4130 PRELIM 2: TAKE-HOME

Due Tuesday April 20: no extensions. *This is an exam. Unlike a normal homework, you are not allowed to work on these problems in groups. However, do not feel that you have to work in complete isolation: you may discuss the problems with the lecturer or the TA if you need help, or if you find the wording of the questions ambiguous. You are also free to use the textbook, lecture notes, and previous homeworks, but you are not supposed to use any other references.*

(1) Let $A \subset \mathbb{R}$. A function $f : A \to \mathbb{R}$ is said to satisfy a *Lipschitz condition* on A if there exists $M \in \mathbb{R}$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in A$.

(a) Show that if f satisfies a Lipschitz condition, then f is uniformly continuous.

(b) Show that the function $f(x) = \sqrt{|x|}$ with domain $[-1, 1]$ is uniformly continuous but does not satisfy a Lipschitz condition.

(2) **[Extra credit.]** Recall that for A, B subsets of $\mathbb{R}$, we say that A is *dense* in B if $A \subset B$ and B is a subset of the closure of A.

(a) Give an example of a set A such that $A \cap \mathbb{Q}$ is not dense in A.

(b) Show that $A \subset B$ is dense in B if and only if for every open set U with $U \cap B \neq \varnothing$, we have $U \cap A \neq \varnothing$.

(c) Let $\xi \in \mathbb{R}$. Show that the set

$$\mathbb{Q}\xi = \{q\xi : q \in \mathbb{Q}\}$$

is dense in $\mathbb{R}$.

(d) Show that every $A \subset \mathbb{R}$ has a countable dense subset.

[TURN OVER.]

(3) In this problem, we will show that the Lagrange form of the remainder term in Taylor's Theorem can be derived from the integral form of the remainder.

(a) Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. Suppose $g(x) \geq 0$ for all $x \in [a, b]$, and that $\int_a^b g(x)dx \neq 0$. Show that there exists $t \in [a, b]$ with

$$\int_a^b f(x)g(x)dx = f(t) \int_a^b g(x)dx.$$

(b) Let $x_0, x \in \mathbb{R}$. Show that there exists a number t between x and x_0 such that

$$\frac{1}{n!} \int_{x_0}^x f(u)(u - x_0)^n du = \frac{f(t)}{(n+1)!}(x - x_0)^{n+1}.$$

(4) For $f : [a, b] \to \mathbb{R}$ and $P = \{x_0, \ldots, x_n\}$ a partition of $[a, b]$, define

$$V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

We say that f has *bounded variation* on $[a, b]$ if there exists a number L such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$, $|V(f, P) - L| < \varepsilon$. In this case, we call L the *first-order variation* of f on $[a, b]$ and denote it by $V_a^b(f)$.

(a) If f is monotone increasing on $[a, b]$, show that $V_a^b(f) = f(b) - f(a)$.

(b) Give an example of a function $f : [0, 1] \to \mathbb{R}$ which does not have bounded variation.

(c) Suppose f is a C^1 function on some open set $U \supset [a, b]$. Show that f has bounded variation on $[a, b]$ and

$$V_a^b(f) = \int_a^b |f'(x)|dx.$$