4130 HOMEWORK 5

Due Tuesday March 9

(1) A subset I of $\mathbb{R}$ is called an *interval* if for all $x, y \in I$ and all $z \in \mathbb{R}$, if $x < z < y$ then $z \in I$.

Show that if I is a bounded interval, then $(\inf I, \sup I) \subset I$. Using this, show that I must be one of the following four intervals:

$$(\inf I, \sup I) \qquad [\inf I, \sup I) \qquad (\inf I, \sup I] \qquad [\inf I, \sup I].$$

(2) For each of the following $S \subset \mathbb{R} \cup \{\pm\infty\}$, state whether there is a sequence whose set of limit-point is S. If there is, find one. If not, give a reason.

(a) $S = \{0\}$.

(b) $S = \{\infty, -\infty\}$.

(c) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$

(d) $S = \mathbb{N}$.

(e) $S = \mathbb{N} \cup \{\infty\}$.

(3) Let $\mathcal{U}$ be the following collection of subsets of $\mathbb{R}$.

$$\mathcal{U} = \{(q, r) : q, r \in \mathbb{Q}, q < r\}.$$

(a) Show that $\mathcal{U}$ is countable.

(b) Show that every open set can be expressed as a union of intervals from $\mathcal{U}$.

(c) Let $\mathcal{X}$ denote the set of all open subsets of $\mathbb{R}$. Show that $|\mathcal{X}| = |\mathbb{R}|$. (Hint: recall that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.)

(d) Show that the set of all *closed* subsets of $\mathbb{R}$ also has cardinality $|\mathbb{R}|$.

[Remark: Let $\mathcal{Y}$ denote the set of all subsets of $\mathbb{R}$ that are either open or closed. One can now show that $|\mathcal{Y}| < |\mathcal{P}(\mathbb{R}) \setminus \mathcal{Y}|$. In other words, loosely speaking, "most" subsets of $\mathbb{R}$ are neither open nor closed.]