PALINDROMIC WIDTH OF WREATH PRODUCTS, METABELIAN GROUPS, AND MAX-N SOLVABLE GROUPS

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Abstract. A group has finite palindromic width if there exists n such that every element can be expressed as a product of n or fewer palindromic words. We show that if G has finite palindromic width with respect to some generating set, then so does $G \wr \mathbb{Z}^r$. We also give a new, self-contained, proof that finitely generated metabelian groups have finite palindromic width. Finally, we show that solvable groups satisfying the maximal condition on normal subgroups (max-n) have finite palindromic width.

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1. Introduction

The *width* of a group G with respect to an (often infinite) generating set A is the minimal n such that every $g \in G$ can expressed as the product of n or fewer elements from A. If no such n exists, the width is infinite. Examples include the primitive width of free groups (e.g. [6]), and the commutator width of a derived subgroup, or more generally the verbal width of a verbal subgroup with respect to any given word ([21] is a survey).

This paper concerns *palindromic width*. Suppose G is a group with generating set X. Write $\mathrm{PW}(G, X)$ for the width of G with respect to the set of palindromic words on $X \cup X^{-1}$ — the words that read the same forwards as backwards.

We give bounds on palindromic width in a variety of settings. Here is the first. (We view G and the $\mathbb{Z}^r$-factor as subgroups of $G \wr \mathbb{Z}^r$ in the standard way.)

Theorem 1.1. *If G is a group with finite generating set A, then*

$$\mathrm{PW}(G \wr \mathbb{Z}^r, A \cup S) \leqslant 3r + \mathrm{PW}(G, A)$$

where S is the standard generating set of $\mathbb{Z}^r$. Better, when $r = 1$,

$$\mathrm{PW}(G \wr \mathbb{Z}, A \cup \{t\}) \leqslant 2 + \mathrm{PW}(G, A)$$

where t is a generator of $\mathbb{Z}$.

The upper bound of our next theorem is a corollary.

Theorem 1.2. *The palindromic width of*

$$\mathbb{Z} \wr \mathbb{Z} = \left\langle a, t \;\middle|\; \left[a, a^{t^k}\right] = 1 \ (k \in \mathbb{Z}) \right\rangle$$

with respect to a, t is 3.

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The heart of our proof of Theorem 1.1 is a result (Lemma 2.1) on expressing finitely supported functions from $\mathbb{Z}^r$ to a group as a pointwise product of two such functions both exhibiting certain symmetry. We develop this result (in Section 4.1) to more elaborate results on expressing finitely supported functions from $\mathbb{Z}^r$ to a ring as the sum of what we call *skew-symmetric* finitely supported functions. This led us to a new proof of the following theorem which we have since discovered was proved by Bardakov & Gongopadhyay not long prior.

Theorem 1.3 (Bardakov & Gongopadhyay [8])**.** *The palindromic width of any metabelian group with respect to any finite generating set is finite.*

Our proof is self-contained. Bardakov & Gongopadhyay use a result of Akhavan-Malayeri and Rhemtulla [3], which in turn uses a result from the unpublished PhD thesis of Stroud [22], details of which may also be found in [21]. However they established more, namely that free abelian-by-nilpotent groups have finite palindromic width. In their sequel [4], Bardakov & Gongopadhyay have investigated lower bounds for the palindromic width of nilpotent groups and abelian–by–nilpotent groups. Also using work of Akhavan-Malayeri concerning the nature of commutators [2], E. Fink claims that the wreath product of a finitely generated free group with a finitely generated free abelian group, and hence also the wreath product of any finitely generated group with a finitely generated free abelian group, has finite palindromic width, [15].

Boundedly generated groups provide many examples with finite palindromic width. A group G is boundedly generated when there exist $a_1, \ldots, a_k \in G$ such that every element can be expressed as $a_1^{r_1} \cdots a_k^{r_k}$ for some $r_1, \ldots, r_k \in \mathbb{Z}$. In such groups, $\mathrm{PW}(G, \{a_1, \ldots, a_k\}) \leqslant k$. They include all finitely generated solvable minimax groups [17] (and so all finitely generated nilpotent or, more generally, polycyclic groups), a non-finitely presentable example of Sury [23], and $\mathrm{SL}_n(\mathbb{Z})$ for $n \geqslant 3$ with respect to elementary matrices [11] and generalizations [18, 24]. All finitely presented, torsion-free, abelian-by-cyclic groups (and so all solvable Baumslag–Solitar groups) are boundedly generated: by [10] (see also [14, §1.1]) they have presentations

$$\langle\, t, a_1, \ldots, a_m \mid a_i a_j = a_j a_i,\ t a_i t^{-1} = w_i(a_1, \ldots, a_m),\ \forall i, j \,\rangle$$

and each element can be represented as $t^{-i} a_1^{r_1} \cdots a_m^{r_m} t^j$ for some non-negative integers i, j and some $r_1, \ldots, r_m \in \mathbb{Z}$. And

$$\mathbb{Z}_2 * \mathbb{Z}_2 \;=\; \langle x, y \mid x^2 = y^2 = 1 \rangle$$

is boundedly generated as every element is expressible as $(xy)^l$, $(xy)^l x$, or $y(xy)^l$ for some $l \in \mathbb{Z}$. Passing to or from subgroups of finite index preserves bounded generation [9, Exercise 4.4.3], as does passing to a quotient.

There are finitely generated metabelian groups which are not boundedly generated, for example, $\mathbb{Z} \wr \mathbb{Z}$ [13]. So:

Corollary 1.4. *There are finitely generated groups with finite palindromic width (with respect to all finite generating sets) which are not boundedly generated.*

A group G satisfies the maximal condition for normal subgroups (max-n) if for every normal subgroup N of G, there is a finite subset which normally generates N. Finitely generated (abelian-by-polycyclic)-by-finite groups are examples [16]. We extend Theorem 1.3 to:

Theorem 1.5. *If G is a finitely generated solvable group satisfying max-n, then G has a finite generating set B such that* $\mathrm{PW}(G, B)$ *is finite.*

We understand that this result has been proved independently by Bardakov and Gongopadhyay [5], where they in fact show that a finitely generated solvable group which is abelian-by-(max-n) has finite palindromic width. This therefore includes the finitely generated solvable groups of derived length 3. In the same paper they also provide a different proof of Theorem 1.2, first showing that $\mathbb{Z}\wr\mathbb{Z}$ has commutator width 1.

There are groups known to have infinite palindromic width: rank-r free groups $F(x_1,\dots,x_r)$ for all $r \geqslant 2$ [6] with respect to $\{x_1,\dots,x_r\}$ and, with the sole exception of $\mathbb{Z}_2 * \mathbb{Z}_2$, all free products $\ast_{i=1}^m G_i$ of non-trivial groups with respect to $\bigcup_i G_i$, [7]. The proofs in [6] and [7] are novel in that they use quasi-morphisms.

The structure of the paper is as follows. In Section 2 we consider the palindromic width of groups $G \wr \mathbb{Z}^r$, proving Theorem 1.1. Section 3 gives the precise value for the palindromic width of $\mathbb{Z} \wr \mathbb{Z}$ (Theorem 1.2). Our proof for the result concerning metabelian groups is contained in Section 4, while Section 5 deals with solvable groups satisfying max-n. We conclude with a discussion of open questions in Section 6.

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2. The palindromic width of $G \wr \mathbb{Z}^r$

Suppose G is a group with finite generating set A. The group $G\wr\mathbb{Z}^r = (\bigoplus_{\mathbb{Z}^r} G) \rtimes \mathbb{Z}^r$, where we view $\bigoplus_{\mathbb{Z}^r} G$ as the group of finitely supported functions $\mathbb{Z}^r \to G$ under coordinatewise multiplication, and elements $\mathbf{v}$ of the $\mathbb{Z}^r$-factor act on $\bigoplus_{\mathbb{Z}^r} G$ by the *shift operation*: $f^{\mathbf{v}}(\mathbf{x}) = f(\mathbf{x}-\mathbf{v})$ for all $\mathbf{x} \in \mathbb{Z}^r$. Note that $G \wr \mathbb{Z}^r$ is generated by the union of $A \times \{\mathbf{0}\}$ and $\{1\} \times B$ where B is the standard basis $B = \{\mathbf{e}_1,\dots,\mathbf{e}_r\}$ for $\mathbb{Z}^r$.

2.1. An example from $\mathbb{Z} \wr \mathbb{Z}$.

Define $\Delta_1 : \mathbb{Z} \to \mathbb{Z}$ to be 1 at 0 and 0 elsewhere, and $\mathbf{0} : \mathbb{Z} \to \mathbb{Z}$ to be everywhere 0. The standard generating set for $\mathbb{Z} \wr \mathbb{Z}$ is $\{a, t\}$ where $a = (\Delta_1, 0)$ and $t = (\mathbf{0}, 1)$.

Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function whose non-zero values are given in the following table.

x	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$f(x)$	0	0	0	3	-1	4	0	0	1	5	0	0	0	0	2
$g(x)$	0	-2	-2	1	0	4	-1	-2	-1	4	0	1	-2	-2	0
$h(x)$	0	2	2	2	-1	0	1	2	2	1	0	-1	2	2	2

We will explain how to express $(f, 7)$ as a product of three palindromes. The ideas apparent here are the core of the proof of our upper bounds on the palindromic width of general $G \wr \mathbb{Z}^r$.

The table above also shows the non-zero values of functions $g, h : \mathbb{Z} \to \mathbb{Z}$ such that $f = g + h$ with g being symmetric about zero (that is, $g(x) = g(-x)$ for all x) and h being symmetric about $\frac{1}{2}$ (that is, $h(x) = h(1-x)$ for all x).

Here is how these g and h were found. We sought g supported on $\{-7,-6,\dots,6,7\}$ and h supported on $\{-6,-5,\dots,6,7\}$, which was reasonable since the support of f is a subset of each these sets and they are symmetric about 0 and $\frac{1}{2}$, respectively.

Since -7 is outside the support of h we have $h(-7) = 0$. We deduced from the requirement that $f = g + h$ that $g(-7) = 0$. From here, symmetry of g about 0 and of h about $\frac{1}{2}$, together with $f = g + h$, determined the remaining values taken by g and h. The symmetry allowed us to "jump" to the other side of 0 or $\frac{1}{2}$. Once there we applied $f = g + h$ and were ready to "jump" again.

Specifically in this example symmetry of g about 0 gave $g(7) = g(-7) = 0$, and then $h(7) = f(7) - g(7) = 2$. Then, using the symmetry of h about $\frac{1}{2}$, we found $h(-6) = h(6) = 2$, and then $g(-6) = -2$ (as $f = g + h$). We continued in this way until the table was complete.

As we will shortly explain, the palindromes

$$\begin{aligned} w_g &:= t^{-6}a^{-2}ta^{-2}tat^2a^4ta^{-1}ta^{-2}ta^{-1}ta^4t^2ata^{-2}ta^{-2}t^{-6} \\ w_h &:= t^{-6}a^2ta^2ta^2ta^{-1}t^2ata^2ta^2tat^2a^{-1}ta^2ta^2ta^2t^{-6}. \end{aligned}$$

represent $(g, 0)$ and $(h, 1)$, respectively. So the product $w_g w_h t^6$ of three palindromes represents $(f, 7)$.

How we obtained w_g and w_h is best explained using the *lamplighter model* of $\mathbb{Z} \wr \mathbb{Z}$. View the real line as an infinite street and imagine a lamp at each integer point. Each lamp has $\mathbb{Z}$–many states. A *lamp configuration* is an assignment of a state (an integer) to each lamp with all but finitely many lamps assigned zero. (Equivalently a lamp configuration is a finitely supported function $\mathbb{Z} \to \mathbb{Z}$.) Elements of $\mathbb{Z} \wr \mathbb{Z}$ are represented by a lamp configuration together with a choice of lamp by which a lamplighter is imagined to stand. The identity is represented by the street in darkness, all lamps are in state zero, with the lamplighter at position zero.

The generators t and a of $\mathbb{Z} \wr \mathbb{Z}$ act in the following manner: applying t moves the lamplighter one step right (that is, in the positive direction); applying a adds one to the state of the bulb at the location of the lamplighter.

Group elements that can be represented by palindromes on $\{a^{\pm 1}, t^{\pm 1}\}$ can be recognised as those for which the lamp configuration is symmetric about some point $m/2$, where $m \in \mathbb{Z}$, and such that the lamplighter finishes at position m.

For example, in the instance of $(g, 0)$ we begin by sending the lamplighter from zero to the leftmost extreme of the support of g. This is 6 steps left, so is an application of t^{-6}. Next we change the state of the bulb here to $g(-6) = -2$, so we apply a^{-2}. Now we proceed right one step at a time by applying t. After each step we adjust the state of the bulb according to the function g by applying a or a^{-1} the appropriate number of times. When we reach the rightmost extreme of the support of g, we will have a lamp configuration which is symmetric about 0. To finish, and give ourselves a palindrome, we need to repeat the first steps we took in reaching the left-most point of the support from the origin, namely we apply t^{-6} again.

2.2. **The upper bounds.** Suppose G is a group with generating set A. In the example above we expressed $f : \mathbb{Z} \to \mathbb{Z}$ as a sum of two symmetric functions. We will generalise this to expressing functions $f : \mathbb{Z}^r \to G$ as products of symmetric functions.

Let $(A \cup A^{-1})^*$ denote the free monoid on the set $A \cup A^{-1}$—that is, the (finite) words on A and A^{-1}. For u in this free monoid, let $\overline{u}$ denote the same word written in reverse. Let ε denote the empty word. For $\gamma \in G$, define $\Delta_\gamma : \mathbb{Z}^r \to (A \cup A^{-1})^*$ by $\Delta_\gamma(\mathbf{0}) = \gamma$ and $\Delta_\gamma(\mathbf{x}) = \varepsilon$ for all $\mathbf{x} \neq \mathbf{0}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_r$ be the standard generating set for $\mathbb{Z}^r$.

Lemma 2.1. *Suppose $f : \mathbb{Z}^r \to (A \cup A^{-1})^*$ is finitely supported. Then*

$$f \;=\; \Delta_\gamma f_0 f_1 \dots f_r$$

in $\bigoplus_{\mathbb{Z}^r} G$ for some $\gamma \in G$ and some finitely supported

$$f_0, f_1, \dots, f_r : \mathbb{Z}^r \to (A \cup A^{-1})^*$$

such that $f_0(\mathbf{x}) = \overline{f_0(-\mathbf{x})}$ and $f_i(\mathbf{x}) = \overline{f_i(\mathbf{e}_i - \mathbf{x})}$ for $i = 1, \dots, r$.

Proof. We induct on r. For $r = 1$, take $n > 0$ such that $f(j) = \varepsilon$ for all j for which $|j| > n$. Define g and h as follows. First set $h(-n) := \varepsilon$. Then, for $i = n, \dots, 1$, define

$$\begin{array}{rclcl} \overline{g(i)} & := & g(-i) & := & f(-i)h(-i)^{-1} \\ h(i) & := & \overline{h(1-i)} & := & g(i)^{-1}f(i) \end{array}$$

and finish by setting $g(0) := \varepsilon$ and $g(j) = h(j) = \varepsilon$ for all j for which $|j| > n$. Then take $\gamma := f(0)h(0)^{-1}$. By construction, $f_0 := g$ and $f_1 := h$ have the required properties.

Now suppose $r > 1$. Let $n > 0$ be such that f is supported on $\{-n, \dots, n-1, n\}^r$. We first define functions $g, h : \mathbb{Z}^r \to (A \cup A^{-1})^*$ such that $f = gh$, with h carrying the required symmetry about $\frac{1}{2}\mathbf{e}_r$ and the symmetry of g about the origin being satisfied everywhere except in the codimension 1 subspace orthogonal to $\mathbf{e}_r$.

Consider $\mathbb{Z}^r$ as $\mathbb{Z}^{r-1} \times \mathbb{Z}$, where the 1–dimensional factor is the span of $\mathbf{e}_r$. For $\mathbf{x} \in \mathbb{Z}^{r-1}$, recursively define $g(\mathbf{x}, i)$ and $h(\mathbf{x}, i)$ for $i = 1, \dots, n$ and $g(-\mathbf{x}, i)$ and $h(-\mathbf{x}, i)$ for $i = 0, -1, \dots, -n$ as follows. First set $h(-\mathbf{x}, -n) := \varepsilon$. Then, for $i = n, \dots, 1$, define

$$\begin{array}{rclcl} \overline{g(\mathbf{x}, i)} & := & g(-\mathbf{x}, -i) & := & f(-\mathbf{x}, -i)h(-\mathbf{x}, -i)^{-1} \\ h(\mathbf{x}, i) & := & \overline{h(-\mathbf{x}, 1-i)} & := & g(\mathbf{x}, i)^{-1}f(\mathbf{x}, i) \end{array}$$

and

$$g(\mathbf{x}, 0) := f(\mathbf{x}, 0)h(\mathbf{x}, 0)^{-1}.$$

Since g and h are supported on $\{-n, \dots, n-1, n\}^r$, this defines them everywhere on $\mathbb{Z}^r$. Let $S := \{(\mathbf{x}, 0) \mid \mathbf{x} \in \mathbb{Z}^{r-1}\}$. Note that $h(\mathbf{x}, i) = \overline{h(-\mathbf{x}, 1-i)}$ for all $(\mathbf{x}, i) \in \mathbb{Z}^r$ and so $f_r := h$ satisfies the condition required by the lemma. However, we only know that $g(\mathbf{x}, i) = \overline{g(-\mathbf{x}, -i)}$ for all $(\mathbf{x}, i) \in \mathbb{Z}^r \smallsetminus S$, and so g cannot serve as f_0 as it stands.

By the inductive hypothesis, the restriction of g to S can be expressed as the product of Δ_γ, for some $\gamma \in G$, and suitably symmetric functions $f_0, f_1, \dots, f_{r-1} : \mathbb{Z}^{r-1} \to (A \cup A^{-1})^*$. Extend these to be functions on $\mathbb{Z}^r$ by defining them to be ε outside of S. Note that they retain their symmetry. The product $f = \Delta_\gamma f_0 f_1 \dots f_{r-1} f_r$ is therefore the required expression. □

We are now ready to prove the upper bounds of Theorem 1.1. We will show that if $x_1, \dots, x_r$ are generators for $\mathbb{Z}^r$, which will we now write multiplicatively, then $\mathrm{PW}(G \wr \mathbb{Z}^r, A \cup \{x_1, \dots, x_r\}) \leqslant 3r + \mathrm{PW}(G, A)$.

We will explain how to write an arbitrary element $(f, x_1^{t_1} \dots x_r^{t_r}) \in G \wr \mathbb{Z}^r$ as a product of at most $3r + \mathrm{PW}(G, A)$ palindromes. Choose $n > 0$ so that f is supported on

$\{-n, \dots n-1, n\}^r$. Iteratively define palindromes u_i, for $i = 1, \dots, r$, as follows:

$$\begin{aligned} u_1 &:= x_1^{2n}, \\ u_2 &:= (u_1 x_2 u_1^{-1} x_2)^n u_1, \\ &\vdots \\ u_r &:= (u_{r-1} x_r u_{r-1}^{-1} x_r)^n u_{r-1}. \end{aligned}$$

So u_r defines a Hamiltonian path which snakes around the cube $[0, 2n]^r$. Set $u := x_1^{-n} \dots x_r^{-n} u_r x_r^{-n} \dots x_1^{-n}$. Insert the words $f(\mathbf{x})$ for all $\mathbf{x} \in \{-n, \dots, n-1, n\}^r$ into u as follows. For each such $\mathbf{x}$ there exists a unique prefix $u(\mathbf{x})$ of u_r such that $x_1^{-n} \dots x_r^{-n} u(\mathbf{x}) = \mathbf{x}$. Rewrite u by inserting $f(\mathbf{x})$ after $x_1^{-n} \dots x_r^{-n} u(\mathbf{x})$, for each such $\mathbf{x}$. Denote the resulting word by u^f.

Suppose $g : \mathbb{Z}^r \to (A \cup A^{-1})^*$ satisfies $g(\mathbf{x}) = \overline{g(-\mathbf{x})}$ for all $\mathbf{x} \in \mathbb{Z}^r$. Then u^g will be a palindrome and will represent $(g, 0)$ in $G \wr \mathbb{Z}^r$.

We wish to produce words in a similar manner for a function $h : \mathbb{Z}^r \to (A \cup A^{-1})^*$ which satisfies $h(\mathbf{x}) = \overline{h(\mathbf{e}_i - \mathbf{x})}$, for all $\mathbf{x} \in \mathbb{Z}^r$. After permuting the basis vectors, we may assume $i = 1$. Define palindromes v_i as follows:

$$\begin{aligned} v_1 &:= x_1^{2n+1}, \\ v_2 &:= (v_1 x_2 v_1^{-1} x_2)^n v_1, \\ &\vdots \\ v_r &:= (v_{r-1} x_r v_{r-1}^{-1} x_r)^n v_{r-1}, \end{aligned}$$

where n, as before, is taken so that $\mathrm{Supp}(h) \subseteq \{-n, \dots, n-1, n\}^r$. Set $v := x_1^{-n} \dots x_r^{-n} v_r x_r^{-n} \dots x_1^{-n} x_1^{-1}$, which is not a palindrome, but rather is the product of two palindromes, the second being x_1^{-1}.

We rewrite v, as we did for u, by inserting $h(\mathbf{x})$ at the appropriate points to give a word v^h. The symmetric properties held by h mean that when we insert $h(\mathbf{x})$ and $h(\mathbf{e}_1 - \mathbf{x})$ into the appropriate places of v we will still have a product of two palindromes, with the second palindrome being x_1^{-1} as was originally the case. Thus the word v^h will be the product of two palindromes and moreover will represent $(h, 0)$ in $G \wr \mathbb{Z}^r$.

Express the function f as per Lemma 2.1 as $f = \Delta_\gamma f_0 f_1 \dots f_r$. In $G \wr \mathbb{Z}^r$ the element $(f_0, 0)$ is represented by the palindrome u^{f_0}, and each $(f_i, 0)$, for $i = 1, \dots, r$, is represented by the product v^{f_i} of two palindromes. Let π be a product of at most $\mathrm{PW}(G, A)$ palindromes representing γ. Then

$$(f, 0) \;=\; \pi u^{f_0} v^{f_1} \dots v^{f_r}$$

is the product of at most $\mathrm{PW}(G, A) + 2r + 1$ palindromes. Finally, we post-multiply by $x_r^{t_r} \dots x_1^{t_1}$ to obtain a word for $(f, x_1^{t_1} \dots x_r^{t_r})$. Since the second palindrome of v^{f_r} is x_r^{-1}, this is absorbed into the first palindrome of $x_r^{t_r} \dots x_1^{t_1}$. Thus we obtain a word representing $(f, x_1^{t_1} \dots x_r^{t_r})$ which is the product of $\mathrm{PW}(G, A) + 3r$ or fewer palindromes.

When $r = 1$ we can modify our proof of Lemma 2.1 to obtain a stronger upper bound for $\mathrm{PW}(G \wr \mathbb{Z}, A \cup \{t\})$ as follows.

We construct g and h from f as in Lemma 2.1, but with one difference. We absorb one palindrome from an expression for γ into $g(0)$, which had been taken to be the empty word in the proof of the lemma.

Suppose that $f(0)h(0)^{-1} = w_1 \dots w_k$, where k is minimal such that each w_i is a palindrome. Set $g(0) = w_k$, which is allowed since w_k is a palindrome. Then

$\gamma := f(0)h(0)^{-1}g(0)^{-1}$ can be expressed as the product of $k-1$ palindromes. In particular, $k \leqslant \mathrm{PW}(G, A)$, leading to:

$$\mathrm{PW}(G \wr \mathbb{Z}, A \cup \{t\}) \ \leqslant \ 2 + \mathrm{PW}(G, A).$$

3. The palindromic width of $\mathbb{Z} \wr \mathbb{Z}$ is at least 3

Here we show that $\mathrm{PW}(\mathbb{Z} \wr \mathbb{Z}, \{a, t\}) \geqslant 3$ and so complete our proof of Theorem 1.4.

Let $f : \mathbb{Z} \to \mathbb{Z}$ have support $\{0, 1\}$ and suppose $f(0) \neq f(1)$. We will show that $(f, 3)$ cannot be expressed as the product of two palindromic words.

First note that $(g, r) \in \mathbb{Z} \wr \mathbb{Z}$ can be expressed as a palindrome if and only if $g(x) = g(r - x)$ for all $x \in \mathbb{Z}$. Indeed, given a palindrome on $\{a, t\}$, the lamp configuration obtained from this word must be symmetric about $\frac{1}{2}r$ (see Section 2.1), implying that the corresponding function must be symmetric about this point, as required. Conversely, if g is symmetric about $\frac{1}{2}r$, then we can construct a palindrome in which the lamplighter will run first to the smallest lamp in the support, and then run in the positive direction, turning on all lamps to the appropriate configuration, and finishing off by running to r. The reader may check that the word obtained from this journey is indeed a palindrome.

Suppose there exists $p, q \in \mathbb{Z}$ and $g, h : \mathbb{Z} \to \mathbb{Z}$ such that g is symmetric about $\frac{1}{2}p$, h is symmetric about $\frac{1}{2}q$ and $(f, 3) = (g, p)(h, q)$. Let h_0 be the shift of h by p—that is, $h_0(x) = h(x - p)$ for $x \in \mathbb{Z}$. Thus $p + q = 3$ and

$$g(x) + h_0(x) = f(x) \tag{1}$$

for all $x \in \mathbb{Z}$. We will show that at least one of g or h_0 (hence h) must have infinite support.

We claim that $g(x) = g(x + 3)$ except possibly when $x \in \{-3, -2, p - 1, p\}$. We use the equalities:

$$g(x) \ = \ g(p - x) \ = \ -h_0(p - x) \ = \ -h_0(x + 3) \ = \ g(x + 3) \tag{2}$$

which follow from, in order, firstly symmetry of g through $\frac{1}{2}p$, secondly equation (1) assuming $f(p - x) = 0$, thirdly symmetry of h_0 through $p + \frac{1}{2}q$, and finally a second application of equation (1) assuming $f(x + 3) = 0$. As $\mathrm{Supp}(f) = \{0, 1\}$, this can fail only if $x \in \{-3, -2, p - 1, p\}$.

Similarly,

$$h_0(x) \ = \ h_0(3 + p - x) \ = \ -g(3 + p - x) \ = \ -g(x - 3) \ = \ h_0(x - 3) \tag{3}$$

provided $3+p-x$ and $x-3$ are not in $\{0, 1\}$—that is, provided $x \notin \{3, 4, p+2, p+3\}$.

First suppose $p < 0$. Then $g(x) = 0$ for all $x \geqslant 0$, otherwise it will have infinite support. Symmetry of $g(x)$ about $\frac{1}{2}p$ then implies that $g(x) = 0$ for $x \leqslant p$. Applying equation (1) gives $\mathrm{Supp}(h_0) \subseteq \{p+1, \ldots, 1\}$ and $h_0(x) = f(x)$ for $x = 1, 2$. Equation (3) implies that

$$h_0(x) \ = \ \begin{cases} f(0) & \text{if } x \equiv 0 \pmod 3, \\ f(1) & \text{if } x \equiv 1 \pmod 3, \\ 0 & \text{if } x \equiv 2 \pmod 3 \end{cases}$$

for $x \in \{p+1, \ldots, 1\}$: the values of h_0 at 0, 1 and 2, determine those at -3, -2 and -1, respectively, and then -6, -5 and -4, and so on until the value at p which

we cannot deduce from that at $p+3$. But then h_0 cannot be symmetric about any point: if a function $\mathbb{Z} \to \mathbb{Z}$ repeats a length-3 (or greater) pattern of distinct values on an interval of length at least 3, and is zero elsewhere, then f cannot be symmetric. (In the case $p = -1$, we use the fact that $\mathrm{Supp}(h_0) = \{0, 1\}$ and $f(0) \neq f(1)$.)

Now suppose $p \geqslant 0$. Then by equation (3), for $x \leqslant 1$, if $h_0(x) \neq 0$ then $h_0(x-3) \neq 0$. Thus, finite support of h_0 implies that $h_0(x) = 0$ for $x \leqslant 1$. Symmetry about $p + \frac{1}{2}q = \frac{1}{2}p + \frac{3}{2}$ then gives $h_0(x) = 0$ for $x \geqslant p+2$. In particular, by equation (1), $g(x) = f(x)$ for $x \leqslant 1$ and $x \geqslant p+2$, hence $\mathrm{Supp}(g) \subseteq \{0, \ldots, p+1\}$. Also, by (2), for $x \in \{0, \ldots, p+1\}$, we have

$$g(x) = \begin{cases} f(0) & \text{if } x \equiv 0 \pmod 3; \\ f(1) & \text{if } x \equiv 1 \pmod 3; \\ 0 & \text{if } x \equiv 2 \pmod 3. \end{cases}$$

As before, such a function cannot be symmetric. (When $p = 0$ we use that $f(1) \neq 0$.)

This covers all possible values of p, so $(f, 3)$ cannot be expressed as the product of two palindromes.

4. Finite palindromic width of metabelian groups

In this section we give our proof of Theorem 1.3.

Let $F = F(x_1, \ldots, x_r)$ be a free group on r generators and F'' be its second derived subgroup. Then F/F'' is the *free metabelian group of rank* r.

The property of having finite palindromic width passes to quotients. Indeed, if G is a group with generating set X and $\overline{G}$ is a quotient, then

$$\mathrm{PW}(\overline{G}, \overline{X}) \leqslant \mathrm{PW}(G, X),$$

where $\overline{X}$ is the image of X under the quotient map. So, it will suffice to prove Theorem 1.3 for finitely generated *free* metabelian groups with respect to their standard generating sets. More precisely, we will prove

that the palindromic width of the free metabelian group F/F'' of rank r with respect to $x_1, \ldots, x_r$ is at most $2^{r-1}r(r+1)(2r+3) + 4r + 1$. Bardakov & Gongopadhyay give a better bound of $5r$ in [8].

Our proof begins with a pair of lemmas in Section 4.1 on expressing finitely supported functions on $\mathbb{Z}^r$ as the sum of what we call *skew-symmetric* finitely supported functions. Then in Section 4.2 we determine a relationship between skew-symmetric functions and palindromes in the subgroup F'/F'' of the free metabelian group, which leads to the theorem.

4.1. Skew-symmetric functions on $\mathbb{Z}^r$.

When dealing with $G \wr \mathbb{Z}$ in Section 2, we saw how palindromes were closely related to the symmetry of functions $f : \mathbb{Z} \to \mathbb{Z}$. However, when instead investigating the free metabelian groups, we shall relate palindromes to what we call *skew-symmetric* functions on $\mathbb{Z}^r$. When $r = 1$ these are the functions that are translates of odd functions. In general, we say that a function f from $\mathbb{Z}^r$ to a ring R is *skew-symmetric* if there exists $\mathbf{p} \in \frac{1}{2}\mathbb{Z}^r$ such that for all $\mathbf{x} \in \mathbb{Z}^r$, $f(\mathbf{x}) = -f(2\mathbf{p} - \mathbf{x})$—that is, its values at $\mathbf{x}$ and at the reflection of $\mathbf{x}$ in $\mathbf{p}$ sum to zero. Note that, when $\mathbf{p} \in \mathbb{Z}^r$, this condition at $\mathbf{x} = \mathbf{p}$ is that $2f(\mathbf{p}) = 0$.

Let $\mathbf{e}_1, \dots, \mathbf{e}_r$ denote the standard basis of unit-vectors for $\mathbb{R}^r$. The following lemma is for a ring R and is written additively as appropriate for its forthcoming application, but we remark that the proof given works with R replaced by any abelian group.

Lemma 4.1. *For all $r \geqslant 1$ and all $\mathbf{p} \in \frac{1}{2}\mathbb{Z}^r$, every finitely supported function $f : \mathbb{Z}^r \to R$ such that $\sum_{\mathbf{x}\in\mathbb{Z}^r} f(\mathbf{x}) = 0$ is the sum of $r+1$ finitely supported functions skew-symmetric about $\mathbf{p}, \mathbf{p} + \frac{1}{2}\mathbf{e}_1, \dots, \mathbf{p} + \frac{1}{2}\mathbf{e}_r$.*

Proof. We will prove the result when every entry in $\mathbf{p}$ is 0 or $\frac{1}{2}$. This suffices as, if the result holds for a given f, then it holds for all its translates.

Take an integer $n > 0$ such that f is supported on $\{-n, \dots, n-1, n\}^r$. We will define functions $g, h : \mathbb{Z}^r \to R$, both supported on $\{-n, \dots, n-1, n\}^r$, such that $f = g + h$. View f, g and h as functions $\mathbb{Z}^{r-1} \times \mathbb{Z} \to R$. Let $\overline{\mathbf{p}}$ be the projection of $\mathbf{p}$ to the $\mathbb{Z}^{r-1}$–factor. So $\mathbf{p}$ is $(\overline{\mathbf{p}}, 0)$ or $(\overline{\mathbf{p}}, \frac{1}{2})$.

First suppose $\mathbf{p} = (\overline{\mathbf{p}}, 0)$. For $\mathbf{x} \in \mathbb{Z}^{r-1}$, recursively define $g(\mathbf{x}, i)$ and $h(\mathbf{x}, i)$ for $i = 1, \dots, n$ and $g(2\overline{\mathbf{p}} - \mathbf{x}, i)$ and $h(2\overline{\mathbf{p}} - \mathbf{x}, i)$ for $i = 0, -1, \dots, -n$ as follows. First set $h(2\overline{\mathbf{p}} - \mathbf{x}, -n) := 0$. Then, for $i = n, \dots 1$, define

$$\begin{array}{rclcl} -g(\mathbf{x}, i) & := & g(2\overline{\mathbf{p}} - \mathbf{x}, -i) & := & f(2\overline{\mathbf{p}} - \mathbf{x}, -i) - h(2\overline{\mathbf{p}} - \mathbf{x}, -i) \\ -h(\mathbf{x}, i) & := & h(2\overline{\mathbf{p}} - \mathbf{x}, -i+1) & := & -f(\mathbf{x}, i) + g(\mathbf{x}, i) \end{array}$$

and

$$g(2\overline{\mathbf{p}} - \mathbf{x}, 0) := f(2\overline{\mathbf{p}} - \mathbf{x}, 0) - h(2\overline{\mathbf{p}} - \mathbf{x}, 0).$$

(In the case $r = 1$, the $\mathbf{x}$ and $2\overline{\mathbf{p}} - \mathbf{x}$ terms are absent.) As g and h are supported on $\{-n, \dots, n-1, n\}^r$, this defines them everywhere on $\mathbb{Z}^r$. Let $S := \{(\mathbf{x}, 0) \mid \mathbf{x} \in \mathbb{Z}^{r-1}\}$. Observe that h is skew-symmetric about $(\overline{\mathbf{p}}, \frac{1}{2}) = \mathbf{p} + \frac{1}{2}\mathbf{e}_r$ and that g is *nearly* skew-symmetric about $(\overline{\mathbf{p}}, 0) = \mathbf{p}$, the condition only (possibly) failing on S.

Suppose, on the other hand, $\mathbf{p} = (\overline{\mathbf{p}}, \frac{1}{2})$. Then define $S := \{\, (\mathbf{x}, 1) \mid \mathbf{x} \in \mathbb{Z}^{r-1} \,\}$ and define g and h similarly in such a way that h is again skew-symmetric about $(\overline{\mathbf{p}}, \frac{1}{2}) = \mathbf{p}$, but this time g is skew-symmetric about $(\overline{\mathbf{p}}, 1) = \mathbf{p} + \frac{1}{2}\mathbf{e}_r$ except possibly on S. Explicitly, for $\mathbf{x} \in \mathbb{Z}^{r-1}$ first set $g(2\overline{\mathbf{p}} - \mathbf{x}, -n) := 0$. Then, for $i = n, \dots, 1$, define

$$\begin{array}{rclcl} -h(\mathbf{x}, i+1) & := & h(2\overline{\mathbf{p}} - \mathbf{x}, -i) & := & f(2\overline{\mathbf{p}} - \mathbf{x}, -i) - g(2\overline{\mathbf{p}} - \mathbf{x}, -i) \\ -g(\mathbf{x}, i+1) & := & g(2\overline{\mathbf{p}} - \mathbf{x}, -i+1) & := & -f(\mathbf{x}, i+1) + h(\mathbf{x}, i+1) \end{array}$$

and

$$\begin{array}{rclcl} -h(\mathbf{x}, 1) & := & h(2\overline{\mathbf{p}} - \mathbf{x}, 0) & := & f(2\overline{\mathbf{p}} - \mathbf{x}, 0) - g(2\overline{\mathbf{p}} - \mathbf{x}, 0) \\ -g(\mathbf{x}, 1) & & & := & -f(\mathbf{x}, 1) + h(\mathbf{x}, 1). \end{array}$$

Suppose that $r = 1$. Then $\mathbf{p}$ is 0 or $\frac{1}{2}$. We claim that in the former case g is skew-symmetric about 0 and in the latter g is skew-symmetric about 1. This follows from the construction of g, which immediately gives $g(i) = -g(-i)$ when $\mathbf{p} = 0$ and gives $g(i) = -g(-i+1)$ when $\mathbf{p} = \frac{1}{2}$, and the calculations:

$$g(0) = f(0) - h(0) = f(0) + f(1) - g(1) = \cdots = \sum_{x\in\mathbb{Z}} f(x) = 0 \quad \text{when} \quad \mathbf{p} = 0,$$

$$g(1) = f(1) - h(1) = f(1) + f(0) - g(0) = \cdots = \sum_{x\in\mathbb{Z}} f(x) = 0 \quad \text{when} \quad \mathbf{p} = \frac{1}{2}.$$

This gives the base case of an induction on r.

Suppose $r > 1$. Redefine g on S to be everywhere zero. When $\mathbf{p} = (\overline{\mathbf{p}}, 0)$, this will make g skew-symmetric about $\mathbf{p}$ while h is skew-symmetric about $\mathbf{p} + \frac{1}{2}\mathbf{e}_r$; and

when $\mathbf{p} = (\overline{\mathbf{p}}, \frac{1}{2})$, it makes g skew-symmetric about $\mathbf{p} + \frac{1}{2}\mathbf{e}_r = (\overline{\mathbf{p}}, 1)$ while h is skew-symmetric about $\mathbf{p}$. However, $f = g + h$ may now fail on S (and only on S). When $\mathbf{p} = (\overline{\mathbf{p}}, 0)$,

$$\begin{aligned}\sum_{\mathbf{x}\in\mathbb{Z}^{r-1}} (f-g-h)(\mathbf{x},0) &= \sum_{\mathbf{x}\in\mathbb{Z}^{r}} (f-g-h)(\mathbf{x}) \\ &= \sum_{\mathbf{x}\in\mathbb{Z}^{r}} f(\mathbf{x}) - \sum_{\mathbf{x}\in\mathbb{Z}^{r}} g(\mathbf{x}) - \sum_{\mathbf{x}\in\mathbb{Z}^{r}} h(\mathbf{x}) \\ &= 0 - 0 - 0\end{aligned}$$

since g and h are skew-symmetric and, by hypothesis, $\sum_{\mathbf{x}\in\mathbb{Z}^r} f(\mathbf{x}) = 0$. Similar calculations show $\sum_{\mathbf{x}\in\mathbb{Z}^{r-1}} (f-g-h)(\mathbf{x},1) = 0$ when $\mathbf{p} = (\overline{\mathbf{p}}, \frac{1}{2})$. Thus, by induction, $f - g - h$ can be expressed as a sum of r functions that are skew-symmetric about $\mathbf{p}, \mathbf{p} + \frac{1}{2}\mathbf{e}_1, \ldots, \mathbf{p} + \frac{1}{2}\mathbf{e}_{r-1}$. So we have the result. □

In Section 4.2 we will need to express a function $f : \mathbb{Z}^r \to R$ as a sum of functions which are skew-symmetric about $\mathbf{p}, \mathbf{p} + \mathbf{e}_1, \ldots, \mathbf{p} + \mathbf{e}_r$. To do so, we invoke hypotheses that are stronger than those for Lemma 4.1.

Consider the set of vectors $\mathcal{D} = \{\varepsilon_1\mathbf{e}_1 + \ldots + \varepsilon_r\mathbf{e}_r \mid \varepsilon_i = 0, 1\}$ and the 2^r double-size grids $2\mathbb{Z}^r + \mathbf{v}$, one for each $\mathbf{v} \in \mathcal{D}$, which partition $\mathbb{Z}^r$.

Lemma 4.2. *For all $r \geqslant 1$ and all $\mathbf{p} \in \mathbb{Z}^r$, every finitely supported function $f : \mathbb{Z}^r \to R$ such that $\sum_{\mathbf{x}\in 2\mathbb{Z}^r+\mathbf{v}} f(\mathbf{x}) = 0$ for all $\mathbf{v} \in \mathcal{D}$ is the sum of $r + 1$ finitely supported functions skew-symmetric about $\mathbf{p}, \mathbf{p} + \mathbf{e}_1, \ldots, \mathbf{p} + \mathbf{e}_r$.*

Proof. We can express f as the sum

$$f(\mathbf{x}) = \sum_{\mathbf{v}\in\mathcal{D}} f_{\mathbf{v}}(\mathbf{x})$$

where $f_{\mathbf{v}}(\mathbf{x}) = f(\mathbf{x})$ if $\mathbf{x} \in 2\mathbb{Z}^r + \mathbf{v}$ and 0 otherwise. Let $\varphi_{\mathbf{v}} : \mathbb{Z}^r \to \mathbb{Z}^r$ be given by $\varphi_{\mathbf{v}}(\mathbf{x}) = 2\mathbf{x} + \mathbf{v}$. For each $\mathbf{v} \in \mathcal{D}$ we can apply Lemma 4.1 to $f \circ \varphi_{\mathbf{v}} : \mathbb{Z}^r \to \mathbb{Z}^r$, writing each $f \circ \varphi_{\mathbf{v}}$ as the sum of $r + 1$ functions:

$$f \circ \varphi_{\mathbf{v}} = \overline{f}_{\mathbf{v},0} + \ldots + \overline{f}_{\mathbf{v},r}$$

where $\overline{f}_{\mathbf{v},i}$ is skew-symmetric about $\frac{1}{2}\mathbf{p} - \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{e}_i$ (taking $\mathbf{e}_0$ to be the zero-vector). For each $\mathbf{v} \in \mathcal{D}$ and each $i \in \{0, \ldots, r\}$, we can write $\overline{f}_{\mathbf{v},i} = f_{\mathbf{v},i} \circ \varphi_{\mathbf{v}}$, where $f_{\mathbf{v},i}$ has support contained in $2\mathbb{Z}^r + \mathbf{v}$ and is skew-symmetric about $\varphi_{\mathbf{v}}(\frac{1}{2}\mathbf{p} - \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{e}_i) = \mathbf{p} + \mathbf{e}_i$. Note that $f_{\mathbf{v}} = f_{\mathbf{v},0} + \ldots + f_{\mathbf{v},r}$. Thus, since the sum of a family of functions which are all skew-symmetric about the same point will itself be skew-symmetric about that point, f is the sum of $r + 1$ skew-symmetric functions about $\mathbf{p}, \mathbf{p} + \mathbf{e}_1, \ldots, \mathbf{p} + \mathbf{e}_r$. □

4.2. **The palindromic width of free metabelian groups.** In the following, $F = F(x_1, \ldots, x_r)$ is the free group of rank r and $[x_i, x_j]$ denotes $x_i x_j x_i^{-1} x_j^{-1}$. We view $x_1, \ldots, x_r$ as also generating F/F' and F/F'', and we identify $x_i \in F/F' \cong \mathbb{Z}^r$ with the basis vector $\mathbf{e}_i$. Given a word w on $x_1^{\pm 1}, \ldots, x_r^{\pm 1}$, we will denote the same word read backwards by $\overline{w}$. So w is a palindrome if and only if w is the same word as $\overline{w}$.

While, for the sake of conciseness, we do not use it here, for visualizing the arguments in this section, we recommend the interpretation of elements of F/F'' as *flows* on the Cayley graph of $\mathbb{Z}^r$—see [12, 19, 20, 25] for further details.

The following lemma shows that considering F'/F'' suffices for obtaining an upper bound on the palindromic width of F/F''.

Lemma 4.3. *If every $g \in F'/F''$ is the product of ℓ or fewer palindromic words on $x_1^{\pm 1}, \dots, x_r^{\pm 1}$, then*

$$\mathrm{PW}(F/F'', \{x_1, \dots, x_r\}) \;\leqslant\; \ell + r.$$

Proof. A set of coset representatives for F'/F'' in F/F'' can be identified with $(F/F'')/(F'/F'') \cong F/F' \cong \mathbb{Z}^r$, which has palindromic width r with respect to $x_1, \dots, x_r$. □

The group F'/F'' is the normal closure of $Y := \{\, [x_i, x_j] \mid 1 \leqslant i < j \leqslant r \,\}$ in F/F''. Enumerate Y as $Y = \{\rho_1, \dots, \rho_m\}$, where $m = r(r+1)/2$. For all $h \in F'/F''$, there exist finitely supported functions $f_1, \dots, f_m : F/F' \to \mathbb{Z}$ such that

$$h \;=\; \prod_{u \in F/F'} u \rho_1^{f_1(u)} \dots \rho_m^{f_m(u)} u^{-1}. \tag{4}$$

Remark 4.4. The reason the product here is expressed as being over F/F' is that if words u and v on $x_1^{\pm 1}, \dots, x_r^{\pm 1}$ are equal in F/F', then $u\rho u^{-1} = v\rho v^{-1}$ in F/F'' for all $\rho \in Y$ as elements of F' commute modulo F''. For the same reason, the order in which the product is evaluated does not effect the element of F'/F'' it represents.

Lemma 4.5. *For $k = 1, \dots, m$ and $\rho_k = [x_i, x_j]$, if f_k is skew-symmetric about $-\frac{1}{2}(\mathbf{e}_i + \mathbf{e}_j)$, then h can be represented by a palindrome on $x_1^{\pm 1}, \dots, x_r^{\pm 1}$.*

Proof. For $k = 1, \dots, m$, let

$$h_k \;:=\; \prod_{u \in F/F'} u \rho_k^{f_k(u)} u^{-1}. \tag{5}$$

Suppose that $\rho_k = [x_i, x_j]$ and that f_k is skew-symmetric about $-\frac{1}{2}(\mathbf{e}_i + \mathbf{e}_j)$. The support of f_k consists of pairs of elements $\mathbf{u}$ and $-\mathbf{u} - \mathbf{e}_i - \mathbf{e}_j$. Enumerate these so that

$$\mathrm{Supp}(f_k) \;=\; \{\mathbf{u}_1, -\mathbf{u}_1 - \mathbf{e}_i - \mathbf{e}_j, \dots, \mathbf{u}_n\ , -\mathbf{u}_n - \mathbf{e}_i - \mathbf{e}_j\}.$$

Suppose u and v are words on $x_1^{\pm 1}, \dots, x_r^{\pm 1}$ representing $\mathbf{u}$ and $-\mathbf{u} - \mathbf{e}_i - \mathbf{e}_j$, respectively. Then $v^{-1} = x_i x_j \bar{u}$ and $v = \overline{u^{-1}} x_j^{-1} x_i^{-1}$ in F/F' and $f_k(v) = -f_k(u)$. So (see Remark 4.4)

$$v \rho_k^{f_k(v)} v^{-1} \;=\; \overline{u^{-1}} x_j^{-1} x_i^{-1} \rho_k^{-f_k(u)} x_i x_j \overline{u}$$

in F'/F''. Thus, if u_i is a word on $x_1^{\pm 1}, \dots, x_r^{\pm 1}$ representing $\mathbf{u}_i$, then

$$\left(u_1 \rho_k^{f_k(u_1)} u_1^{-1}\right) \dots \left(u_n \rho_k^{f_k(u_n)} u_n^{-1}\right) \left(\overline{u_n^{-1}} x_j^{-1} x_i^{-1} \rho_k^{-f_k(u_n)} x_i x_j \overline{u_n}\right) \dots$$
$$\dots \left(\overline{u_1^{-1}} x_j^{-1} x_i^{-1} \rho_k^{-f_k(u_1)} x_i x_j \overline{u_1}\right)$$

represents h_k in F'/F''. But then, as

$$x_j^{-1} x_i^{-1} \rho_k x_i x_j \;=\; x_j^{-1} x_i^{-1}\ x_i x_j x_i^{-1} x_j^{-1}\ x_i x_j \;=\; x_i^{-1} x_j^{-1} x_i x_j \;=\; \overline{\rho_k^{-1}}$$

in F (and so in F'/F''), the palindrome

$$\left(u_1 \rho_k^{f_k(u_1)} u_1^{-1}\right) \dots \left(u_n \rho_k^{f_k(u_n)} u_n^{-1}\right) \left(\overline{u_n^{-1}}\, \overline{\rho_k^{f_k(u_n)}}\, \overline{u_n}\right) \dots \left(\overline{u_1^{-1}}\, \overline{\rho_k^{f_k(u_1)}}\, \overline{u_1}\right)$$

represents h_k in F'/F''. Let $g_k = \left(u_1 \rho_k^{f_k(u_1)} u_1^{-1}\right) \dots \left(u_n \rho_k^{f_k(u_n)} u_n^{-1}\right)$. Then, since conjugates of commutators commute in F'/F'', the palindrome

$$g_m \dots g_1 \overline{g_1} \dots \overline{g_m}$$

represents h in F'/F''. □

Corollary 4.6. *If there exists $\mathbf{p} \in \mathbb{Z}^r$ such that each f_k is skew-symmetric about $\mathbf{p} - \frac{1}{2}(\mathbf{e}_i + \mathbf{e}_j)$ for each k, then $\mathbf{p}^{-1}h\mathbf{p}$ can be represented by a palindrome.*

Next, as $F/F' \cong \mathbb{Z}^r$, given suitable conditions on each of the functions f_k, we will be able to use Lemmas 4.2 and 4.5 to reorder the product (4) representing h in F'/F'', to express h as a product of boundedly many palindromes. Recall that $\mathcal{D} = \{\varepsilon_1\mathbf{e}_1 + \ldots + \varepsilon_r\mathbf{e}_r \mid \varepsilon_i = 0, 1\}$.

Lemma 4.7. *Suppose $h \in F'/F''$ is such that $\sum_{\mathbf{x}\in 2\mathbb{Z}^r+\mathbf{v}} f_k(\mathbf{x}) = 0$ for all $\mathbf{v} \in \mathcal{D}$ and all k. Then in F'/F'', h is a product of $3r+1$ or fewer palindromes on $x_1^{\pm 1}, \ldots, x_r^{\pm 1}$.*

Proof. Let h_k be as in (5). By Lemma 4.2, f_k is the sum of $r+1$ skew-symmetric functions, $f_k = f_k^{(0)} + \ldots + f_k^{(r)}$, where $f_k^{(0)}$ is skew-symmetric about $-\frac{1}{2}(\mathbf{e}_i + \mathbf{e}_j)$ and $f_k^{(\alpha)}$ is skew-symmetric about $-\frac{1}{2}(\mathbf{e}_i + \mathbf{e}_j) + \mathbf{e}_\alpha$ for $\alpha = 1, \ldots, r$. Let

$$h^{(\alpha)} := \prod_{u\in F/F'} u\rho_1^{f_1^{(\alpha)}(u)} \ldots \rho_m^{f_m^{(\alpha)}(u)} u^{-1}.$$

Then $h = h^{(0)} \cdots h^{(r)}$ in F'/F''.

Let x_0 denote the identity and $\mathbf{e}_0$ the zero-vector. For $\alpha = 0, \ldots, r$, by Corollary 4.6, $h^{(\alpha)} = x_\alpha p^{(\alpha)} x_\alpha^{-1}$ in F'/F'' for some palindrome $p^{(\alpha)}$, since $f_k^{(\alpha)}$ is skew-symmetric about $\mathbf{e}_\alpha - \frac{1}{2}(\mathbf{e}_i + \mathbf{e}_j)$. Then

$$p^{(0)} \left(x_1 p^{(1)} x_1^{-1}\right) \ldots \left(x_r p^{(r)} x_r^{-1}\right)$$

represents h in F'/F'' and is the product of $3r+1$ palindromes. □

Next we prove a version of Lemma 4.7 free of the hypotheses on f_k.

Lemma 4.8. *Every $h \in F'/F''$ is a product of*

$$2^{r-1}r(r+1)(2r+3) + 3r + 1$$

or fewer palindromes on $x_1^{\pm 1}, \ldots, x_r^{\pm 1}$.

Proof. For $k = 1, \ldots, m$ and $\mathbf{v} = \varepsilon_1\mathbf{e}_1 + \ldots + \varepsilon_r\mathbf{e}_r \in \mathcal{D}$, let $D_{k,\mathbf{v}} := \sum_{\mathbf{x}\in 2\mathbb{Z}^r+\mathbf{v}} f_k(\mathbf{x})$ and, if $\rho_k = [x_i, x_j]$, define the *battlement* words

$$q_{k,\mathbf{v}} \;:=\; x_1^{\varepsilon_1} \ldots x_r^{\varepsilon_r}\; (x_j x_i x_j^{-1} x_i)^{D_{k,\mathbf{v}}} x_i^{-2D_{k,\mathbf{v}}}\; x_r^{-\varepsilon_r} \ldots x_1^{-\varepsilon_1}.$$

Each $q_{k,\mathbf{v}}$ is the product of at most $2r+3$ palindromes. Indeed, for $D \in \mathbb{N}$, $(x_j x_i x_j^{-1} x_i)^D$ is the product of two palindromes: x_j and $x_i x_j^{-1} x_i x_j \ldots x_j x_i x_j^{-1} x_i$. Enumerate $\mathcal{D}$ as $\{\mathbf{v}_1, \ldots, \mathbf{v}_{2^r}\}$. Define

$$q \;:=\; q_{1,\mathbf{v}_1} \ldots q_{1,\mathbf{v}_{2^r}} \ldots q_{m,\mathbf{v}_1} \ldots q_{m,\mathbf{v}_{2^r}}.$$

In F,

$$q_{k,\mathbf{v}} \;=\; x_1^{\varepsilon_1} \ldots x_r^{\varepsilon_r}\; \rho_k^{-1}\left(x_i^2 \rho_k^{-1} x_i^{-2}\right) \cdots \left(x_i^{2D_{k,\mathbf{v}}-2} \rho_k^{-1} x_i^{-(2D_{k,\mathbf{v}}-2)}\right)\; x_r^{-\varepsilon_r} \ldots x_1^{-\varepsilon_1},$$

a product of $D_{k,\mathbf{v}}$ conjugates of ρ_k^{-1} by elements of $2\mathbb{Z}^r + \mathbf{v}$. So, multiplying by q corrects each function f_k suitably so that hq can be represented by a word as in (4) to which Lemma 4.7 applies.

Each of the $r(r+1)2^{r-1}$ battlement words comprising q costs at most $2r+3$ palindromes. So q, and hence also q^{-1}, is a product of $2^{r-1}r(r+1)(2r+3)$ or fewer palindromes. Since hq can be expressed as the product of $3r+1$ or fewer palindromes in F'/F'', the result follows. □

Proof of Theorem 1.3. Combine Lemmas 4.3 and 4.8. □

5. Finite palindromic width of solvable groups satisfying max-n

The normal closure $\langle\langle X\rangle\rangle$ of a subset X of a group G is the smallest normal subgroup of G containing X. So G satisfies the *maximal condition on normal subgroups* (or *max-n*) if for every $N \trianglelefteq G$, there exists a finite $X \subseteq G$ such that $N = \langle\langle X\rangle\rangle$.

Proof of Theorem 1.5. Let G be a solvable group of derived length d satisfying max-n with finite generating set A. Suppose that the d–th derived subgroup of G is $\langle\langle A_d\rangle\rangle$ for some finite $A_d \subseteq G$. Extend A to the possibly larger, but still finite, generating set $B = A \cup A_1 \cup \cdots \cup A_{d-1}$ of G. The following result gives an expression for an element of the derived subgroup G' of G.

Lemma 5.1 (Akhavan-Malayeri [1])**.** *There exists $K > 0$, depending on the size of B, such that any element of G' can be expressed as the product of K or fewer commutators of the form $[g, b]$, or their conjugates, where $b \in B$.*

The following two observations can both be found in [8, Lemma 2.5]. Each commutator $[g, b]$ is the product of three palindromes, namely

$$gbg^{-1}b^{-1} = (gb\overline{g})(\overline{g}^{-1}g^{-1})(b^{-1}).$$

Conjugation increases palindromic length by at most 1. Indeed, if $g = g_1 \dots g_{2k}$, where each g_i is a palindrome and g_{2k} is possibly the empty word, then, for $h \in G$,

$$hgh^{-1} = (hg_1\overline{h})(\overline{h}^{-1}g_2h^{-1})(hg_3\overline{h})\dots(\overline{h}^{-1}g_{2k}h^{-1}).$$

So, every element of G' may be written as the product of at most $4K$ palindromes. Finally, G/G' is a finitely generated abelian group, so has palindromic width equal to the size of a minimal generating set. So $\mathrm{PW}(G, B)$ is finite. □

6. Open questions

Quantitative results concerning the palindromic width of free nilpotent groups with respect to particular generating sets have recently been established [8]. However the relationship between palindromic width and the choice of finite generating set remains unclear. In particular:

Question 6.1. *Is there a group G with finite generating sets X and Y such that $\mathrm{PW}(G, X)$ is finite, but $\mathrm{PW}(G, Y)$ is infinite?*

A difficultly here may be a shortage of known obstructions to palindromic width being finite. The quasi-morphism approach in [6, 7] does not appear to transfer readily to other groups.

Question 6.2. *Do finitely generated solvable groups of higher derived length have finite palindromic width with respect to some (or all) finite generating sets?*

The methods used in this paper for proving Theorem 1.3 have the potential to be applied to a larger class of finitely generated groups of higher derived length. In particular, one may consider generalising Lemma 4.1 to functions $f : G \to R$ where G is not abelian. For example, taking G to be polycyclic seems to be a suitable area to experiment. However, if the factor groups of the derived series of G include infinite-rank abelian groups then it is not clear whether this will be possible.

Consider a group G with finite commutator width. If, with respect to some finite generating set A, every commutator has finite palindromic length, then $\mathrm{PW}(G, A) < \infty$. After all G/G' is a finitely generated abelian group, and so modulo G' every element of G has palindromic with at most $|A|$. This approach (but specialized to particular commutators) is the basis of Bardakov & Gongopadhyay's proof of Theorem 1.3 and our proof of Theorem 1.5. It motivates:

Question 6.3 (Bardakov & Gongopadhyay [8])**.** *Does a group G have finite palindromic width with respect to some finite generating set precisely when it has finite commutator width?*

Precise values of $\mathrm{PW}(G, A)$ appear generally elusive. In the context of this paper an instance one might pursue is:

Question 6.4. *What is the palindromic width of the free metabelian group F/F'' of rank r with respect to its standard set of r generators?*

Finally we ask:

Question 6.5. *For which normal subgroups N of a finite-rank free group F does F/N having finite palindromic width imply the same of F/N'?*

In Section 4 we answered this affirmatively when $N = F'$. The elements of F/N' can be described as flows on a Cayley graph of F/N [12]. If these flows are suitably symmetric, then they determine a palindromic element of F/N'.

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