# 4130 HOMEWORK 2

**Due Thursday February 11**

(1) In this exercise, we will show that the ordered field $\mathbb{Q}$ is not complete.

   (a) Suppose $q \in \mathbb{Q}$ and $q^2 < 2$. Let $n \in \mathbb{N}$ and suppose that $n > \max\{\frac{2|q|+1}{2-q^2}, 1\}$. Show that $(q + \frac{1}{n})^2 < 2$.

   (b) Suppose $r \in \mathbb{Q}$ and $r^2 > 2$. Let $n \in \mathbb{N}$ and suppose $n > \frac{2r}{r^2-2}$. Show that $(r - \frac{1}{n})^2 > 2$.

   (c) Using the results of (a) and (b), together with the fact that there is no $s \in \mathbb{Q}$ with $s^2 = 2$ (do not prove this), show that $\mathbb{Q}$ is not complete. (Hint: show that $S = \{x \in \mathbb{Q} : 0 < x^2 < 2\}$ is bounded above but has no least upper bound.)

(2) By looking in some books or on the internet, find two examples of ordered fields other than $\mathbb{Q}$ and $\mathbb{R}$, including one which does not satisfy the archimedean property.

(3) Show that any convergent sequence of rational numbers has a unique limit.

(4) Section 2.1.3 # 1.

(5) Section 2.1.3 # 8.