Geometric notions of space complexity for the word problem

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May 2006

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Filling length as SPACE

Γ a group with finite presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$

w a word representing 1

$\mathrm{FL}(w)$ is the minimal L such that w can be converted to the empty word ε through words of length at most L by

– applying relators

– freely reducing

– freely expanding.

Filling length function $\mathrm{FL} : \mathbb{N} \to \mathbb{N}$

$$\mathrm{FL}(n) = \max \{ \mathrm{FL}(w) \mid w = 1 \text{ in } \Gamma \text{ and } \ell(w) \leq n \}$$

Example

$\langle a, b \mid a^{-1}b^{-1}ab \rangle$

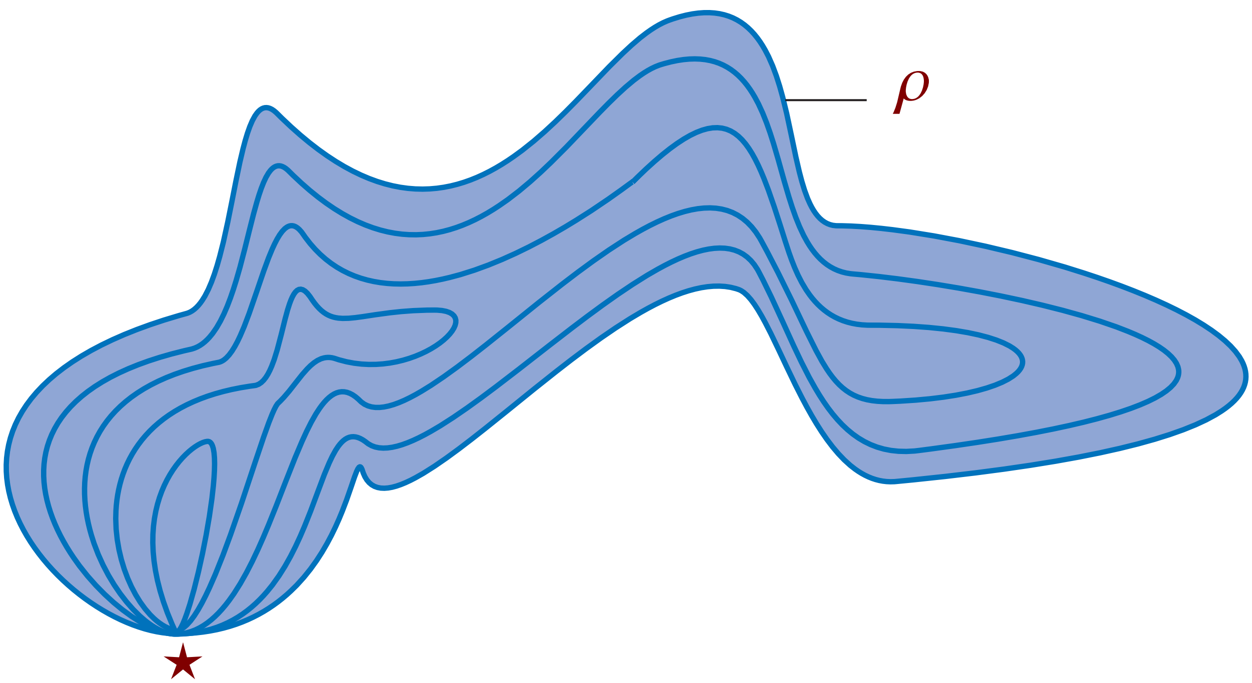
$$baba^{-2}bab^{-3}$$
$$\downarrow$$
$$baba^{-2}abb^{-3}$$
$$\downarrow$$
$$baba^{-1}b^{-1}b^{-1}$$
$$\downarrow$$
$$bb^{-1}$$
$$\downarrow$$
$$\varepsilon$$

$$\mathrm{FL}(n) \simeq n$$

Filling length via geometry

For a loop ρ in a simply connected metric space X,

$$\mathrm{FL}(\rho) = \inf \left\{ L \;\middle|\; \begin{array}{l} \exists \text{ a based null–homotopy of } \rho \\ \text{through loops of length } \leq L \end{array} \right\}$$

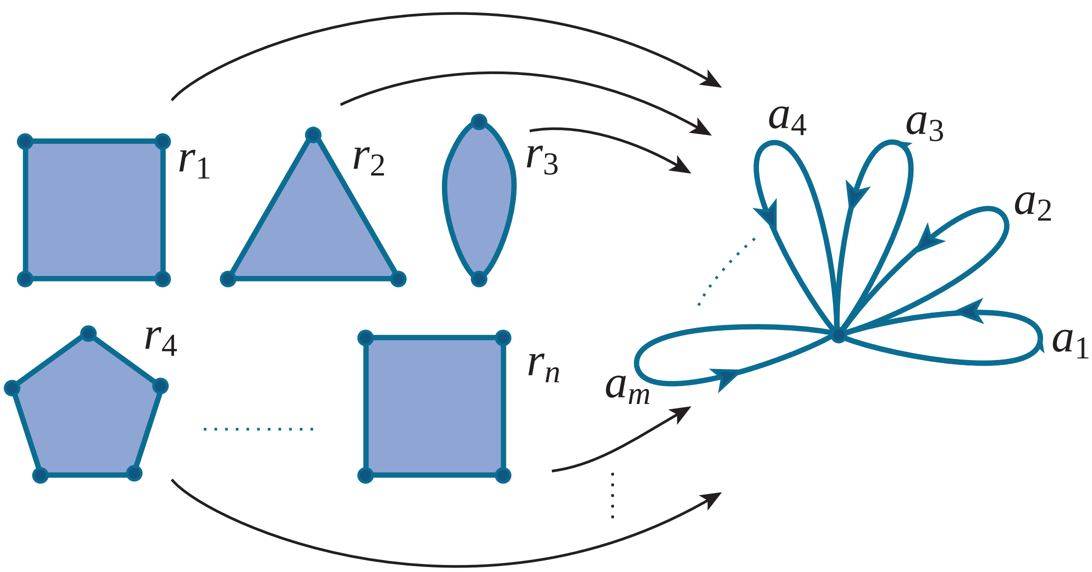


$$\mathrm{FL}(\ell) = \sup \{ \mathrm{FL}(\rho) \mid \text{loops } \rho \text{ of length at most } \ell \}$$

The **Cayley 2-complex** of

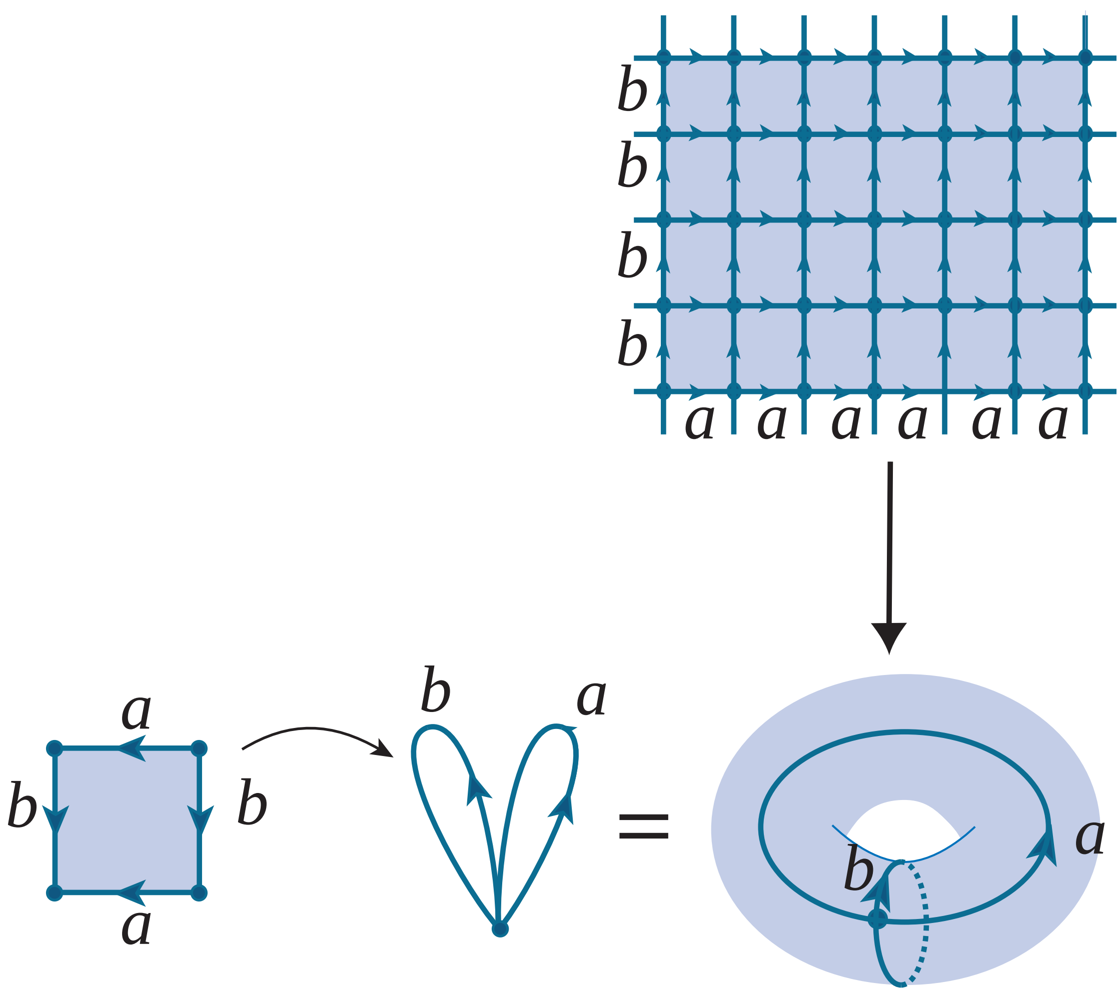
$$\langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$$

is the universal cover of



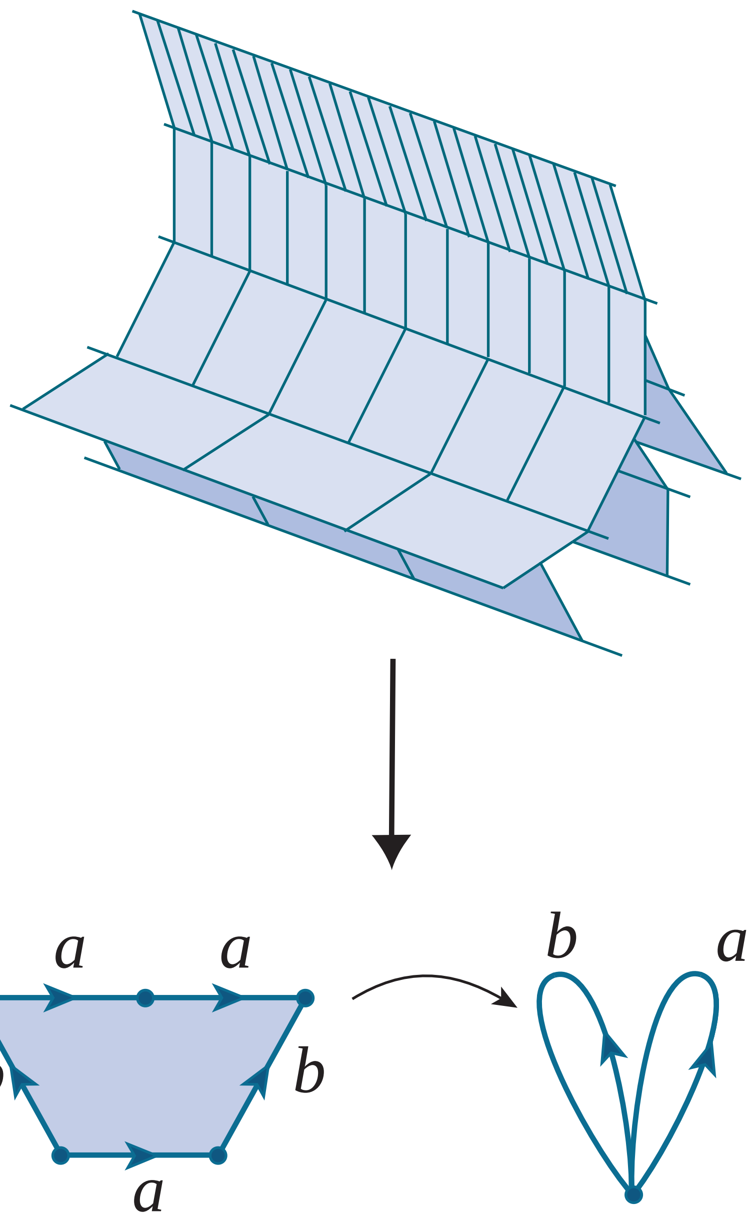
Example

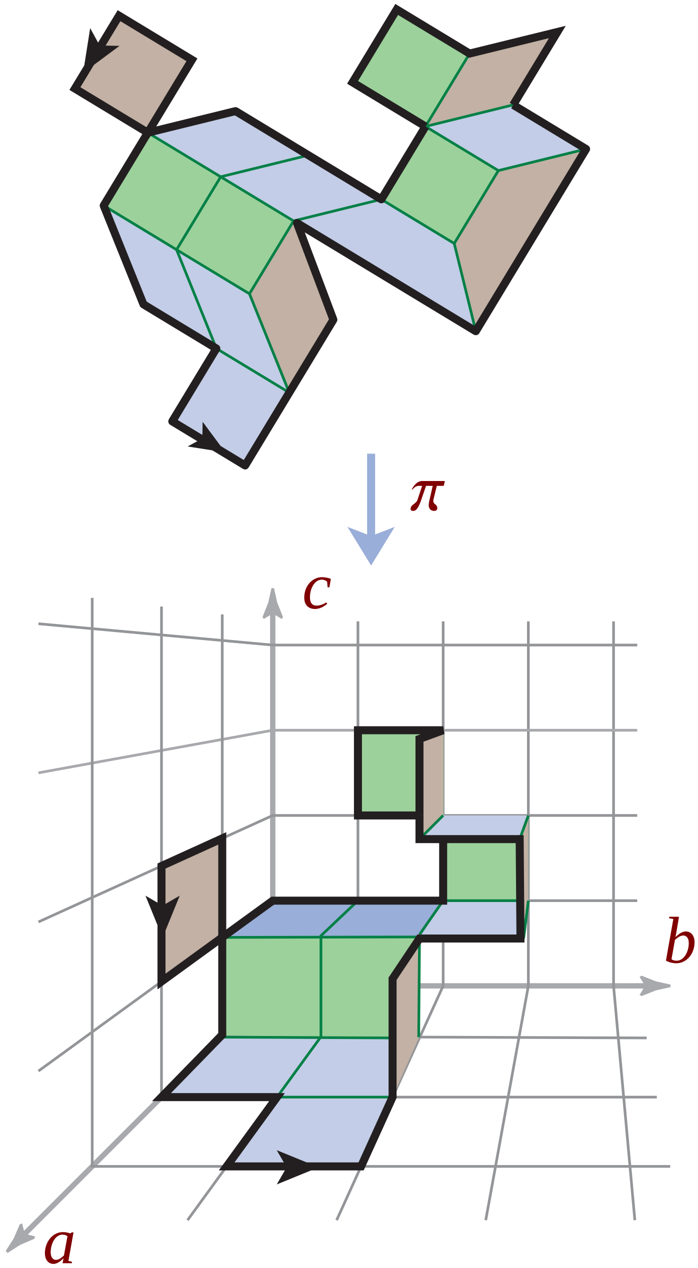
$$\langle\, a, b \;\mid\; [a, b]\,\rangle \;=\; \mathbb{Z}^2$$



Example

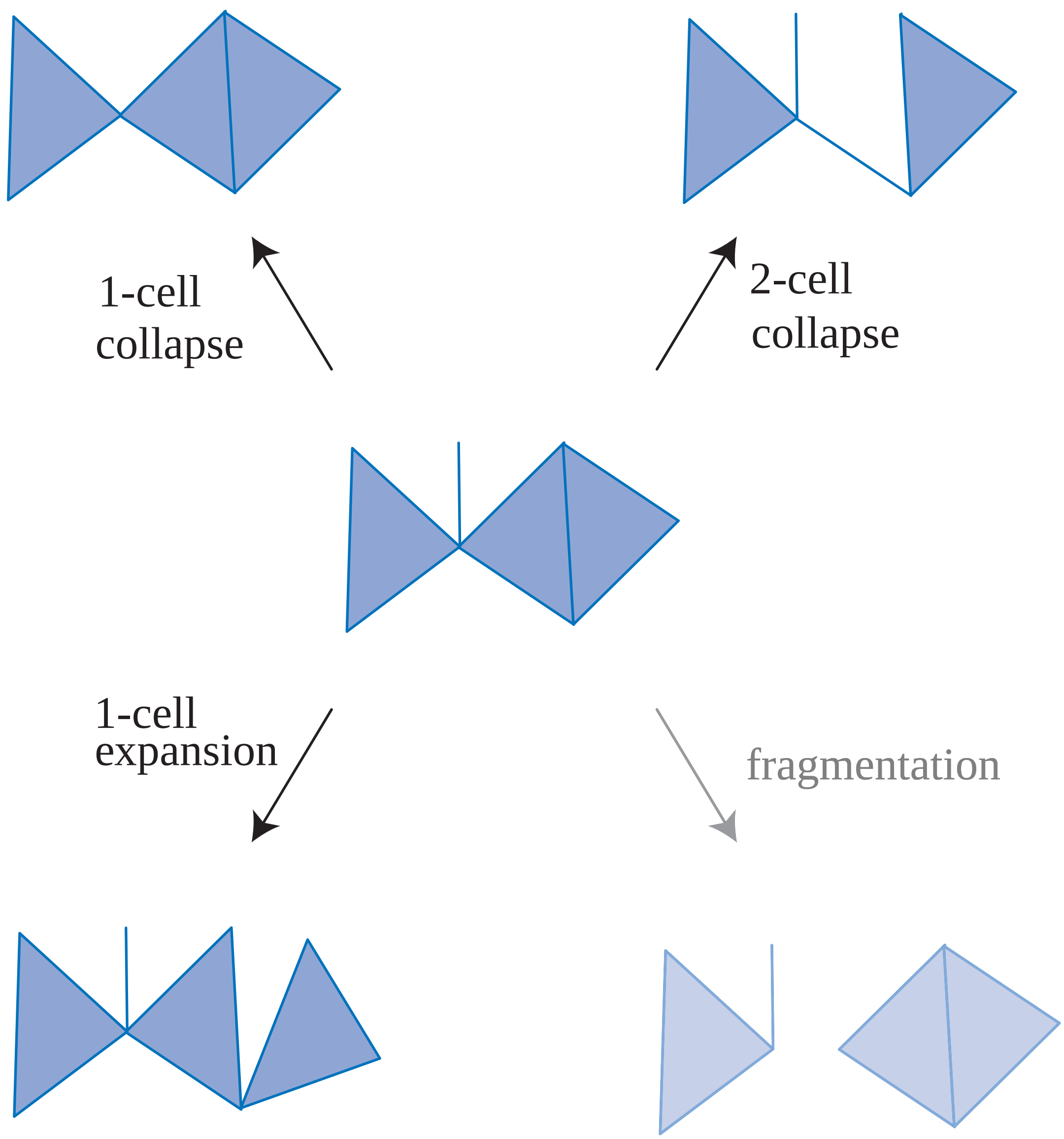
$$\langle\, a, b \mid b^{-1}ab = a^2 \,\rangle$$



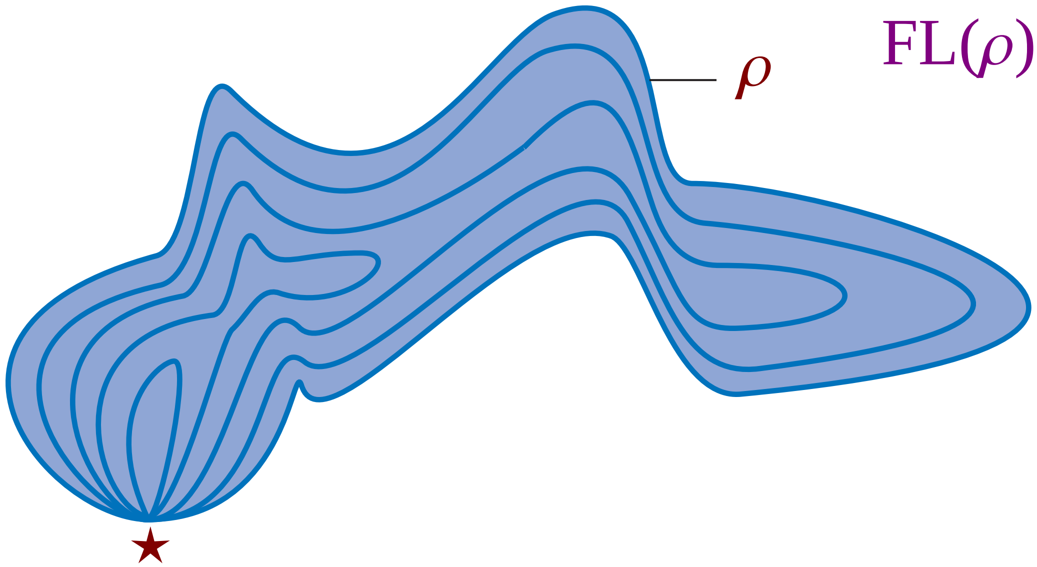


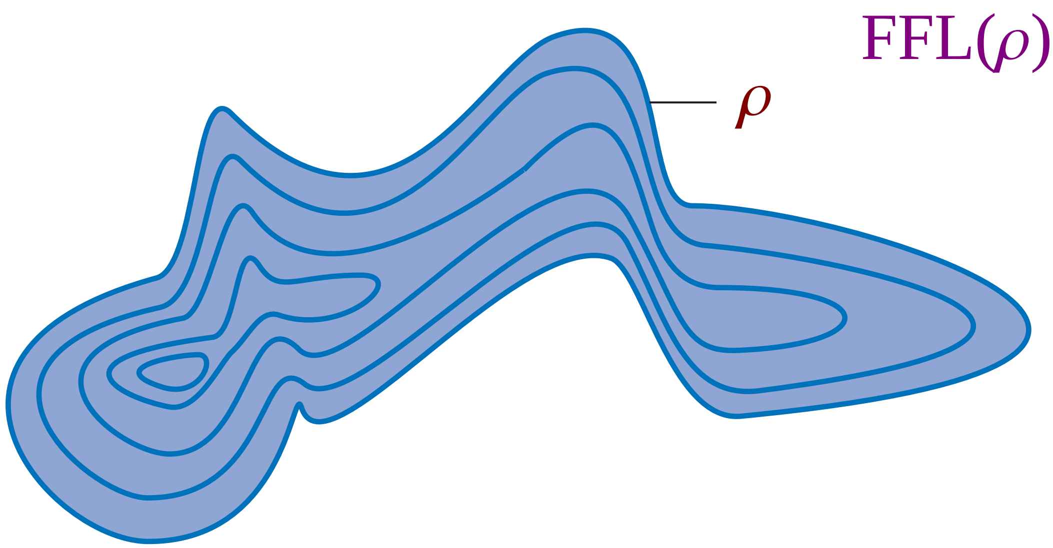
$$\mathbb{Z}^3 = \langle a, b, c \mid [a, b], [b, c], [c, a] \rangle$$

Combinatorial null-homotopy moves



Does allowing free null-homotopies change filling length?





I.e. allowing cyclic conjugation

FFL = Free filling length

Theorem. There is a finitely presented group $\mathcal{P}$ with a family of words w_n representing 1, such that

$$\begin{aligned} \ell(w_n) &\simeq n \\ \mathrm{FFL}(w_n) &\simeq n \\ \mathrm{FL}(w_n) &\simeq 2^n. \end{aligned}$$

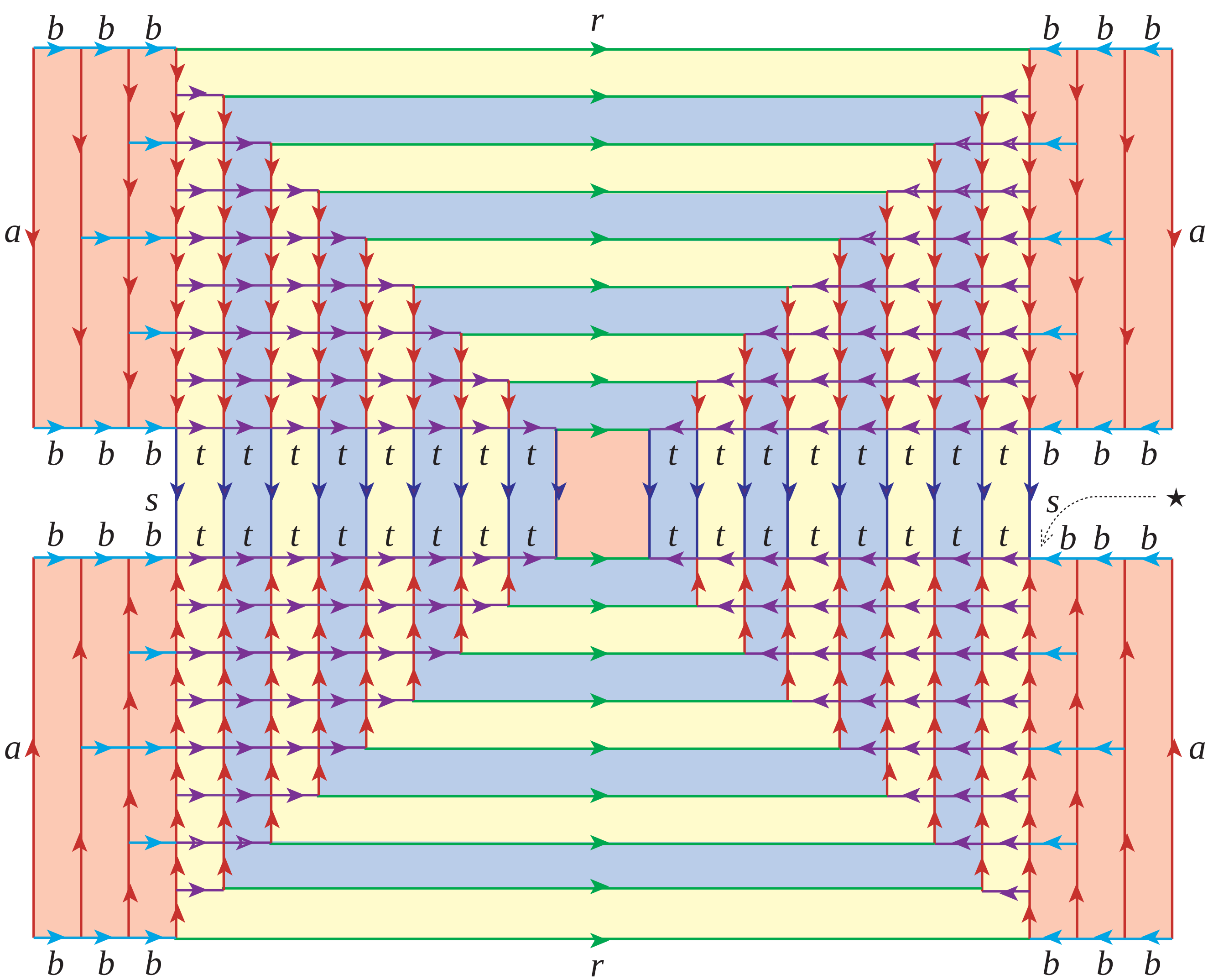
Theorem. There is a closed Riemannian manifold with a family of null-homotopic loops ρ_n such that

$$\begin{aligned} \ell(\rho_n) &\simeq n \\ \mathrm{FFL}(\rho_n) &\simeq n \\ \mathrm{FL}(\rho_n) &\simeq 2^n. \end{aligned}$$

Generators: a, b, r, s, t

Relations: $b^{-1}aba^{-2}, [t, a], [r, at], [r, s], [s, t]$

$$w_n \ := \ [s, (b^{-n}a^{-1}b^{n})r(b^{-n}ab^{n})]$$

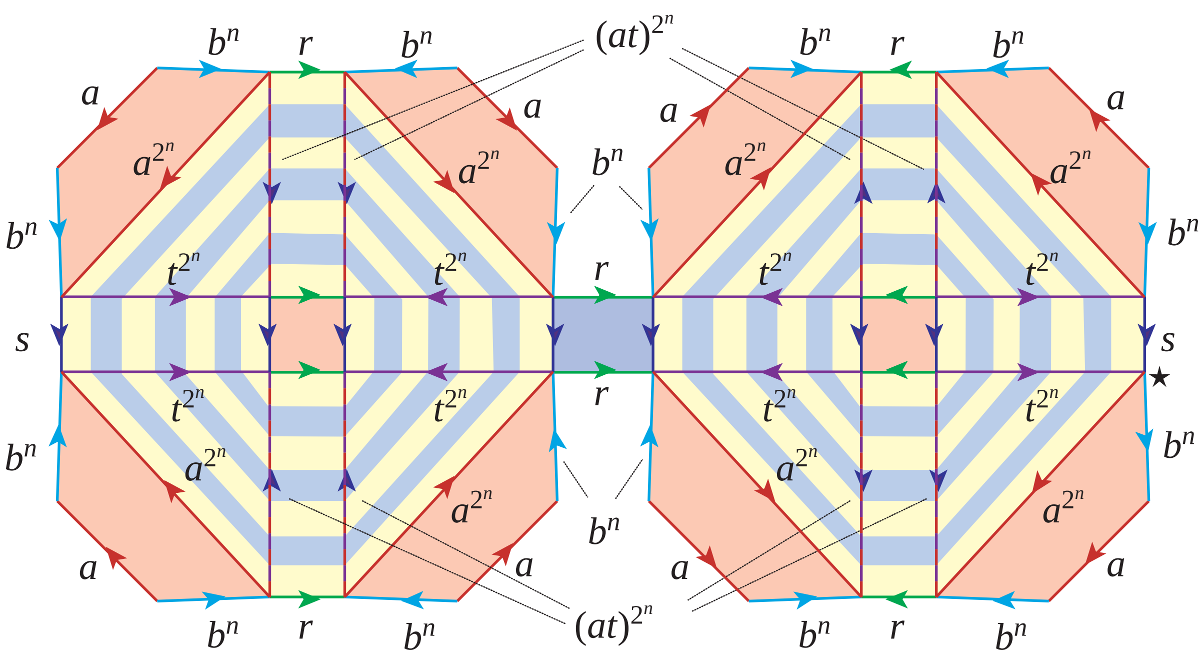


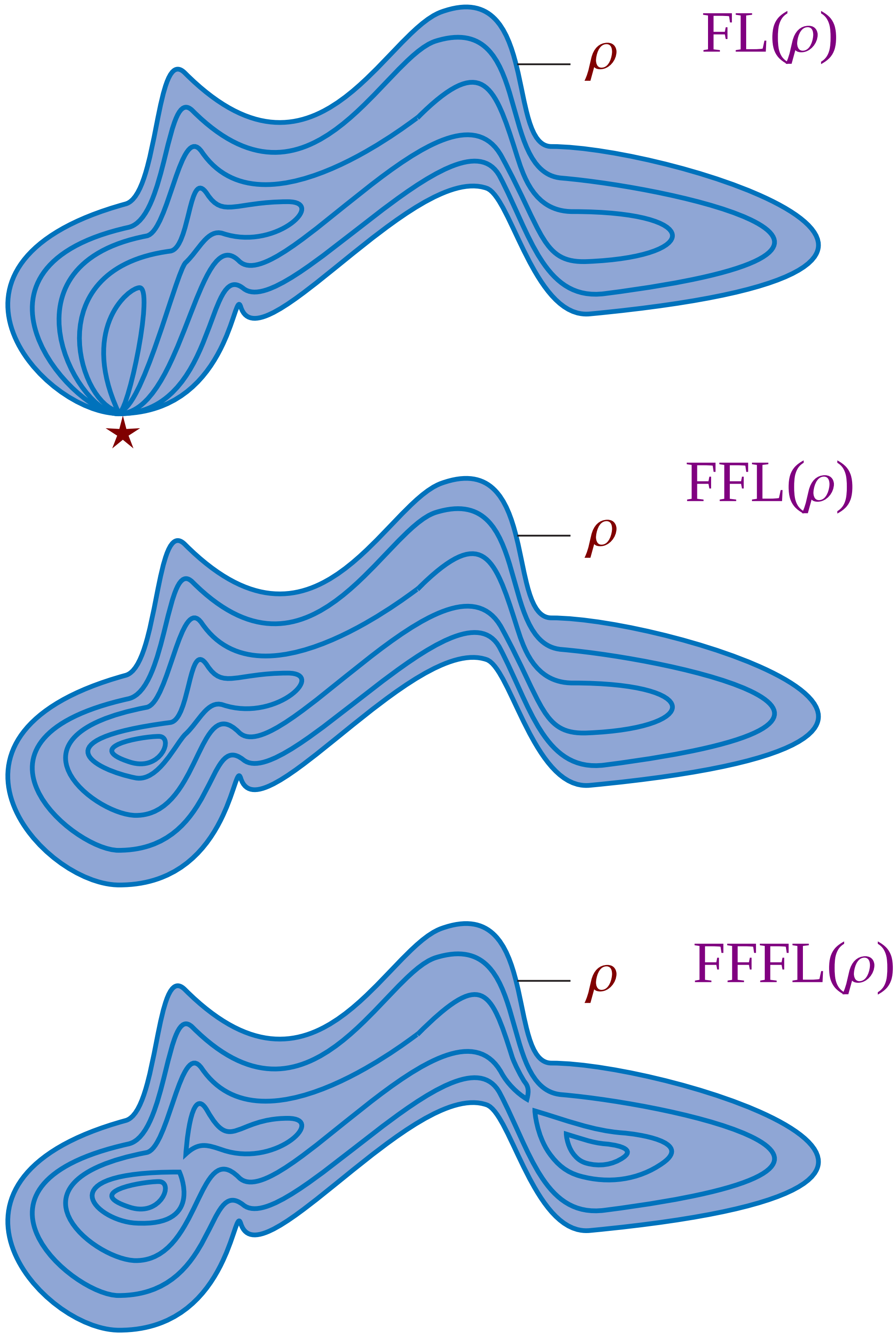
Theorem. The filling functions

$$\mathrm{FL}, \mathrm{FFL}, \mathrm{FFFL} : \mathbb{N} \to \mathbb{N}$$

for $\mathcal{P}$ satisfy

$$\mathrm{FL}(n) \simeq \mathrm{FFL}(n) \simeq 2^n$$
$$\mathrm{FFFL}(n) \simeq n.$$





FFFL = Free and fragmenting filling length

Open problem.
Does there exist a finite presentation for which $\mathrm{FL}(n) \not\simeq \mathrm{FFL}(n)$?