NUMERICALLY COMPUTING THE LYAPUNOV EXPONENTS OF MATRIX-VALUED COCYCLES

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This short note is based on a talk I gave at the student dynamical systems seminar about using your computer to figure out what the Lyapunov exponents of a matrix-valued cocycle are. I will focus only on discrete cocycles, that is, cocycles over $\mathbb{Z}$-actions. It is based on [ER85, §V.C], which also treats the continuous-time case. I tried to make it as self-contained as possible and completely accessible to graduate students of all levels.

1. Theory

Let G be "time" and $T : X \times G \to X$ a dynamical system.

Definition 1. A *linear cocycle over* T is a map $A : G \times X \to GL(n, \mathbb{F})$ which satisfies the *cocycle condition*

$$A(g_1 + g_2, x) = A(g_2, T^{g_1}(x)) \cdot A(g_1, x)$$

for all $g_1, g_2 \in G$ and $x \in X$.

Cocycles act on vector bundles: For a bundle $p : V \to X$ we have

$$T \circ p(q) = p \circ A(q)$$

for all $q \in V$. Let's look at some examples.

Example 1. The most famous and well-known cocycle is the *tangent* or *derivative* cocycle. Let $f : M \to M$ be a diffeomorphism of a manifold M. Then $Df : T_xM \to T_{f(x)}M$ is the tangent cocycle of f acting on the tangent bundle of M, TM. The cocycle condition is satisfied by the chain rule. Studying this cocycle tells you about the contracting/expanding properties of your dynamical system f.

Example 2. Another semi-famous cocycle in dynamical systems is the *Rauzy-Veech-Zorich* (RVZ) cocycle (which is a cocycle over a $\mathbb{Z}$-action) or its continuous-time version, the *Kontsevich-Zorich* (KZ) cocycle. The base dynamics of the RVZ cocycle is a translation flow on a Riemann surface (or an interval exchange transformation) while the base dynamics for the KZ cocycle is the Teichmuller flow on the moduli space of quadratic differentials for Riemann surfaces. Both cocycles act on the (co)homology bundle of Riemann surfaces. I mention this example not only to impress you, but to show you that there are other interesting bundles to consider aside from the tangent bundle. Studying these cocycles tells us much about how flows wind around flat surfaces and about the behavior of ergodic averages.

From now on we will adopt the notational convention $A_x^{(n)} = A(n, x)$.

Definition 2. Let A be a linear cocycle over $f : X \to X$ and let μ be an f-invariant probability measure. Then A is a *measurable cocycle* if

$$\log^+ \|A_x^{(\pm 1)}\| \in L^1(X, \mu),$$

where $\log^+(z) = \max\{0, z\}$.

We now state the main/best theorem about measurable cocycles.

Theorem 1 (Oseledec Multiplicative Ergodic Theorem)**.** *Let A be a measurable cocycle over f acting on the bundle V and μ and f-invariant probability measure. Then*

(1) *There is an A-invariant, pointwise decomposition of V*

$$V = X \times \bigoplus_{i=1}^{k(x)} H_i(x) \tag{1}$$

such that

(2) *The limit*

$$\lim_{n\to\pm\infty} \frac{1}{|n|} \log \frac{\|A_x^{(n)} \cdot v\|}{\|v\|} = \pm\lambda_i(x)$$

exists and is the i^{th} Lyapunov exponent *of A for any nonzero $v \in H_i(x)$ μ-almost everywhere. The dimension of the sub-bundle $H_i(x)$ equals the multiplicity of the Lyapunov exponent λ_i.*

The decomposition (1) is called the *Oseledec decomposition* of V. The Lyapunov exponents measure the infinitesimal expansion of the cocycle along a trajectory. More precisely, there is a fiber-wise splitting of the bundle V

$$p^{-1}(x) \approx \{x\} \times E^s(x) \oplus E^c(x) \oplus E^u(x)$$

where E^s, E^c, and E^u are, respectively, the *stable, center* and *unstable* bundles of A. They measure exponential contraction or expansion for the case of the stable or unstable bundles while the center bundle has some sort of intermediate behavior. If $E^c = \varnothing$ and the stable/unstable bundles do not depend on x, A is *uniformly hyperbolic.* This is a very special case, since the theorem holds only μ-almost everywhere, and we can only expect non-uniform hyperbolicity. If $E^c \neq \varnothing$ the cocycle is called *partially hyperbolic.*

Example 3. Let $X = \mathbb{R}^2/\mathbb{Z}^2$, $A \in SL(2,\mathbb{Z})$, where A has no eigenvalues of modulus 1. Since A is a linear map it is easy to see that the Oseledec decomposition is given by the eigenvectors of A and Lyapunov exponents by the eigenvalues. This system is uniformly hyperbolic.

Since in these notes we are interested in computing the Lyapunov exponents, we will sketch the proof of the second part of the theorem above. We do this because the way of proving this part gives an algorithm to compute the exponents numerically.

Sketch of proof of second part. The clever trick used to prove this is that we can perform a QR-factorization of the cocycle at every iteration. This factorization expresses a matrix M as the product of an orthogonal matrix Q and an upper-triangular matrix R. This factorization is unique if M is non-singular.

To use this for our cocycle, let $A_x^{(1)} = Q_1 \cdot R_1$. Define successively for $k > 1$

$$A'_k = A^{(1)}_{f^{k-1}(x)} Q_{k-1}$$

in order to get the next Q and R:

$$A'_k = A^{(1)}_{f^{k-1}(x)} Q_{k-1} = Q_k R_k.$$

It follows then that the cocycle is

$$A^{(n)}_x = Q_n R_n R_{n-1} \cdots R_1.$$

Since the Q_i are orthogonal (that is, you can think of them as a change of basis), you can assume that the cocycle is given by upper-triangular matrices. By the Birkhoff Ergodic Theorem,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |a_{ii}(f^k(x))| = \lambda_i(x), \tag{2}$$

where a_{ii} is the i^{th} diagonal entry of R_k, exists μ-almost everywhere. These are then the Lyapunov exponents. □

As you can see, the method of proof gives a nice algorithm to approximate the Lyapunov exponents numerically. If you want to code this, everything is figured out for you with the exception of how to perform QR factorization of matrices. There are two major ways of doing this:

Gram-Schmidt Process: You can QR-factorize matrices using this process, but it is quite numerically unstable. Even methods known as *modified* Gram-Schmidt processes perform under-par for this task: they yield matrices Q which are not really orthogonal.

Householder transformations/reflections: Although I will not go over what these are, this is the most reliable way of performing QR factorizations numerically. For more background see [TB97, §II.7]. This method of factorization is well-known and can be obtained from a number of public linear algebra tools libraries, like the Numerical Recipes libraries [PTVF07, §2.10].

2. Practice

Using the algorithms listed in the previous section we can start computing Lyuapunov exponents for as many matrix-valued cocycles as we want. We will only do it for one system here.

Consider a *three-generation Leslie model*[1], which is a non-linear version of an old model from population dynamics introduced by Leslie in the 1940's. To be honest, I am not sure what this model has been used for in the real world and/or to what degree of success, but it's fun to play with it using a computer.

This system is based on modeling a population which has three major generational changes and thus at any time the population is represented by a point in $\mathbb{R}^3_+$. The system is

$$F : (x, y, z) \mapsto \left(f(x + y + z)e^{-\lambda(x+y+z)}, p_1 x, p_2 y \right),$$

where f, λ, p_1, p_2 are some predetermined constants (the p_i is the survival rate of going from the i^{th} generation to the $(i + 1)^{st}$ generation).

[1]Specifically, the three-generation overcompensatory Leslie population model where the fertility rates decay exponentially with population size.

Fix $\lambda = 0.1$, $p_1 = 0.8$, $p_2 = 0.6$, and $f > 0$ a free parameter. This system was studied computationally in [UW04] where it was reported that this system undergoes many qualitative changes as f varies: creation and destruction of strange attractors, crises, period doubling route to chaos, et cetera. It should be pointed out that for all f there is a *global* attractor, that is, all initial conditions in $\mathbb{R}^3_+$ eventually get sucked into a compact region of $\mathbb{R}^3$. In the results below, we pick random points in $\mathbb{R}^3_+$ and compute Lyapunov exponents using the method described above. We compute Lyapunov exponents for this system for varying values of f. The results are described in the figures below.

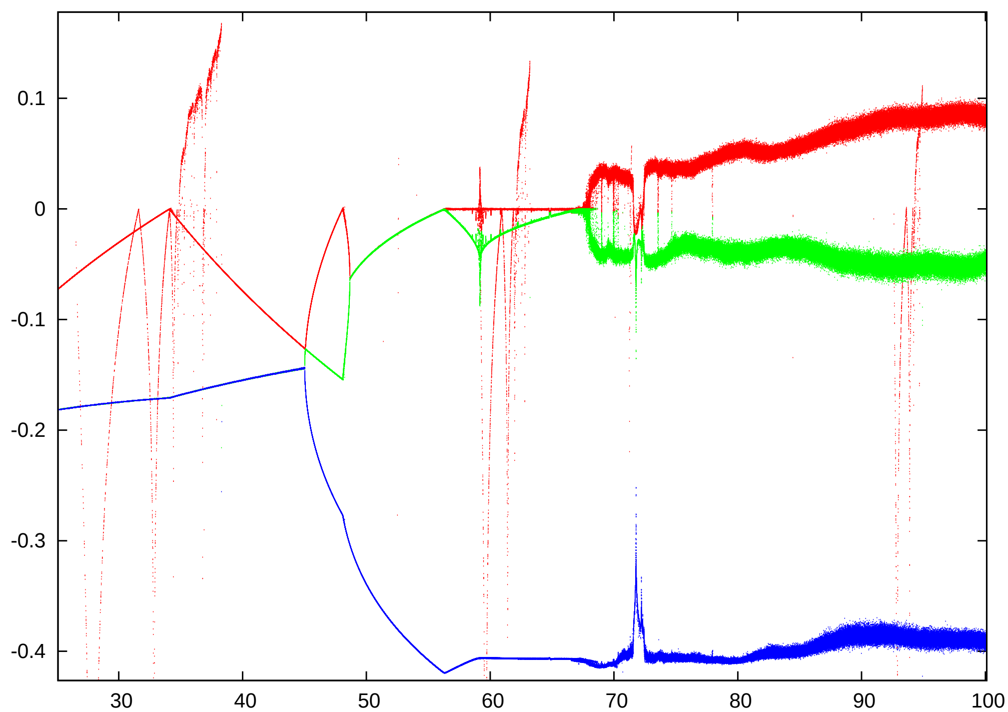

FIGURE 1. Lyapunov exponents for the three-generation Leslie model with parameters $\lambda = 0.1$, $p_1 = 0.8$, $p_2 = 0.6$, and $f > 0$ a free parameter between 25 and 100. In the three intervals where there seem to be multiple Lyapunov exponents, the lowest two Lyapunov exponents are very negative and do not appear in the plot.

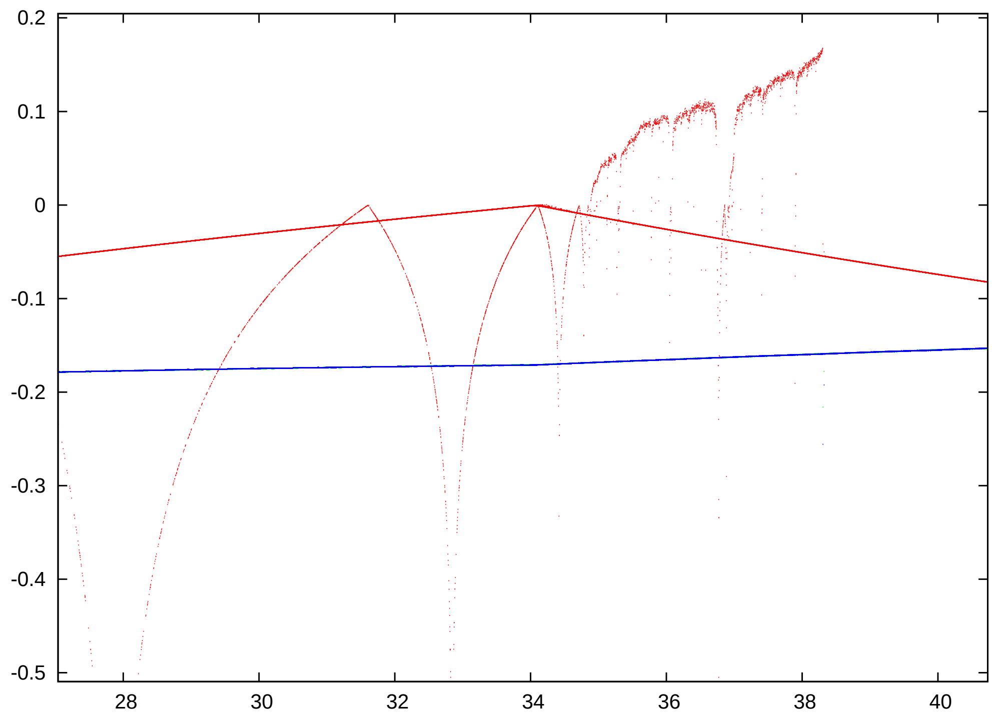

FIGURE 2. For f roughly between 27 and 41 the lowest two Lyapunov exponents are the same and thus the spectrum of the cocycle is not simple. For some values of f the highest exponent seems to oscilate and is positive for some f (thus $E^u \neq \varnothing$ for these values). The oscilation seems to be due to the existence of multiple attractors.

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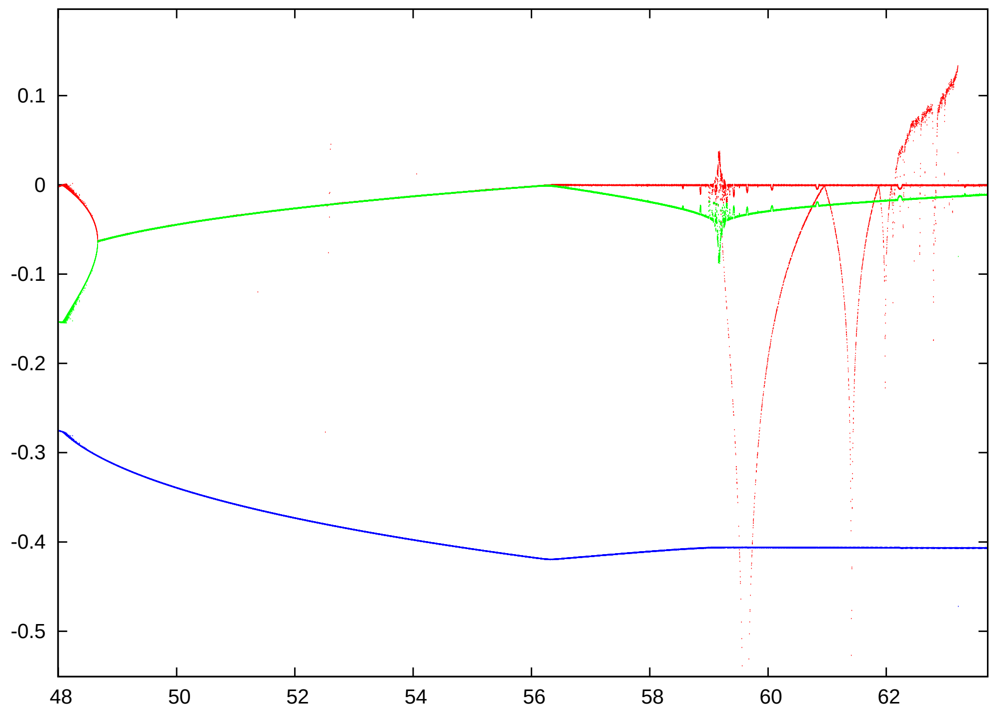

FIGURE 3. The top two Lyapunov exponents merge at first and then diverge.

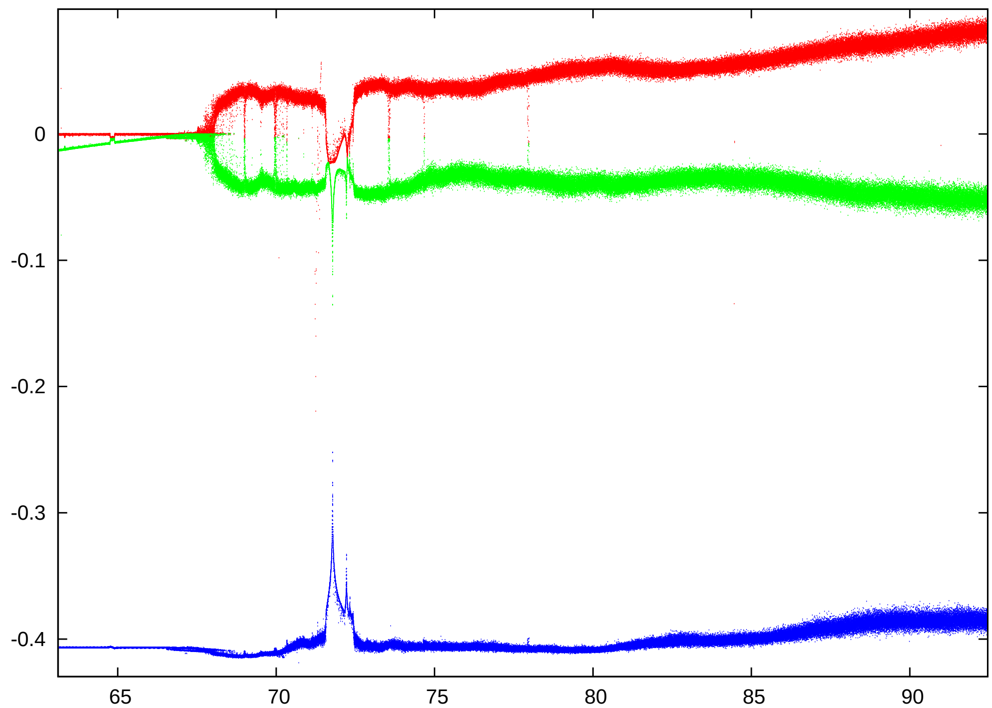

FIGURE 4. The Lyapunov exponents go from having one value to many. My guess is that for these values of f there are very many attractors each with different Lyapunov exponents.