Intrinsic versus extrinsic diameter in finitely presented groups

Work in collaboration with Martin Bridson

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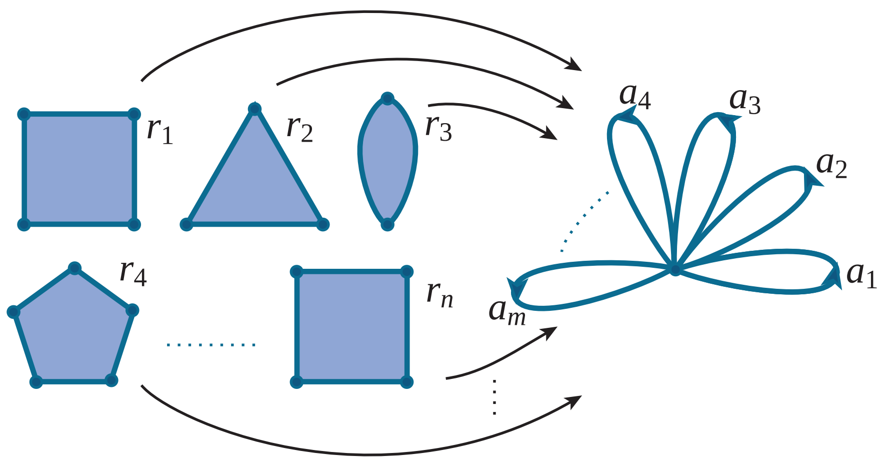
The **Cayley 2-complex** of $\mathcal{P} := \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$ is the universal cover of



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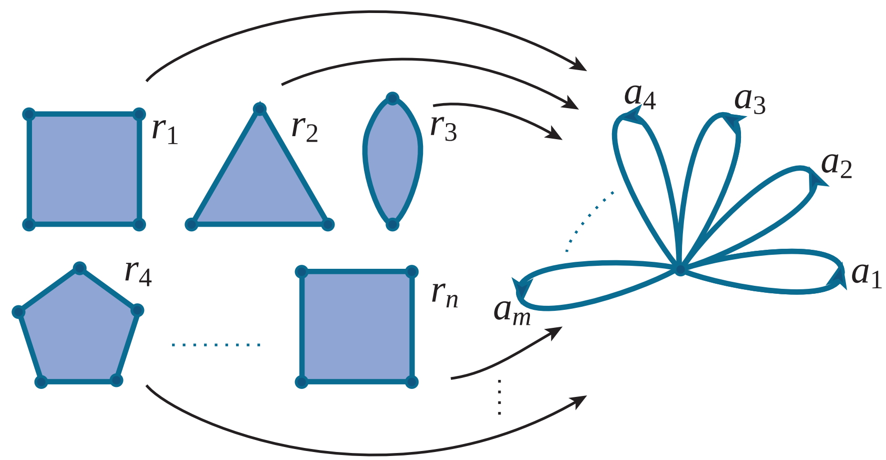
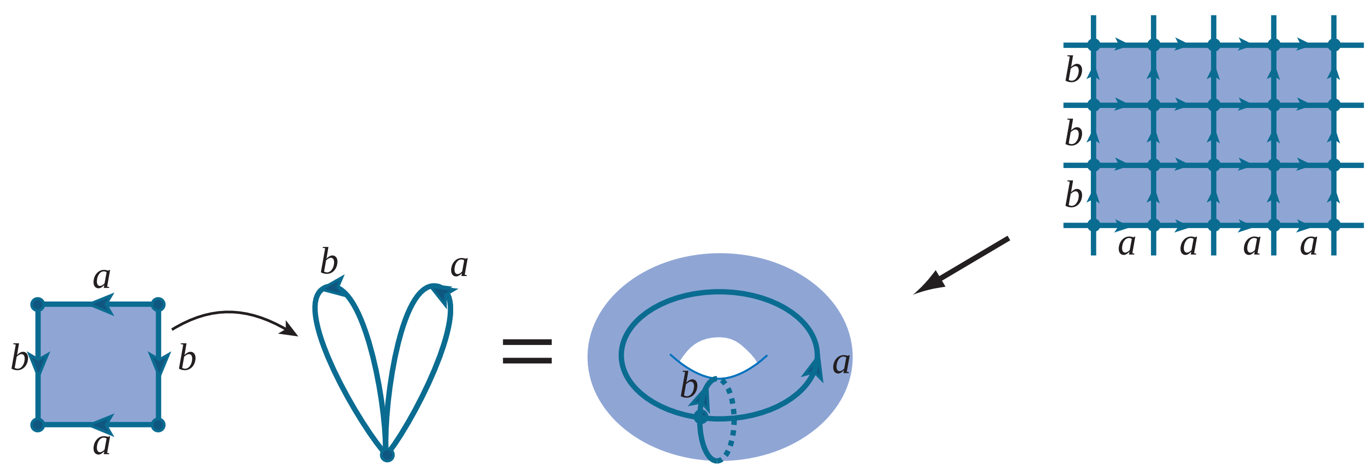
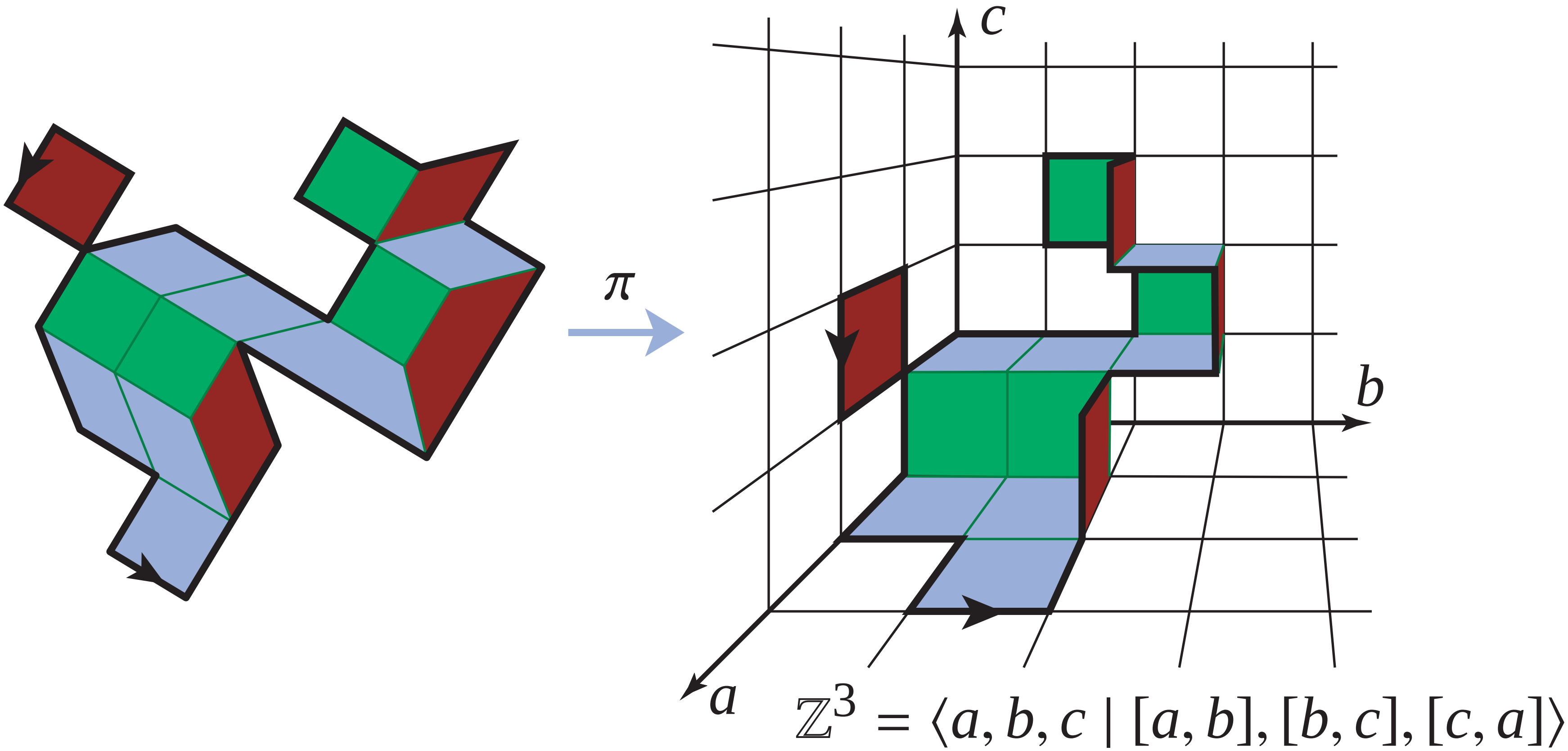


Example $\langle a, b \mid [a, b] \rangle = \mathbb{Z}^2$

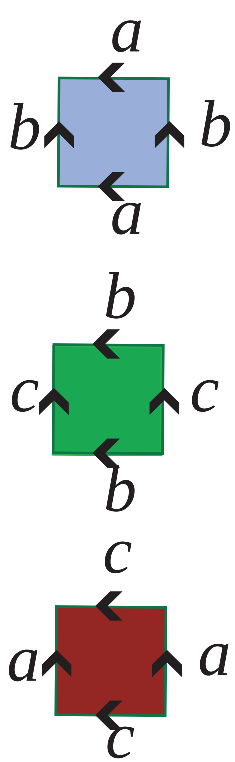


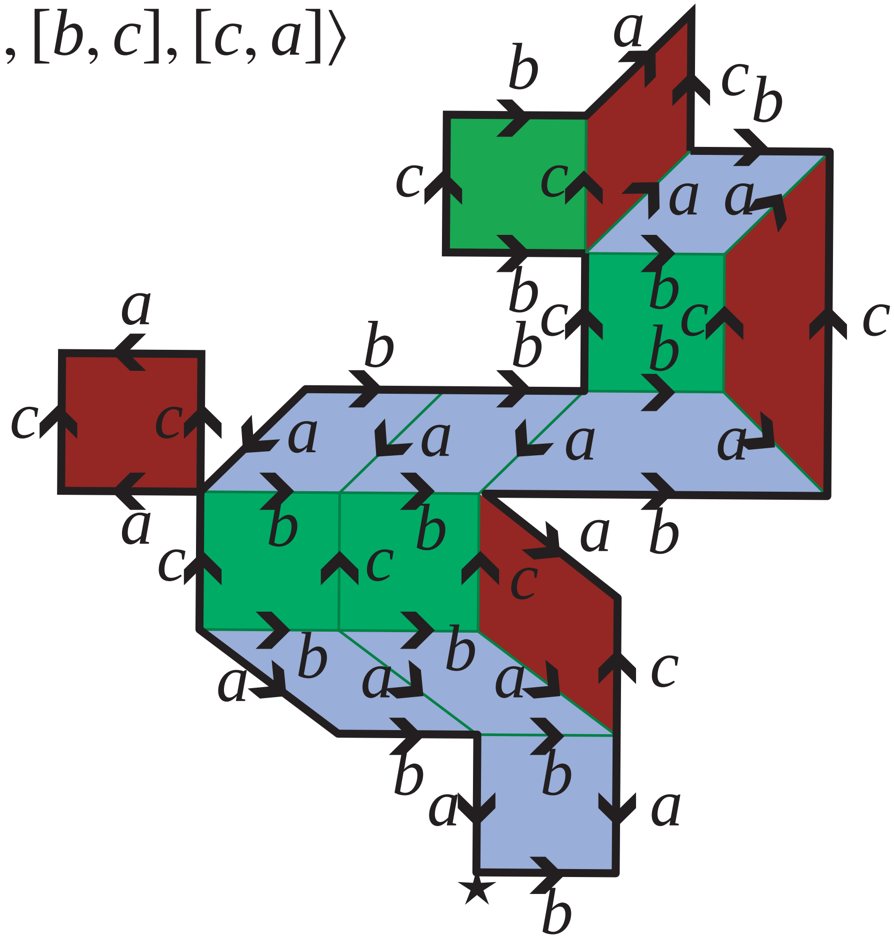

π
c
b
a
$\mathbb{Z}^3 = \langle a, b, c \mid [a,b], [b,c], [c,a] \rangle$

A *van Kampen diagram* for

$$ba^{-1}ca^{-1}bcb^{-1}ca^{-1}b^{-1}c^{-1}bc^{-1}b^{-2}acac^{-1}a^{-1}c^{-1}aba.$$

in $\mathbb{Z}^3 = \langle a, b, c \mid [a,b], [b,c], [c,a] \rangle$





Define the *intrinsic* and *extrinsic diameter* of a van Kampen diagram D by

$$\mathrm{IDiam}(D) \;:=\; \max\{\, \rho(a,b) \mid \text{vertices } a,b \text{ of } D\}$$

$$\mathrm{EDiam}(D) \;:=\; \max\{d(\pi(a),\pi(b)) \mid \text{vertices } a,b \text{ of } D\}$$

where

ρ = combinatorial metric on $D^{(1)}$

d = *word metric* = combinatorial metric on the Cayley graph

$\mathcal{P}$ a finite presentation of a group Γ.

For edge-loops γ in the Cayley 2-complex of $\mathcal{P}$ define

$$\mathrm{IDiam}(\gamma) := \min \{ \mathrm{IDiam}(D) \mid D \text{ a filling of } \gamma \}$$

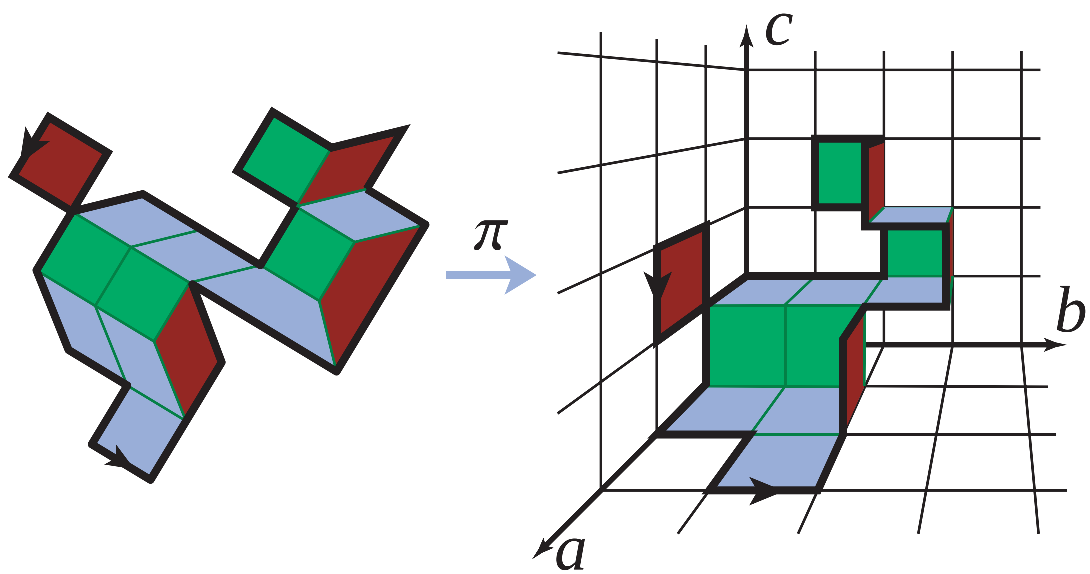
$$\mathrm{EDiam}(\gamma) := \min \{ \mathrm{EDiam}(D) \mid D \text{ a filling of } \gamma \}$$

and for $n \in \mathbb{N}$ define the resulting *filling functions* are

$$\mathrm{IDiam}(n) := \max \{ \mathrm{IDiam}(\gamma) \mid \text{edge-loops } \gamma \text{ of length } \leq n \}$$

$$\mathrm{EDiam}(n) := \max \{ \mathrm{EDiam}(\gamma) \mid \text{edge-loops } \gamma \text{ of length } \leq n \}$$

Question. Does measuring diameter in the Cayley (*extrinsically*) and in van Kampen diagrams (*intrinsically*) give qualitatively different results?



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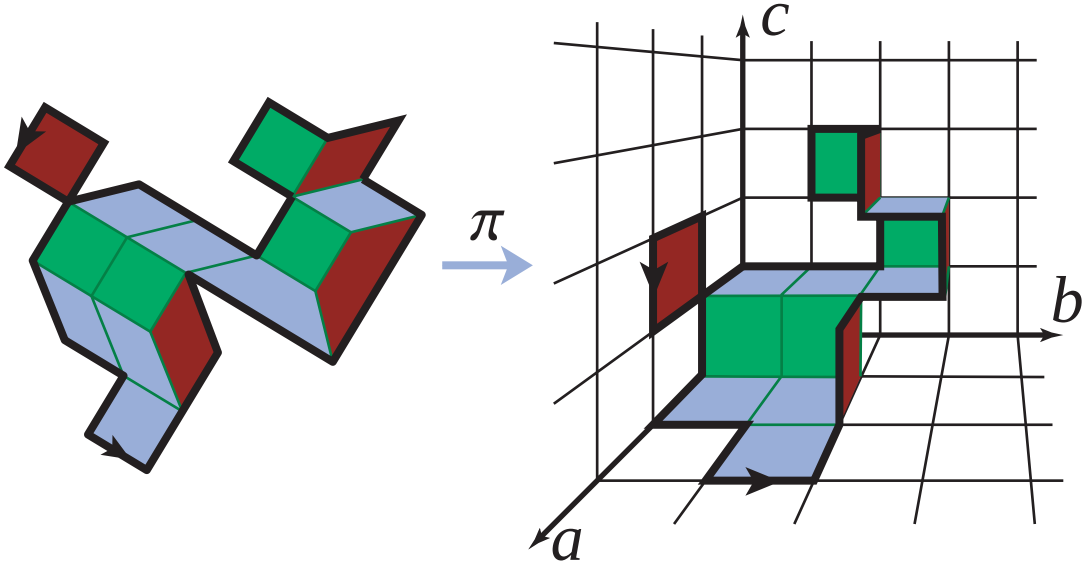

Question. Is there a finite presentation for which

$$\mathrm{IDiam}(n) \not\simeq \mathrm{EDiam}(n)\,?$$

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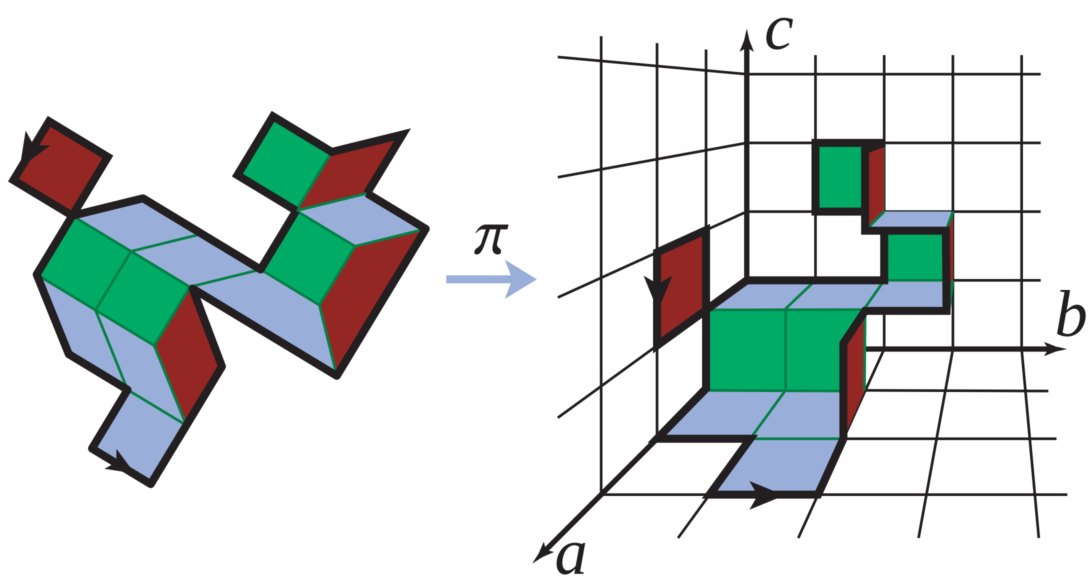


Question. Is there a finite presentation for which

$$\mathrm{IDiam}(n) \not\simeq \mathrm{EDiam}(n)\ ?$$

Theorem. [*Bridson, R.*] Yes!

Theorem. [*Bridson, R.*]
$\forall \alpha > 0$, there is a finite presentation for which

$$n^{\alpha}\,\mathrm{EDiam}(n) = O(\mathrm{IDiam}(n)).$$

Our family of groups:

$$\Psi_{k,m} \;=\; \Phi_k *_{\langle t \rangle} \Gamma_m,$$

amalgamated along an infinite cyclic subgroup $\langle t \rangle$.

Presentation of Γ_m

generators $a_1, \ldots, a_m, \sigma, t, \tau, T$

relations $\sigma^{-1} a_m \sigma = a_m;\ \forall\, i < m,\ \sigma^{-1} a_i \sigma = a_i a_{i+1}$
$\forall\, j,\ [t, a_j] = 1, [t, T], [\tau, T],$
$[\tau, a_m t],\ \forall\, i < m,\ [\tau, a_i]$

Presentation of Φ_k

generators $s_1, \ldots, s_k, f, g \quad \hat{s}_1, \ldots, \hat{s}_k, \hat{f}, \hat{g} \quad b, t$

relations $t^{-1} b s_k = b^3,\ s_k{}^{-1} b t = b^3,\ \hat{s}_k{}^{-1} b \hat{s}_k = b^3$

$\forall i < k,\quad f^{-1} s_k f = s_k,\ f^{-1} s_i f = s_i s_{i+1},\ \hat{f}^{-1} \hat{s}_k \hat{f} = \hat{s}_k,\ \hat{f}^{-1} \hat{s}_i \hat{f} = \hat{s}_i \hat{s}_{i+1}$

$g^{-1} s_k g = s_k,\ g^{-1} s_{k-1} g = s_{k-1}, \hat{g}^{-1} \hat{s}_k \hat{g} = \hat{s}_k,\ \hat{g}^{-1} \hat{s}_{k-1} \hat{g} = \hat{s}_{k-1}$

$\forall i < k-1,\ g^{-1} s_i g = s_i s_{i+1}, \qquad \hat{g}^{-1} \hat{s}_i \hat{g} = \hat{s}_i \hat{s}_{i+1}$

$\forall i \neq j,\quad [s_i, s_j] = 1, \qquad [\hat{s}_i, \hat{s}_j] = 1$

Γ_m-Diagrams with large intrinsic diameter

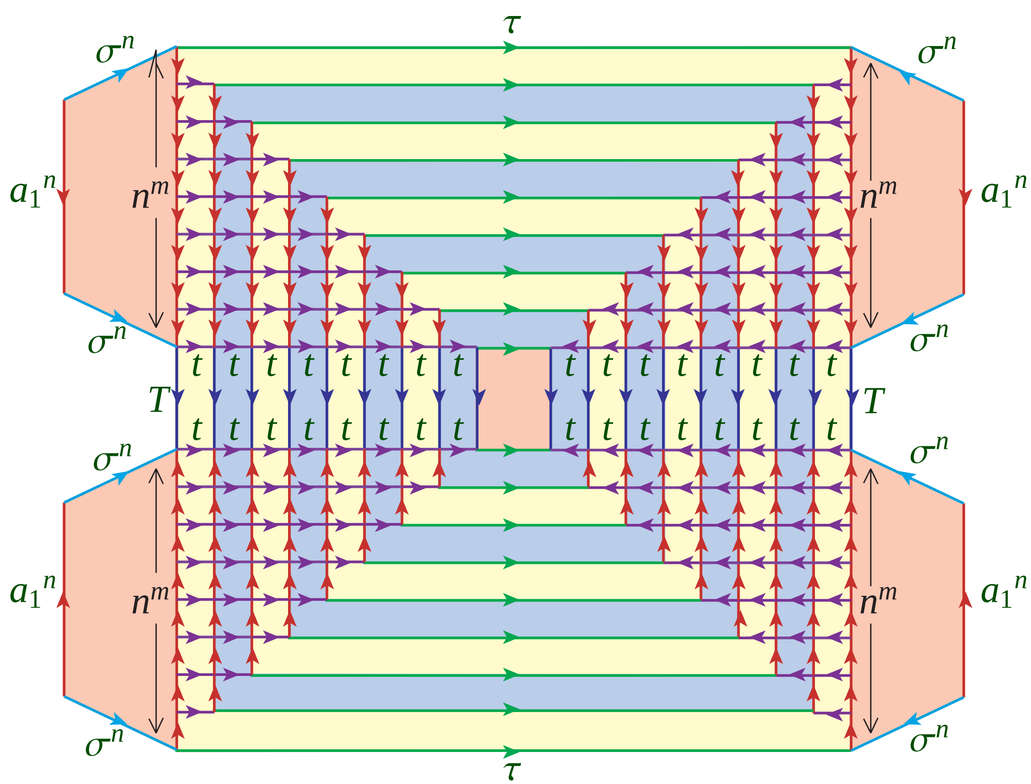
generators $a_1, \ldots, a_m, \sigma, t, \tau, T$

relations $\sigma^{-1} a_m \sigma = a_m; \ \forall\, i < m, \sigma^{-1} a_i \sigma = a_i a_{i+1}$

$\forall\, j, \ [t, a_j] = 1, [t, T], [\tau, T],$

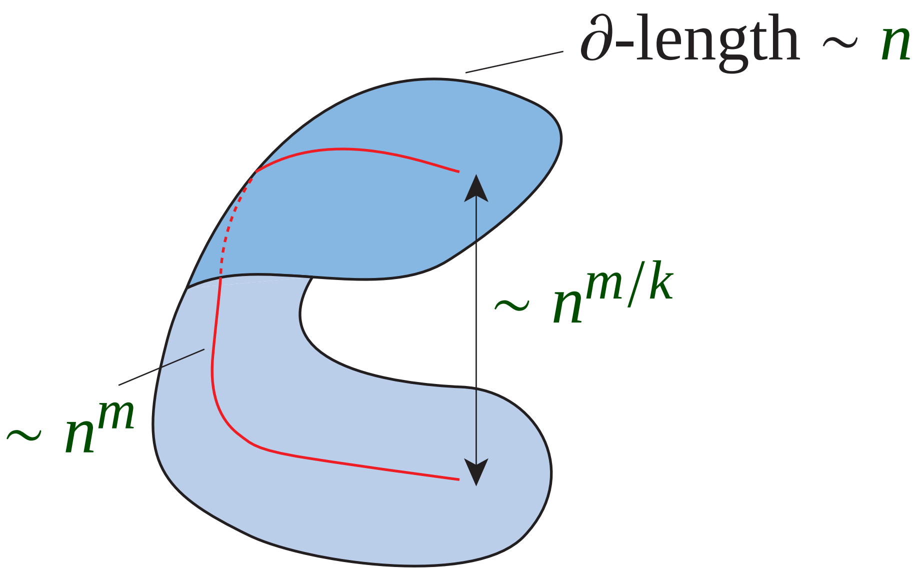
$[\tau, a_m t], \ \forall\, i < m, \ [\tau, a_i]$

Γ_m–diagrams such as



have intrinsic diameter $\sim n^m$ on account of the nested t-rings.

Inserting a shortcut



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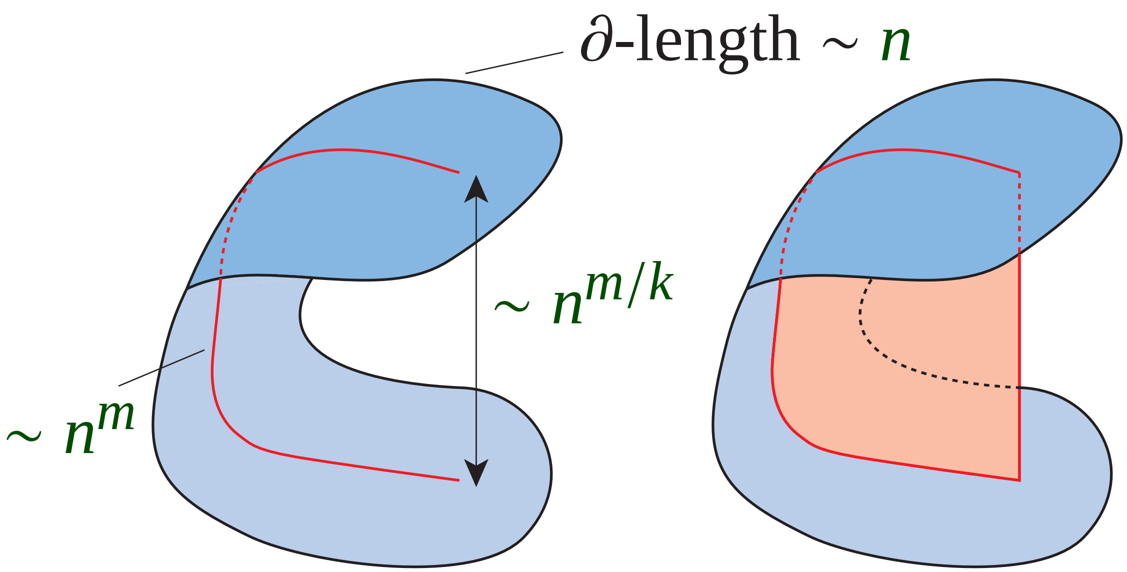

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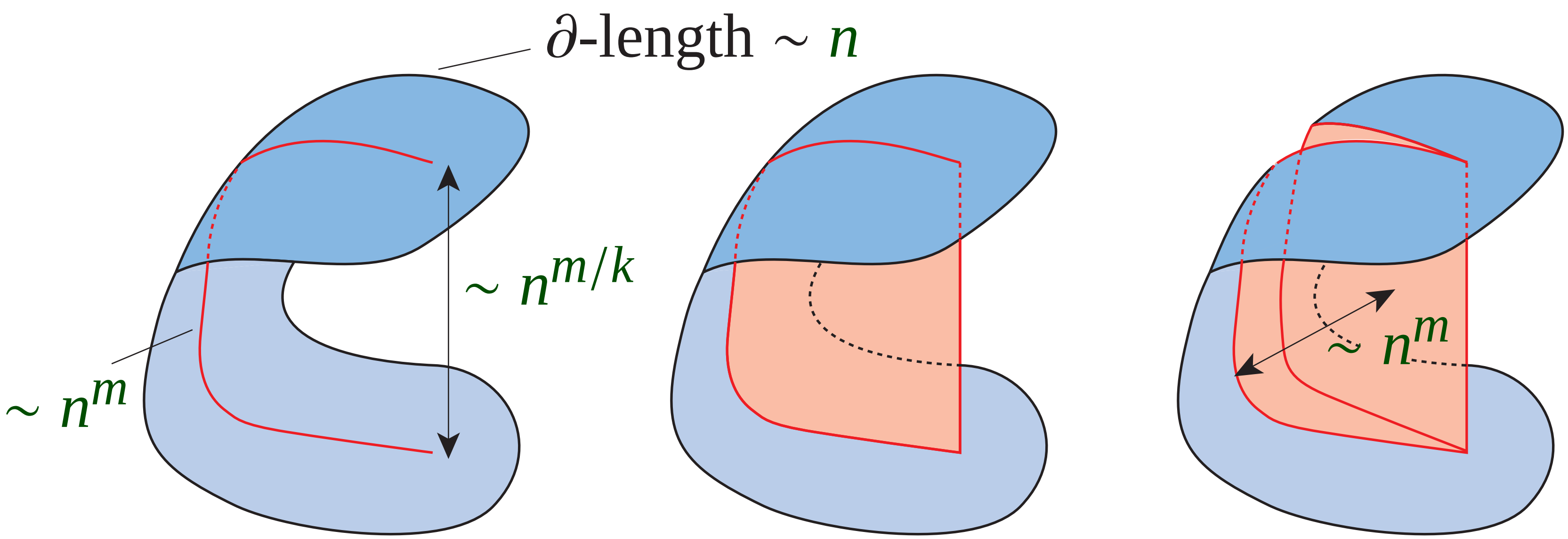
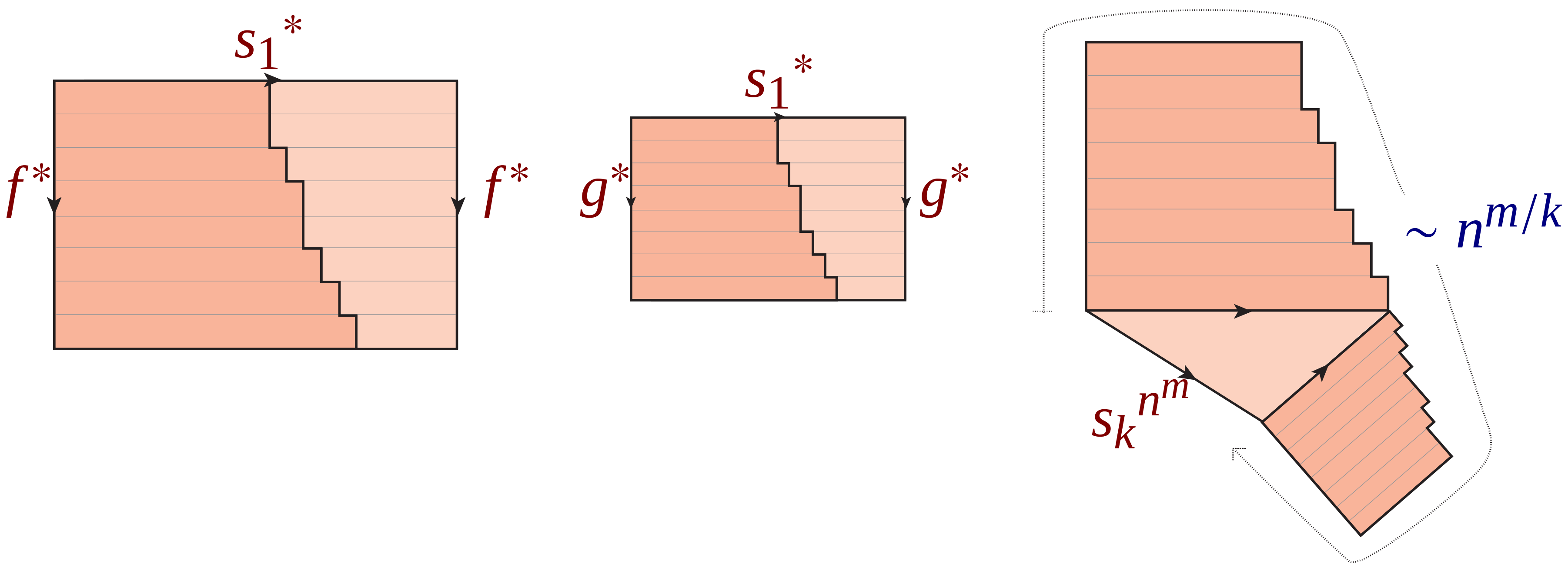


Constructing a shortcut I

The relations for Φ_k include:

$$f^{-1} s_k f = s_k; \; \forall\, i < k, \; f^{-1} s_i f = s_i s_{i+1}$$
$$g^{-1} s_k g = s_k, \; g^{-1} s_{k-1} g = s_{k-1}$$
$$\forall\, i < k-1, \; g^{-1} s_i g = s_i s_{i+1}$$
$$\forall\, i \neq j, \; [s_i, s_j] = 1$$

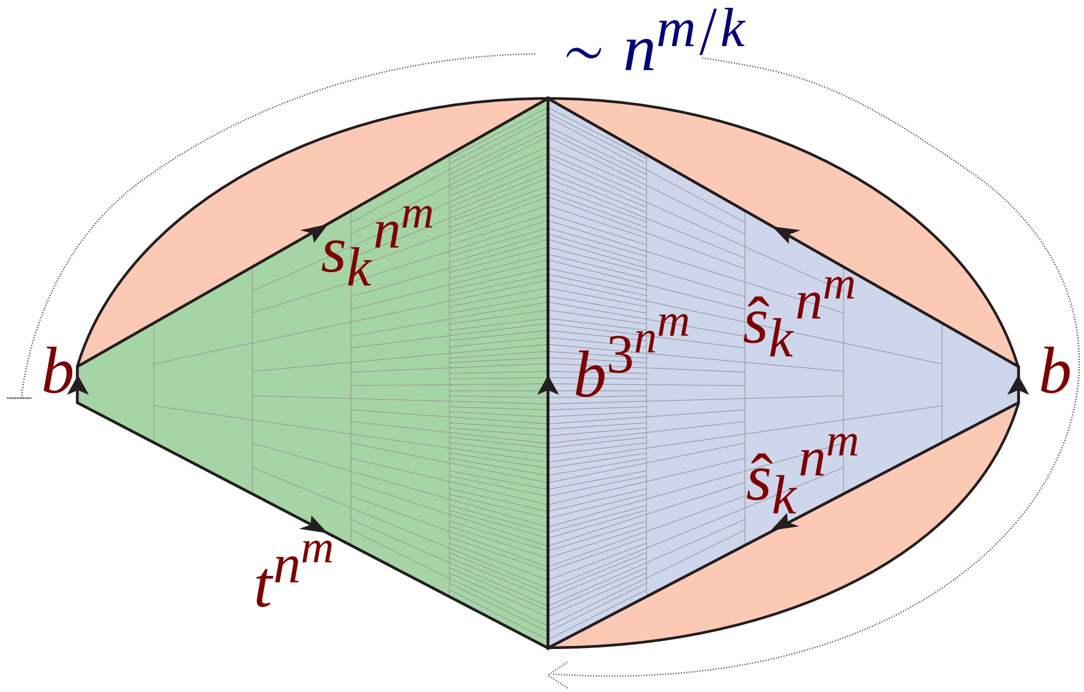
These distort ${s_k}^{n^m}$ to a word of length $\sim n^{m/k}$:



Constructing a shortcut II

Relations for Φ_k include $t^{-1}bs_k = b^3,\ s_k{}^{-1}bt = b^3,\ \hat{s}_k{}^{-1}b\hat{s}_k = b^3$

t^{n^m} is distorted in Φ_k to a word of length $\sim n^{m/k}$ by a *fat* diagram:



This diagram has IDiam $\sim n^m$.

We get* that $\Psi_{k,m}$ has

$$\mathrm{EDiam}(n) \;\preceq\; n^{\max\left\{\frac{m}{k},\, k\right\}}$$

and

$$n^m \;\preceq\; \mathrm{IDiam}(n),$$

and choosing k and m appropriately establishes:

Theorem. $\forall \alpha > 0$, there is a finite presentation $\mathcal{P}$ for which

$$n^{\alpha}\,\mathrm{EDiam}_{\mathcal{P}}(n) \;=\; O(\mathrm{IDiam}_{\mathcal{P}}(n)).$$

* roughly speaking

Theorem. If $\mathcal{P}$ is a finite presentation of the fundamental group Γ of a closed connected smooth Riemannian manifold M then

$$\mathrm{IDiam}_{\mathcal{P}} \simeq \mathrm{IDiam}_M \quad \textit{and} \quad \mathrm{EDiam}_{\mathcal{P}} \simeq \mathrm{EDiam}_M.$$

As all finitely presentable groups can be so realised, deduce –

Corollary. $\forall \alpha > 0$, there exists a closed connected smooth Riemannian manifold M such that

$$l^{\alpha}\,\mathrm{EDiam}_M(l) = O(\mathrm{IDiam}_M(l)).$$