4130 HOMEWORK 2

Due Thursday February 11

(1) In this exercise, we will show that the ordered field $\mathbb{Q}$ is not complete.

(a) Suppose $q \in \mathbb{Q}$ and $q^2 < 2$. Let $n \in \mathbb{N}$ and suppose that $n > \max\{\frac{2|q|+1}{2-q^2}, 1\}$. Show that $(q + \frac{1}{n})^2 < 2$.

If $n > \max\{\frac{2|q|+1}{2-q^2}, 1\}$ then $n > \frac{2|q|+1}{2-q^2}$, so $2 - q^2 > \frac{2|q|+1}{n}$ and thus $2 > q^2 + \frac{2|q|+1}{n}$.

We therefore have:

$$\begin{aligned}
(q + \frac{1}{n})^2 &= q^2 + \frac{2q}{n} + \frac{1}{n^2} \\
&< q^2 + \frac{2q}{n} + \frac{1}{n} \qquad \text{because } n > 1. \\
&\leq q^2 + \frac{2|q| + 1}{n} \qquad \text{because } q \leq |q|. \\
&< 2.
\end{aligned}$$

(b) Suppose $r \in \mathbb{Q}$ and $r^2 > 2$. Let $n \in \mathbb{N}$ and suppose $n > \frac{2r}{r^2-2}$. Show that $(r - \frac{1}{n})^2 > 2$.

Since $n > \frac{2r}{r^2-2}$, we have $r^2 - 2 > \frac{2r}{n}$ and so $r^2 - \frac{2r}{n} > 2$.

We therefore have:

$$\begin{aligned}
(r - \frac{1}{n})^2 &= r^2 - \frac{2r}{n} + \frac{1}{n^2} \\
&> r^2 - \frac{2r}{n} \\
&> 2.
\end{aligned}$$

(c) Using the results of (a) and (b), together with the fact that there is no $s \in \mathbb{Q}$ with $s^2 = 2$ (do not prove this), show that $\mathbb{Q}$ is not complete. (Hint: show that $S = \{x \in \mathbb{Q} : 0 < x^2 < 2\}$ is bounded above but has no least upper bound.)

It is clear that S is bounded above, for example by $b = 2$. We must show that there is no *least* upper bound.

Suppose for a contradiction that $b \in \mathbb{Q}$ is the least upper bound for S. Then $b^2 \neq 2$ by the hint, so either $b^2 < 2$ or $b^2 > 2$. Suppose $b^2 < 2$. Then by (a), there exists $n \in \mathbb{N}$ with $(b + \frac{1}{n})^2 < 2$. Thus, $b < b + \frac{1}{n} \in S$, which contradicts that b is an upper bound for S.

Therefore, we must have $b^2 > 2$. But then, by (b) there exists $n \in \mathbb{N}$ with $(b - \frac{1}{n})^2 > 2$.

We claim that $b - \frac{1}{n}$ is an upper bound for S. Indeed, if $x \in S$ and $x > b - \frac{1}{n}$ then $x^2 > (b - \frac{1}{n})^2 > 2$, which contradicts that $x \in S$. Therefore, if $x \in S$ then $x \leq b - \frac{1}{n}$, and so $b - \frac{1}{n}$ is an upper bound for S.

But $b - \frac{1}{n} < b$, which contradicts that b is the *least* upper bound.

This contradiction shows that S has no least upper bound.

(2) By looking in some books or on the internet, find two examples of ordered fields other than $\mathbb{Q}$ and $\mathbb{R}$, including one which does not satisfy the archimedean property.

There are many possible examples. An ordered field which is not $\mathbb{R}$ or $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Another example, which is not archimedean, is the field $\mathbb{Q}(x)$ of rational functions in a variable x, equipped with an appropriate ordering.

(3) Show that any convergent sequence of rational numbers has a unique limit.

Let $\{x_n\}$ be a sequence of rational numbers and suppose that $\{x_n\}$ converges to $L \in \mathbb{Q}$ and to $M \in \mathbb{Q}$. We must show that $L = M$.

We have the following facts:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N |x_n - L| < \varepsilon.$$

$$\forall \varepsilon > 0 \exists N' \in \mathbb{N} \forall n > N' |x_n - M| < \varepsilon.$$

Let $\varepsilon > 0$. Choose $N_1 \in \mathbb{N}$ such that if $n > N_1$, we have $|x_n - L| < \varepsilon/2$. Choose $N_2 \in \mathbb{N}$ such that if $n > N_2$, we have $|x_n - M| < \varepsilon/2$. Now suppose $n > N_1, N_2$. Then

$$|L - M| = |L - x_n + x_n - M| \leq |x_n - L| + |x_n - M| < \varepsilon$$

by the triangle inequality.

This shows that for all $\varepsilon > 0, |L - M| < \varepsilon$. Thus, we must have $|L - M| = 0$, so $L = M$.

(4) Section 2.1.3 # 1.

Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{Q}$. Our task is to show that there are uncountably many Cauchy sequences equivalent to $\{x_n\}$. Let S be the set of Cauchy sequences equivalent to $\{x_n\}$. It suffices to give an injection $f : \mathcal{P}(\mathbb{N}) \to S$.

Given a subset A of $\mathbb{N}$, define a sequence $\{y_n\}$ by

$$y_n = \begin{cases} x_n + \frac{1}{n} & n \in A \\ x_n & n \notin A. \end{cases}$$

We show that $\{y_n\}$ is a Cauchy sequence. Given $\varepsilon > 0$, choose $N_1 \in \mathbb{N}$ such that if $m, n > N_1$ then $|x_m - x_n| < \varepsilon/3$. Now suppose $N > \max\{N_1, 3/\varepsilon\}$. Then if $m, n > N$, we have:

$$|y_m - y_n| \le |x_m - x_n| + \frac{1}{n} + \frac{1}{m} < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

which shows that $\{y_n\}$ is Cauchy.

Now we show that $\{y_n\} \sim \{x_n\}$. We have $|y_n - x_n| \le \frac{1}{n}$. Thus, given $\varepsilon > 0$, if we choose $N > 1/\varepsilon$, then for $n > N$ we will have $|y_n - x_n| \le \frac{1}{n} < \varepsilon$, which shows that $\{y_n\} \sim \{x_n\}$.

Thus, we have a well-defined function $f : \mathcal{P}(\mathbb{N}) \to S$, defined by $f(A) = \{y_n\}$. We show that f is injective. Indeed, if $A \neq B$ are subsets of $\mathbb{N}$, then either $A \setminus B \neq \varnothing$ or $B \setminus A \neq \varnothing$. Suppose $A \setminus B \neq \varnothing$. Then if $n \in A \setminus B$, the n^{th} term of $f(A)$ is $x_n + \frac{1}{n}$ and the n^{th} term of $f(B)$ is x_n, so $f(A) \neq f(B)$. Similarly, if $B \setminus A \neq \varnothing$ then $f(A) \neq f(B)$. Thus, f is one-to-one.

(5) Section 2.1.3 # 8.

It is possible to have a Cauchy sequence of negative rationals which is equivalent to a Cauchy sequence of positive rationals. An example is

$$\{\frac{1}{n}\} \sim \{\frac{-1}{n}\}.$$

Both of these sequences converge to 0, so they are Cauchy. They are equivalent because, given $\varepsilon > 0$, we can choose $N > 2/\varepsilon$, and then for $n > N$ we have $|1/n - (-1/n)| = 2/n < \varepsilon$.