4130 HOMEWORK 6

Due Thursday April 1

(1) Let $A \subset \mathbb{R}$. A point $x \in A$ is called *isolated* if it is not a cluster point of A.
  (a) Can an open set have an isolated point? Can a closed set have one?
  (b) Give an example of a countable set with no isolated points.

(2) Section 3.3.1 # 8.

(3) Section 4.2.4 # 3. (Recall that an *interval* is, by definition, a subset I of $\mathbb{R}$ such that for all $x, y \in I$ and all $z \in \mathbb{R}$ with $x < z < y$, we have $z \in I$.)

(4) In this question, we will show that every positive real number has an n^{th} root.
  (a) Let $x \in (0, \infty)$ and $n \in \mathbb{N}$. Show that there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha^n < x < \beta^n$.
  (b) Show that there exists $y \in \mathbb{R}$ with $x = y^n$.
  (c) For $x \in [0, \infty)$, show that there exists a *unique* $y \in [0, \infty)$ with $x = y^2$. We denote this y by $\sqrt{x}$.
  (d) Define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Show that f is a continuous function.

(5) Two monasteries, A and B, are joined by exactly one path AB which is 20 miles long. One morning, Brother Albert (a monk) sets out from monastery A at 9 am, arriving at monastery B at 9 pm. The next day, he sets out from monastery B at 9 am, arriving at monastery A at 9 pm. On both journeys, he may have stopped to rest, or even walked backwards for some of the time.
  (a) Prove that there is a point x on the path AB such that Brother Albert was at x at exactly the same time on both days. (Hint: let $f_i(t)$ be the distance of Brother Albert from A at time t on day i, $i = 1, 2$. Apply the Intermediate Value Theorem to a suitable combination of f_1 and f_2.)
  (b) Another monk, Brother Gilbert, has been dabbling in forbidden knowledge. Once per day, by snapping his fingers, he can instantaneously teleport himself to any point within a 3 ft. radius of his current location. Suppose Brother Gilbert makes the same journey as Brother Albert. Does the conclusion from part (a) still hold?

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $a > 0$. We say that f is *periodic with period a* if

$$f(x + a) = f(x)$$

for all $x \in \mathbb{R}$.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is periodic with period a and define

$$g(x) = f(1/x)$$

for $x > 0$.

(a) Show that for all $x > 0$, we have

$$f([x, x + a]) = g\left(\left[\frac{1}{x + a}, \frac{1}{x}\right]\right).$$

(b) Suppose f is not constant. Show that g is not uniformly continuous on $(0, \infty)$. [**Remark:** In particular, taking $f(x) = \sin(x)$ and $a = 2\pi$, we see that $\sin(1/x)$ is not uniformly continuous.]