}(I) \to \{0,1\}$ such that $\omega^{-1}(1)$ is non-empty and:

(i). if $\omega(A) = \omega(B) = 1$ then $\omega(A \cap B) = 1$,
(ii). if $\omega(A) = 1$ and $A \subseteq B \subseteq I$ then $\omega(B) = 1$.

The filter is ***proper*** if $\omega(\phi) = 0$. And a proper filter is an ***ultrafilter*** if

(iii). for any $A \subseteq I$ either $\omega(A) = 1$ or $\omega(I\backslash A) = 1$.

Further ω is called ***non-principal*** if

(iv). $\omega(A) = 1$ for every cofinite subset A of I.

So (as *(iii)*, *(iv)* $\Rightarrow$ $\omega(\phi) = 0$) a ***non-principal ultrafilter*** is a map $\omega : \mathcal{P}(I) \to \{0,1\}$ satisfying *(i)*, *(ii)*, *(iii)* and *(iv)*. Axioms *(i)*-*(iv)* amount to saying ω is a finitely additive probability measure taking values 0 and 1.

Filters on I form a partially ordered set via $\omega \preceq \hat{\omega}$ if and only if $\omega^{-1}(1) \subseteq \hat{\omega}^{-1}(1)$. Notice that a filter is a maximal proper filter if and only if it is an ultrafilter.

Given a set Ω of subsets of I we can form the ***filter*** ω_Ω ***generated by*** Ω:

$$\omega_\Omega^{-1}(1) \;=\; \{A \subseteq I \mid B_1 \cap \ldots \cap B_n \subseteq A \text{ for some } n \geq 1 \text{ and } B_1, \ldots, B_n \in \Omega\}.$$

Observe that ω_Ω is proper if and only if Ω has the property that any finite intersection of sets in Ω is non-empty (the *finite intersection property*). Thus Zorn's Lemma allows us to deduce:

Proposition A.1. *If a set Ω of subsets of I satisfies the finite intersection property then the filter generated by Ω can be extended to an ultrafilter on I.*

In particular, the cofinite subsets of an infinite set I satisfy the finite intersection property, so there exists a non-principal ultrafilter on I. Notice also that there exists a non-principal ultrafilter on I if and only if I is infinite, since for a finite I non-principal implies not proper.

Remarks A.2.

1. For a non-principal ultrafilter ω on a set I, permutations of I induce new non-principal ultrafilters. (But not all non-principal ultrafilters on a countably infinite set I are related by a permutation of I since there are $2^{\aleph_0}$ such permutations and $2^{\left(2^{\aleph_0}\right)}$ non-principal ultrafilters - see [21]).

2. *Restriction*: suppose I is a non-empty set and ω is a non-principal ultrafilter on I. If $J \subseteq I$ with $\omega(J) = 1$, then $\omega\left|_{\mathcal{P}(J)}\right.$ is a non-principal ultrafilter.

3. *Extension*: suppose J is a non-empty subset of I and ω is a non-principal ultrafilter on J. Then by applying Proposition A.1 to the subsets of J of ω-measure 1 together with the cofinite subsets of I we deduce that there is a non-principal ultrafilter extending ω to I.

Definition A.3. Take a non-principal ultrafilter ω on $\mathbb{N}$. Given a sequence (a_n) in $\mathbb{R}$ we say $a \in \mathbb{R}$ is an ω-***ultralimit*** of (a_n) when $\forall \varepsilon > 0,\ \omega\{n : |a - a_n| < \varepsilon\} = 1$. Say ∞ is an *ω-ultralimit* of (a_n) when for all $N > 0$ we have $\omega\{n \mid a_n > N\} = 1$, and similarly $-\infty$ is an ω-ultralimit when for all $N > 0$ we have $\omega\{n \mid a_n < -N\} = 1$.

Remarks A.4.

1. Every ω-ultralimit is also a limit point in the usual sense.

2. Any sequence (a_n) of reals has a unique ω -ultralimit in $\mathbb{R} \cup \{\infty\}$ denoted $\lim_\omega a_n$. (*Sketch proof.* If ∞ or $-\infty$ is an ultralimit then it is the unique ultralimit, else there is a bounded interval containing ω-infinitely many a_n; successively halve this interval always choosing the *unique* half in which there are ω-infinitely many of the a_n.)

3. Given a sequence of reals (a_n) with a limit point a (in the usual sense) there exists a non-principal ultrafilter on $\mathbb{N}$ with $\lim_\omega a_n = a$. This follows from an application of Proposition A.1, taking Ω to be the cofinites together with the sets formed from the indices of the a_n found in neighbourhoods of a.

B Relating the sequence of scalars and the non-principal ultrafilter

Here we prove a proposition which relates the role of the sequence of scalars to that of the non-principal ultrafilter in the definition of an asymptotic cone.

Say that a sequence of scalars $\mathbf{s}$ has ***bounded accumulation*** when there is a bound on the size of the sets $S_r := \{n \mid s_n \in [r, r+1)\}$. (So, for example, the sequence $s_n := \sum_{i=1}^n 1/i$ fails to have bounded accumulation.)

Proposition B.1. *Let* $\mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$ *be an asymptotic cone of a metric space X, for which the sequence of scalars* $\mathbf{s}$ *has bounded accumulation. Then there is a non-principal ultrafilter ω' and a sequence of base points* $\mathbf{e}'$ *such that* $\mathrm{Cone}_\omega(X, \boldsymbol{e}, \boldsymbol{s})$ *and $Cone_{\omega'}(X, \mathbf{e}', \mathbb{N})$ are isometric.*

Proof. Use the following sequence of isometries:

$$\begin{aligned}
\mathrm{Cone}_\omega(X, \mathbf{e}, \mathbf{s}) &\overset{1}{\cong} \mathrm{Cone}_\omega(X, \mathbf{e}, (\lfloor s_n \rfloor)) \\
&\overset{2}{\cong} \mathrm{Cone}_{\omega|_T}(X, (e_n)_{n\in T}, (\lfloor s_n \rfloor)_{n\in T}) \\
&\overset{3}{\cong} \mathrm{Cone}_{\bar{\omega}}\big(X, (e_t)_{t\in\bar{T}}, \bar{T}\big) \\
&\overset{4}{\cong} \mathrm{Cone}_{\omega'}\big(X, \mathbf{e}', \mathbb{N}\big),
\end{aligned}$$

where

1. $\lfloor s_n \rfloor$ denotes the integer part of s_n;

2. $T \subseteq \mathbb{N}$ is a set with $\omega(T) = 1$ that contains at most one element of each $S_r := \{n \mid s_n \in [r, r+1)\}$. Such a set exists because of our hypothesis of bounded accumulation of $\mathbf{s}$;

3. $\bar{T} := \{\lfloor s_n \rfloor : n \in T\} \subseteq \mathbb{N}$. This is in one to one correspondence with T. So $\bar{\omega}$ is obtained from $\omega\,|_T$ by a relabelling;

4. ω' is an extension of $\bar{\omega}$ in such a way that $\omega'(\bar{T}) = 1$. And $\mathbf{e}' = (e'_t)$ is obtained by setting $e'_t := e_t$ when $t \in \bar{T}$ and when $t \notin \bar{T}$ the definition of e'_t is of no consequence.

■

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